UNIVERSITY of RWANDA

COLLEGE OF SCIENCE AND TECHNOLOGY

School of Engineering

Civil, Environmental & Geomatics Engineering (CEGE)

Course Lecturer: Ir. Philbert Habimana (MSc. & BSc.)

Course code: TRE1162

Course name: MECHANICS OF MATERIALS

UNIVERSITY of RWANDA

COLLEGE OF SCIENCE AND TECHNOLOGY

School of Engineering

CEGE

Lecture xviii – *Deflection of beams*

UNIVERSITY of RWANDA

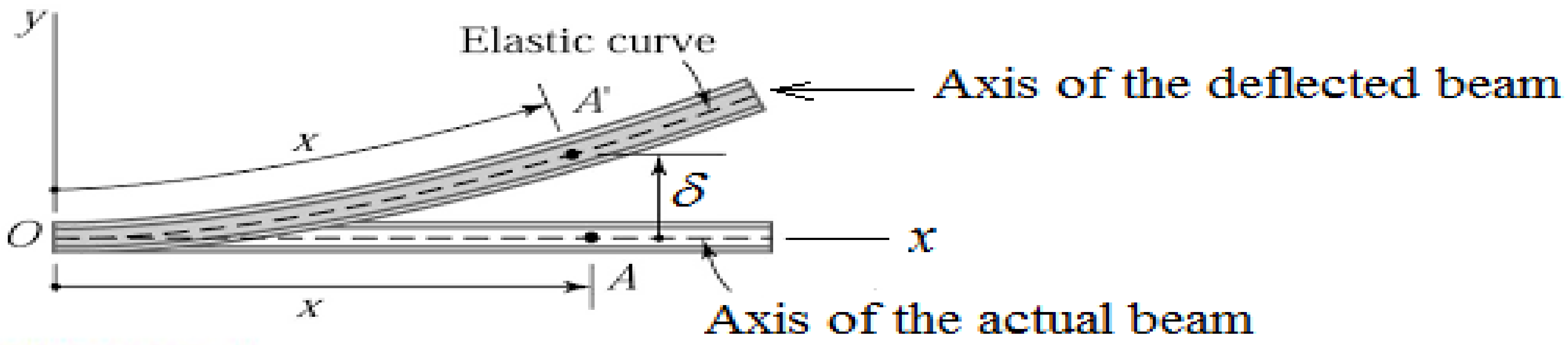
COLLEGE OF SCIENCE AND TECHNOLOGY

School of Engineering

CEGE

xviii. 1. INTRODUCTION

- The deformation of a beam is usually expressed in terms of its **deflection** from its original unloaded position.
- The **deflection** is measured from the **original neutral axis** surface of the beam to the **neutral surface** of the **deformed beam.**
- The **configuration** assumed by the **deformed neutral surface** is known as the **elastic curve** of the beam.



xviii. 1. INTRODUCTION

- **Slope of a beam:** It is the **angle** between the **deflected beam** to the **actual beam** at the same point.
- **Deflection of a beam:** It is defined as the **vertical displacement** of a point on a loaded beam.
- The **maximum deflection** occurs where the slope is **zero**.
- **Curvature** κ is a **measure** of **how sharply** a beam is **bent**, and it is related to the **radius of curvature** R by $\kappa = \frac{1}{R}$.

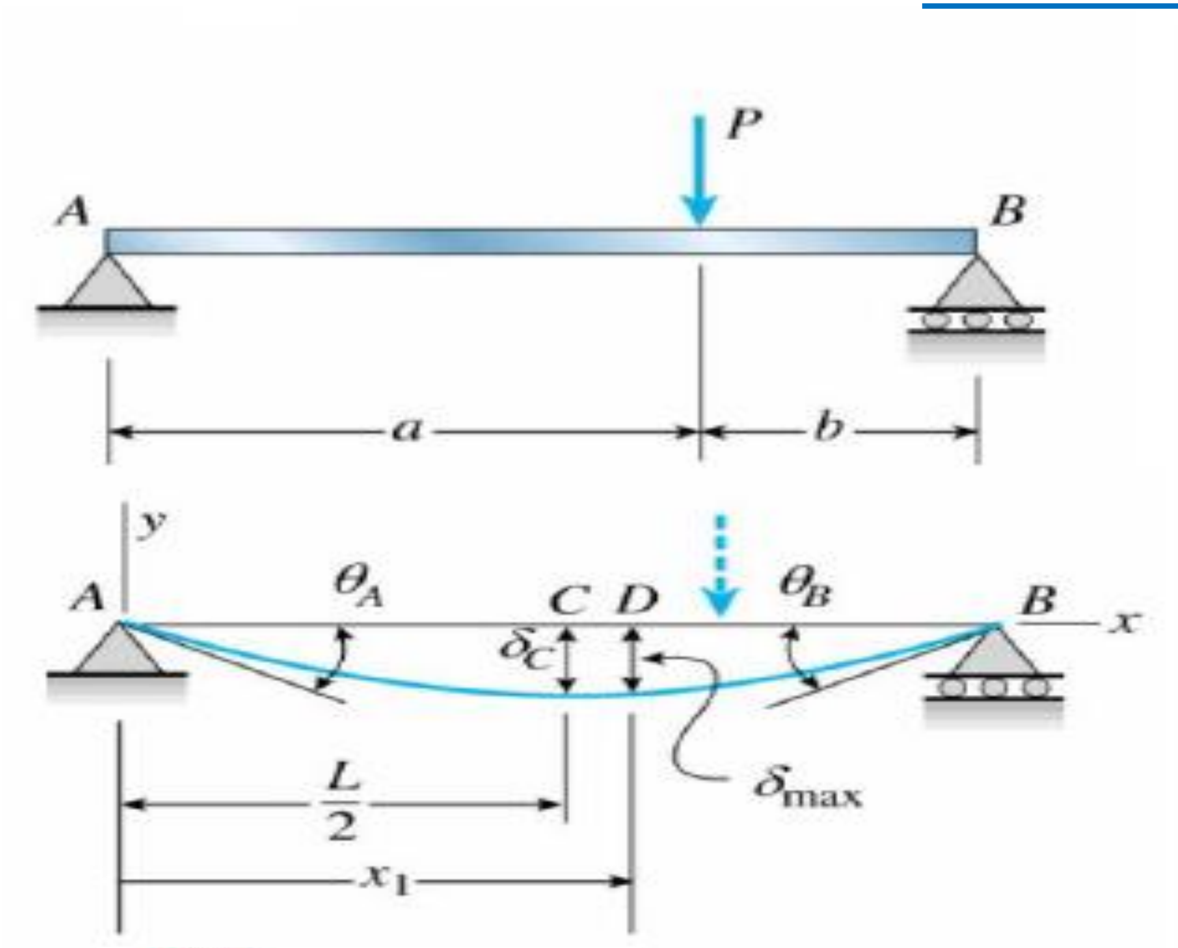
xviii. 1. INTRODUCTION

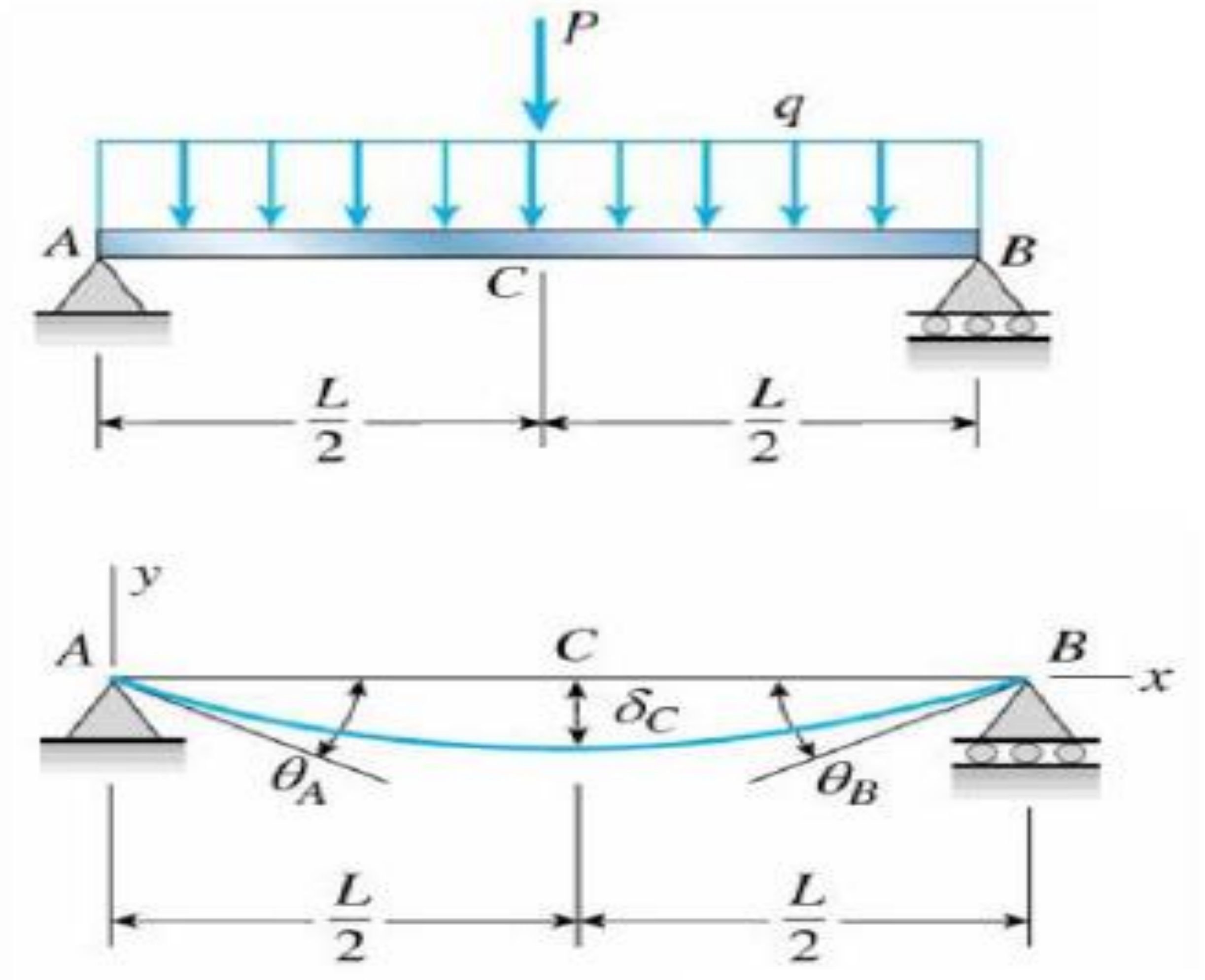
- The position of the maximum deflection is found out by equating the slope equation to zero. Then the value of x is substituted in the deflection equation to calculate the maximum deflection.

- Stiffness is the resistance of an elastic body to deflection or deformation by an applied force. It can be expressed as $\mathrm{k} = \frac{P}{\delta}$

  Where k is the stiffness (N/m)
  P is the applied force (load) (N)
  δ is the deflection (m)

xviii. 1. INTRODUCTION





xviii. 2. RELATIONSHIPS BETWEEN LOADING, SF, BM, SLOPE AND DEFLECTION OF A BEAM

- The following relationships exist between loading, shearing force (SF), bending moment (BM), slope and deflection of a beam.

Deflection is represented by $\boldsymbol{y}$, ν or $\boldsymbol{\delta}$

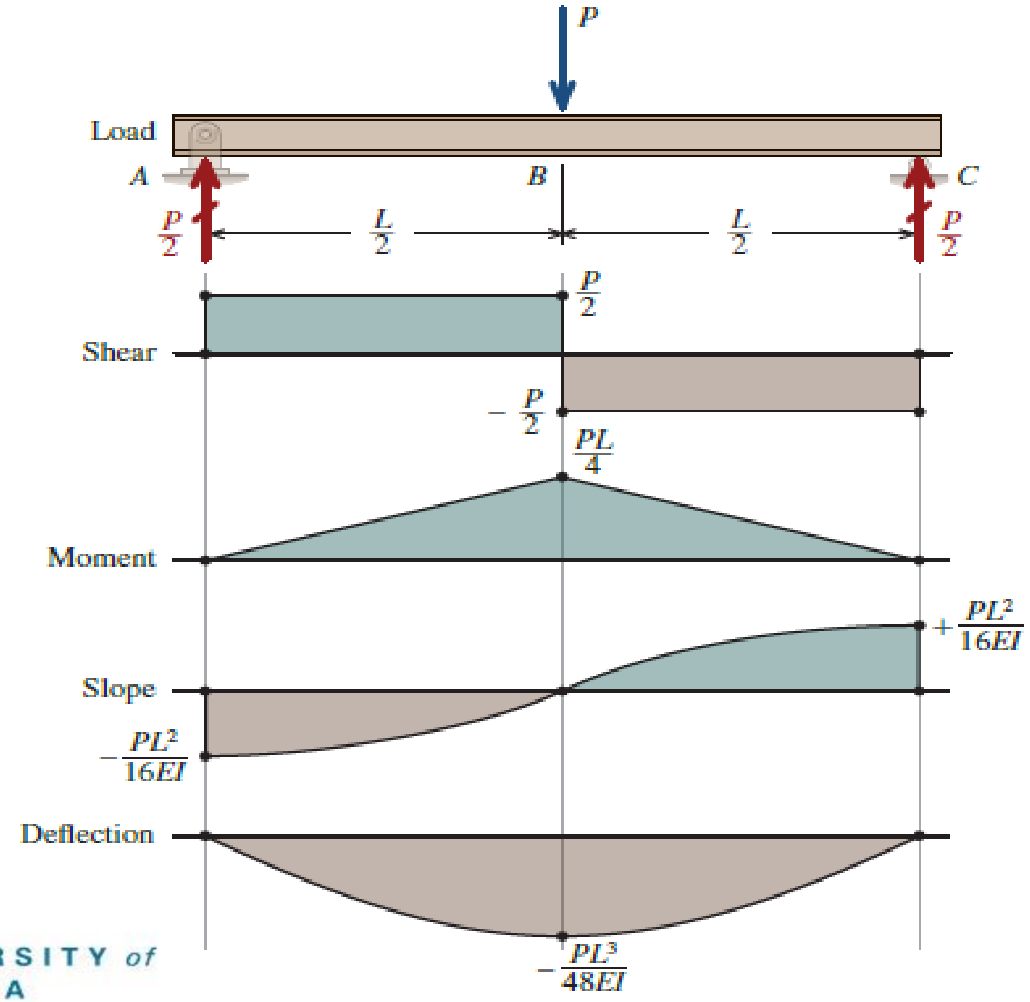
Slope is represented by I or $\boldsymbol{\theta = \frac{dy}{dx}}$

Bending moment is represented by $M = \boldsymbol{EI\frac{d^2y}{dx^2}}$

Shearing force is represented by **Q**, **V** or $\mathbf{T} = \boldsymbol{EI\frac{d^3y}{dx^3}}$

Loading is represented by **p**, **f**, or $\mathbf{W} = \boldsymbol{EI\frac{d^4y}{d4}}$

xviii. 2. RELATIONSHIPS BETWEEN LOADING, SF, BM, SLOPE AND DEFLECTION OF A BEAM



Relationship among beam diagrams.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1)$
2. $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$
3. $\int e^x\,dx = e^x + C$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x \tan x\,dx = \sec x + C$
9. $\int \csc x \cot x\,dx = -\csc x + C$
10. $\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$
11. $\int \cot x\,dx = -\ln|\csc x| + C = \ln|\sin x| + C$
12. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$
14. $\int u\,dv = uv - \int v\,du$ (Integration by Parts)

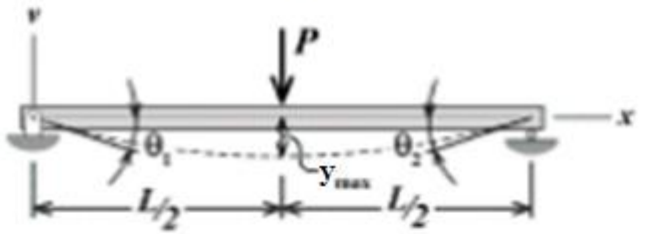
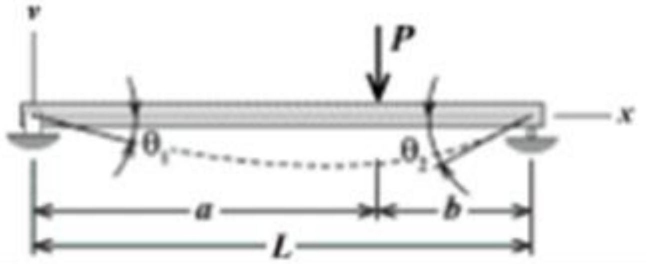
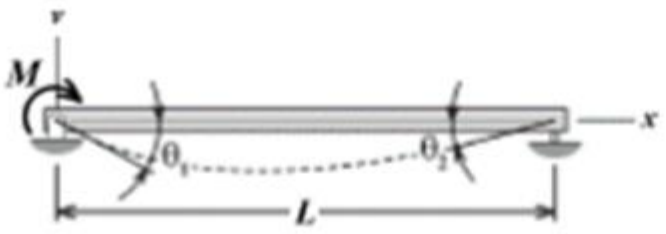
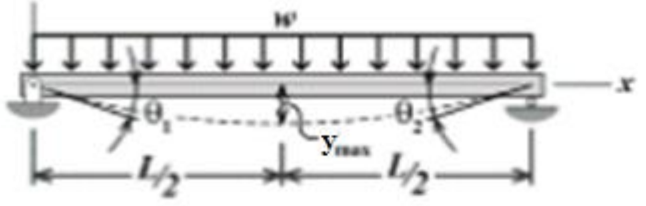
xviii. 3. METHODS FOR DETERMINING DEFLECTION OF A BEAM

- The following methods are use to determine the deflection of a beam:

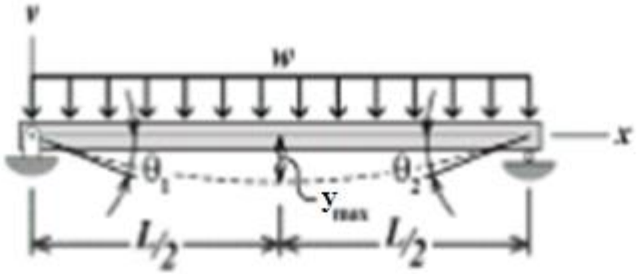
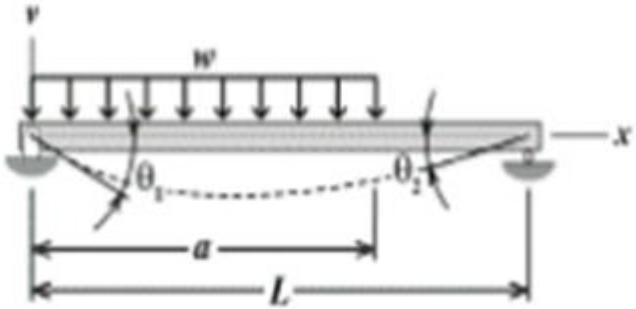
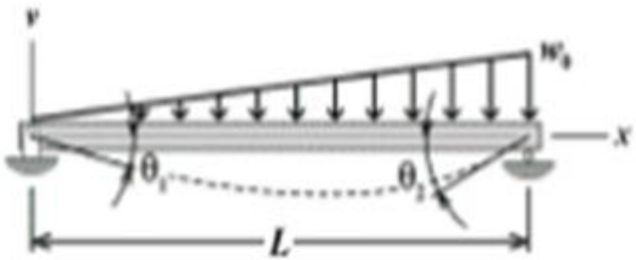
1. Integration of moment equation
2. Integration shear-force or load equations
3. Discontinuity functions
4. Superposition methods

- The current course will focus on integration of moment equation and discontinuity functions methods
- **Various tables of formulae of slope, deflection and elastic curve equations were developed based on direct integration method.**

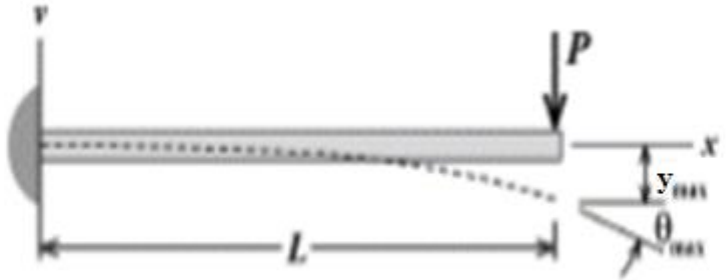
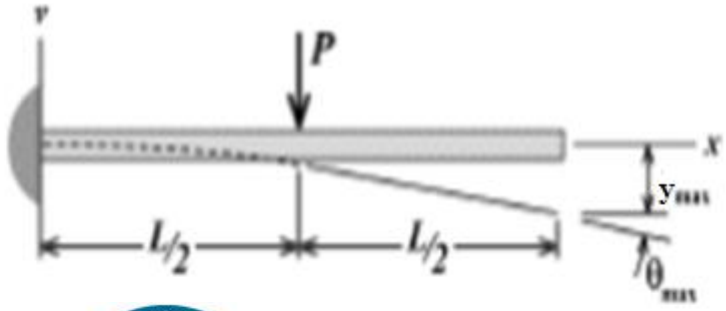
xviii. DEFLECTION OF BEAMS (8 of 98)

SIMPLY SUPPORTED BEAMS			
Beam	**Slope**	**Deflection**	**Elastic Curve**
	1 $\theta_1 = -\theta_2 = -\dfrac{PL^2}{16EI}$	2 $y_{max} = -\dfrac{PL^3}{48EI}$	3 $y = -\dfrac{Px}{48EI}(3L^2 - 4x^2)$ for $0 \le x \le L/2$
	4 $\theta_1 = -\dfrac{Pb(L^2 - b^2)}{6LEI}$ $\theta_2 = +\dfrac{Pa(L^2 - a^2)}{6LEI}$	5 $y = -\dfrac{Pa^2b^2}{3LEI}$ at $x = a$ $y_{max} = -\dfrac{Pb}{9\sqrt{3}EI\ L}(a^2 + 2ab)^{3/2}$	6 $y = -\dfrac{Pbx}{6LEI}(L^2 - b^2 - x^2)$ for $0 \le x \le a$
	7 $\theta_1 = -\dfrac{ML}{3EI}$ $\theta_2 = +\dfrac{ML}{6EI}$	8 $y_{max} = -\dfrac{ML^2}{9\sqrt{3}\ EI}$ at $x = L\left(1 - \dfrac{\sqrt{3}}{3}\right)$	9 $y = -\dfrac{Mx}{6LEI}(2L^2 - 3Lx + x^2)$
	10 $\theta_1 = -\theta_2 = -\dfrac{wL^3}{24EI}$	11 $y_{max} = -\dfrac{5wL^4}{384EI}$	12 $y = -\dfrac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$

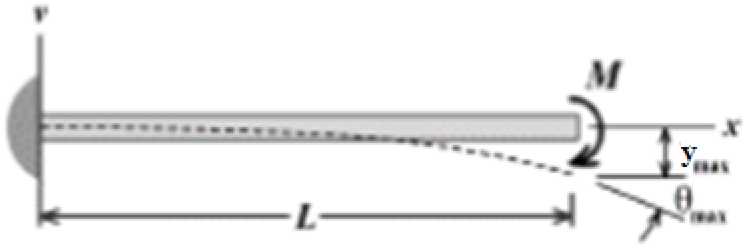
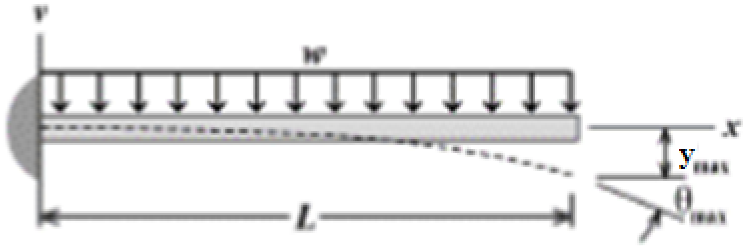
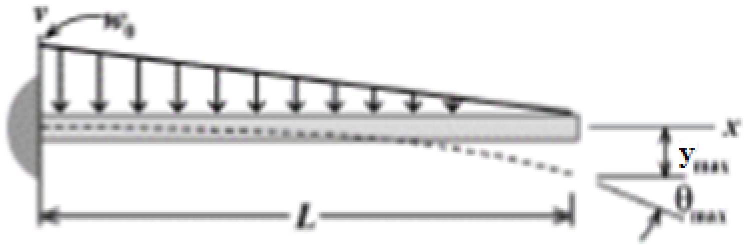
xviii. DEFLECTION OF BEAMS (9 of 98)

SIMPLY SUPPORTED BEAMS			
Beam	**Slope**	**Deflection**	**Elastic Curve**
	10 $\theta_1 = -\theta_2 = -\dfrac{wL^3}{24EI}$	11 $y_{max} = -\dfrac{5wL^4}{384EI}$	12 $y = -\dfrac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$
	13 $\theta_1 = -\dfrac{wa^2}{24LEI}(2L-a)^2$ $\theta_2 = +\dfrac{wa^2}{24LEI}(2L^2-a^2)$	14 $y = -\dfrac{wa^3}{24LEI}(4L^2 - 7aL + 3a^2)$ at $x = a$	$y = -\dfrac{wx}{24LEI}(Lx^3 - 4aLx^2 + 2a^2x^2 + 4a^2L^2$ $-4a^3L + a^4)$ for $0 \le x \le a$ $y = -\dfrac{wa^2}{24LEI}(2x^3 - 6Lx^2 + a^2x + 4L^2x - a^2L)$ 15 for $a \le x \le L$
	16 $\theta_1 = -\dfrac{7w_0L^3}{360EI}$ $\theta_2 = +\dfrac{w_0L^3}{45EI}$	17 $y_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	18 $y = -\dfrac{w_0x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$

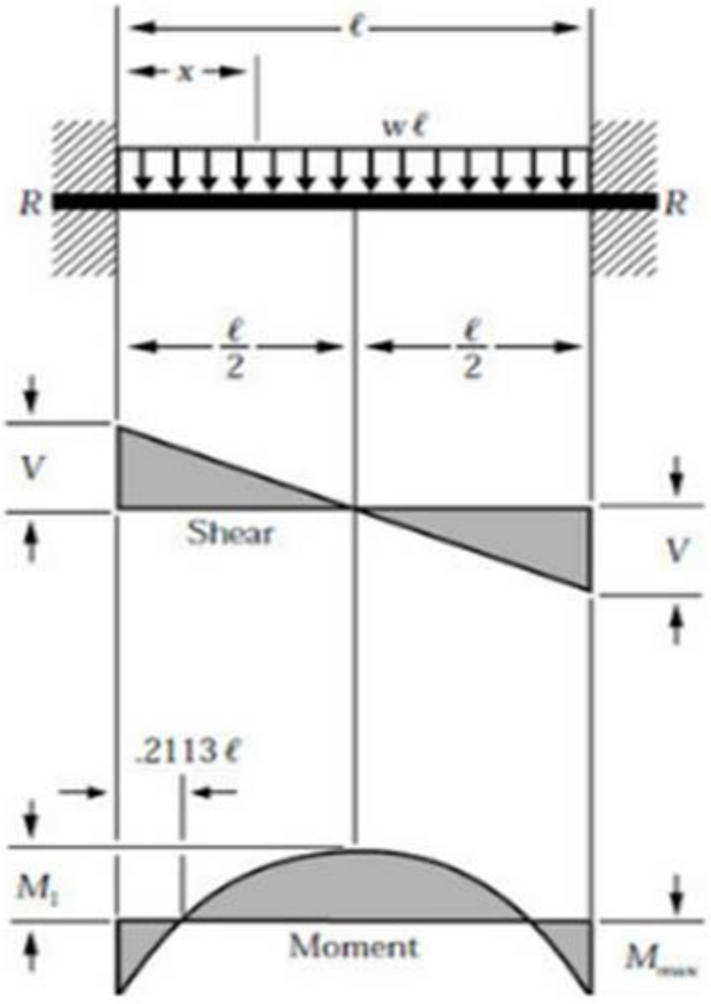
xviii. 4. FORMULAE FOR SLOPE AND DEFLECTION OF BEAMS

CANTILEVER BEAMS			
Beam	Slope	Deflection	Elastic Curve
	19 $\theta_{max} = -\frac{PL^2}{2EI}$	20 $y_{max} = -\frac{PL^3}{3EI}$	21 $y = -\frac{Px^2}{6EI}(3L - x)$
	22 $\theta_{max} = -\frac{PL^2}{8EI}$	23 $y_{max} = -\frac{5PL^3}{48EI}$	24 $y = -\frac{Px^2}{12EI}(3L - 2x) \quad \text{for } 0 \le x \le L/2$ $y = -\frac{PL^2}{48EI}(6x - L) \quad \text{for } L/2 \le x \le L$

CANTILEVER BEAMS

Beam	Slope	Deflection	Elastic Curve
	25 $\theta_{max} = -\frac{ML}{EI}$	26 $y_{max} = -\frac{ML^2}{2EI}$	27 $y = -\frac{Mx^2}{2EI}$
	28 $\theta_{max} = -\frac{wL^3}{6EI}$	29 $y_{max} = -\frac{wL^4}{8EI}$	30 $y = -\frac{wx^2}{24EI}(6L^2 - 4Lx + x^2)$
	31 $\theta_{max} = -\frac{w_0L^3}{24EI}$	32 $y_{max} = -\frac{w_0L^4}{30EI}$	33 $y = -\frac{w_0x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

Beam Fixed at Both Ends – Uniformly Distributed Load



$R = V$ . . . . $= \frac{w\ell}{2}$

V_x . . . . $= w\left(\frac{\ell}{2} - x\right)$

M_{max} (at ends) . . . . $= \frac{w\ell^2}{12}$

M_1 (at center) . . . . $= \frac{w\ell^2}{24}$

M_x . . . . $= \frac{w}{12}(6\ell x - \ell^2 - 6x^2)$

y_{max} (at center) . . . . $= \frac{w\ell^4}{384EI}$

y_x . . . . $= \frac{wx^2}{24EI}(\ell - x)^2$

xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

xviii. 5. 1. INTRODUCTION

- In this section, a method will be presented in which a single function is formulated that incorporates all loads acting on the beam.
- The load function $w\ (x)$ can be integrated twice--first to derive $V(x)$ and a second time to obtain $M(x)$. In addition to that the moment function $M(x)$ can be integrated twice--first to derive $\theta(x)$ and a second time to obtain $\delta(x)$.
- To express the load on the beam in a single function, two types of mathematical operators will be employed.
- **Macaulay** functions will be used to describe distributed loads, and **singularity** functions will be used to represent concentrated forces and concentrated moments.

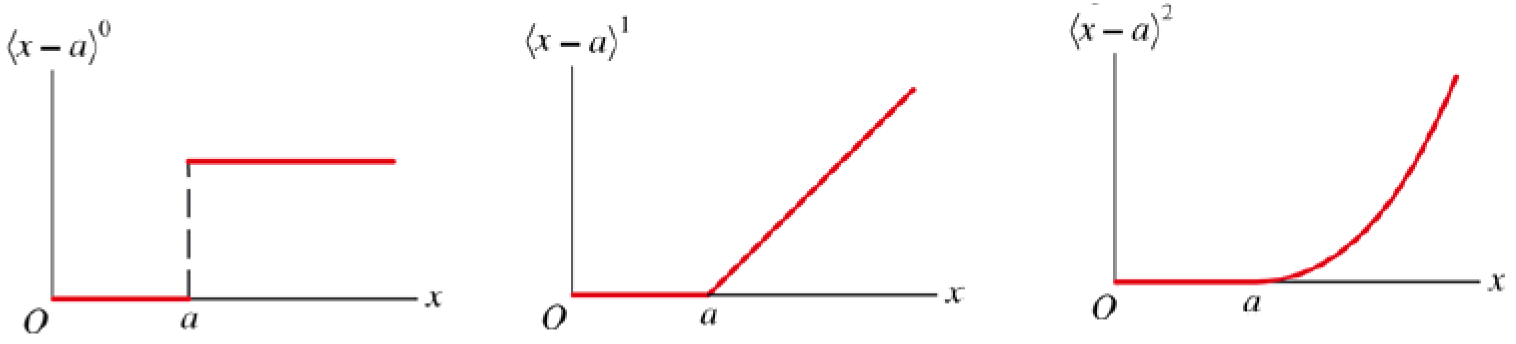
xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

xviii. 5. 2. DISCONTUITY FUNCTIONS

Macaulay functions ($n \geq 0$)

$$\langle x-a \rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

$$\int \langle x-a \rangle^n dx = \frac{\langle x-a \rangle^{n+1}}{n+1} + C$$



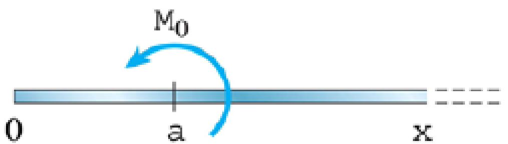
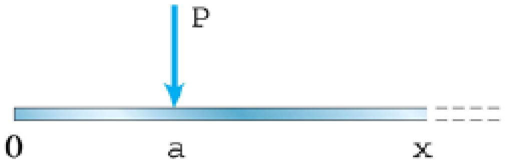
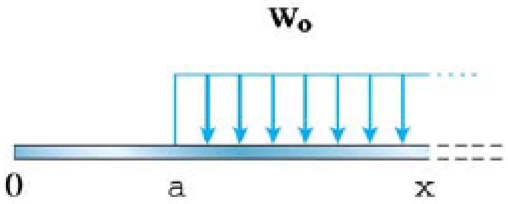
Singularity functions ($n < 0$)

$$\langle x-a \rangle^n = \begin{cases} 0 & \text{for } x \neq a \\ \pm\infty & \text{for } x = a \end{cases}$$

$$\int \langle x-a \rangle^n dx = \langle x-a \rangle^{n+1}$$

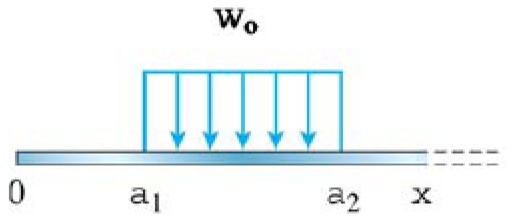
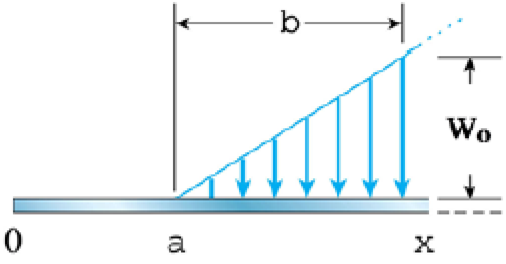
xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

xviii. 5. 2. DISCONTUITY FUNCTIONS

Case	Load on Beam	Discontinuity Expressions
1		$w(x)=M_o\langle x-a\rangle^{-2}$ $V(x)=-M_o\langle x-a\rangle^{-1}$ $M(x)=-M_o\langle x-a\rangle^{0}$
2		$w(x)=P\langle x-a\rangle^{-1}$ $V(x)=-P\langle x-a\rangle^{0}$ $M(x)=-P\langle x-a\rangle$
3		$w(x)=w_o\langle x-a\rangle^{0}$ $V(x)=-w_o\langle x-a\rangle$ $M(x)=-\frac{1}{2}w_o\langle x-a\rangle^{2}$

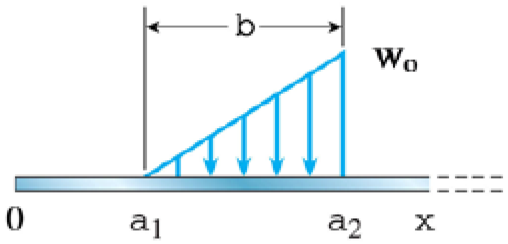
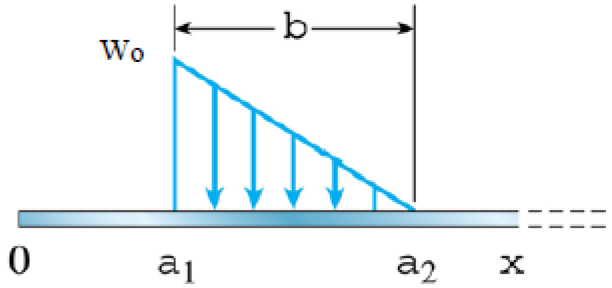
xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

xviii. 5. 2. DISCONTUITY FUNCTIONS

Case	Load on Beam	Discontinuity Expressions
4		$w(x) = w_o\langle x-a_1\rangle^0 - w_o\langle x-a_2\rangle^0$ $V(x) = -w_o\langle x-a_1\rangle + w_o\langle x-a_2\rangle$ $M(x) = -\frac{1}{2}w_o\langle x-a_1\rangle^2 + \frac{1}{2}w_o\langle x-a_2\rangle^2$
5		$w(x) = \frac{w_o}{b}\langle x-a\rangle$ $V(x) = -\frac{1}{2}\frac{w_o}{b}\langle x-a\rangle^2$ $M(x) = -\frac{1}{6}\frac{w_o}{b}\langle x-a\rangle^3$

xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

xviii. 5. 2. DISCONTUITY FUNCTIONS

Case	Load on Beam	Discontinuity Expressions
6		$w(x)=\frac{w_o}{b}\langle x-a_1\rangle-\frac{w_o}{b}\langle x-a_2\rangle-w_o\langle x-a_2\rangle^0$ $V(x)=-\frac{1}{2}\frac{w_o}{b}\langle x-a_1\rangle^2+\frac{1}{2}\frac{w_o}{b}\langle x-a_2\rangle^2+w_o\langle x-a_2\rangle$ $M(x)=-\frac{1}{6}\frac{w_o}{b}\langle x-a_1\rangle^3+\frac{1}{6}\frac{w_o}{b}\langle x-a_2\rangle^3+\frac{1}{2}w_o\langle x-a_2\rangle^2$
7		$w(x)=w_o\langle x-a_1\rangle^0-\frac{w_o}{b}\langle x-a_1\rangle+\frac{w_o}{b}\langle x-a_2\rangle$ $V(x)=-w_o\langle x-a_1\rangle+\frac{1}{2}\frac{w_o}{b}\langle x-a_1\rangle^2-\frac{1}{2}\frac{w_o}{b}\langle x-a_2\rangle^2$ $M(x)=-\frac{1}{2}w_o\langle x-a_1\rangle^2+\frac{1}{6}\frac{w_o}{b}\langle x-a_1\rangle^3-\frac{1}{6}\frac{w_o}{b}\langle x-a_2\rangle^3$

xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

Procedure

1. Identify support types and establish the free body diagram (FBD).
2. Calculate support reactions
3. Identify boundary conditions at support. Zero displacement ($\delta = 0$) occurs at all pin and roller supports, and zero slope and zero displacement ($\theta = 0$, $\delta = 0$) occur at fixed supports.
4. Establish the x-axis so that it extends to the right (in the last section possible on the beam under study) and has its origin at the beam's left end.
5. Use discontinuity functions to express the loading w *or the moment* M as a function of x. Make sure to follow the sign convention for each loading as it applies for each question.

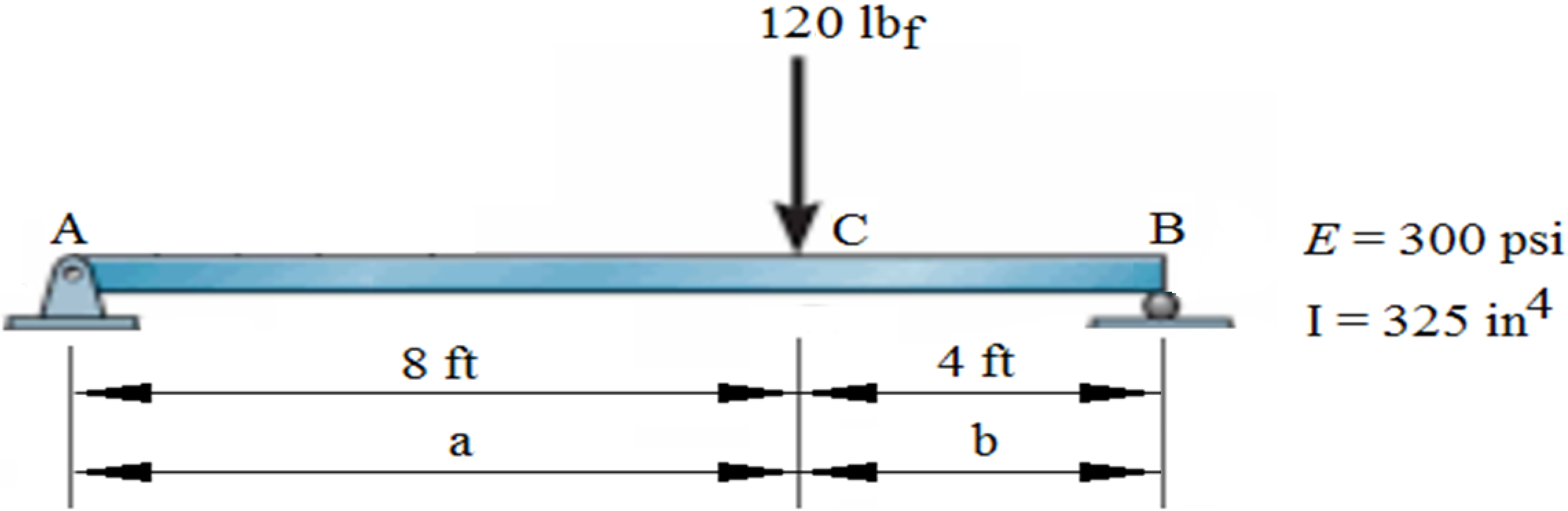
xviii. 5. DISCONTUITY FUNCTIONS FOR DETERMING SLOPE AND DEFLECTION OF BEAMS

Procedure

6. Substitute w into $EI\frac{d^4y}{dx^4} = -w\,(x)$ or M into the moment-curvature relation $EI\frac{d^2y}{dx^2} = M\,(x)$, and integrate to obtain the equation for the beam's slope, $\theta = EI\frac{dy}{dx}$ and deflection $\delta = EIy$.
7. Evaluate the constants of using the boundary conditions, and substitute these constants into the slope and deflection equations to obtain the final results.
8. Remember that when the slope and deflection equations are evaluated at any point on the beam, a positive slope is counterclockwise, and a positive deflection is upward.

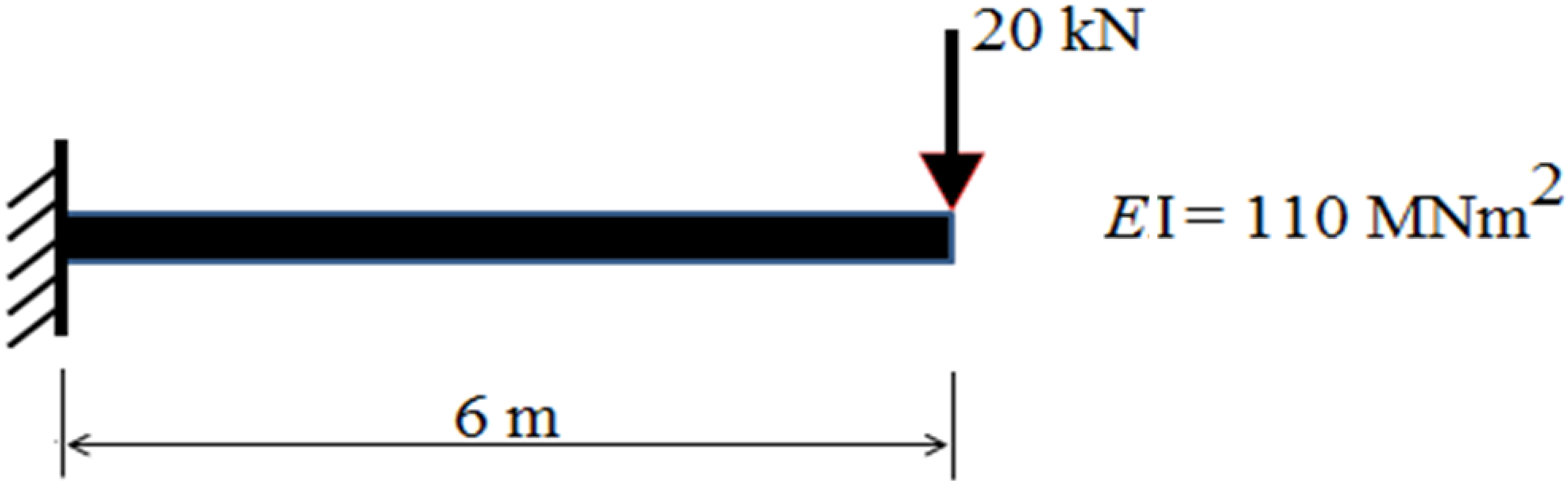
GIVEN EXAMPLES

1. Determine the deflection under the load and the maximum deflection for the simply supported beam shown in figure below.



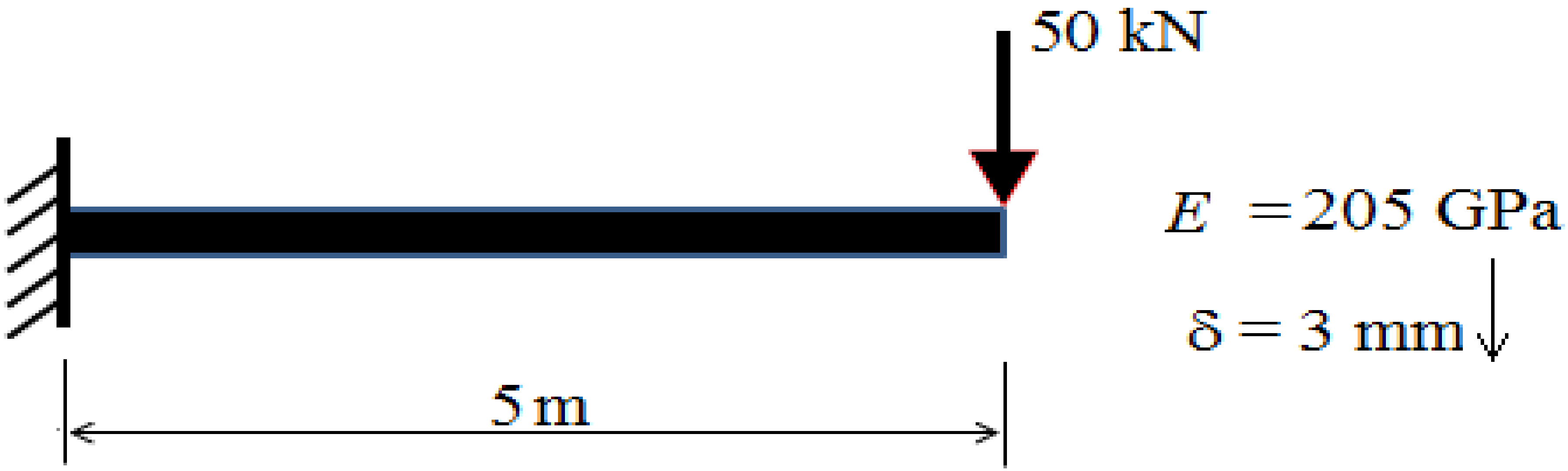
GIVEN EXAMPLES

2. Determine the maximum slope and maximum deflection for the cantilever beam shown in figure below.



GIVEN EXAMPLES

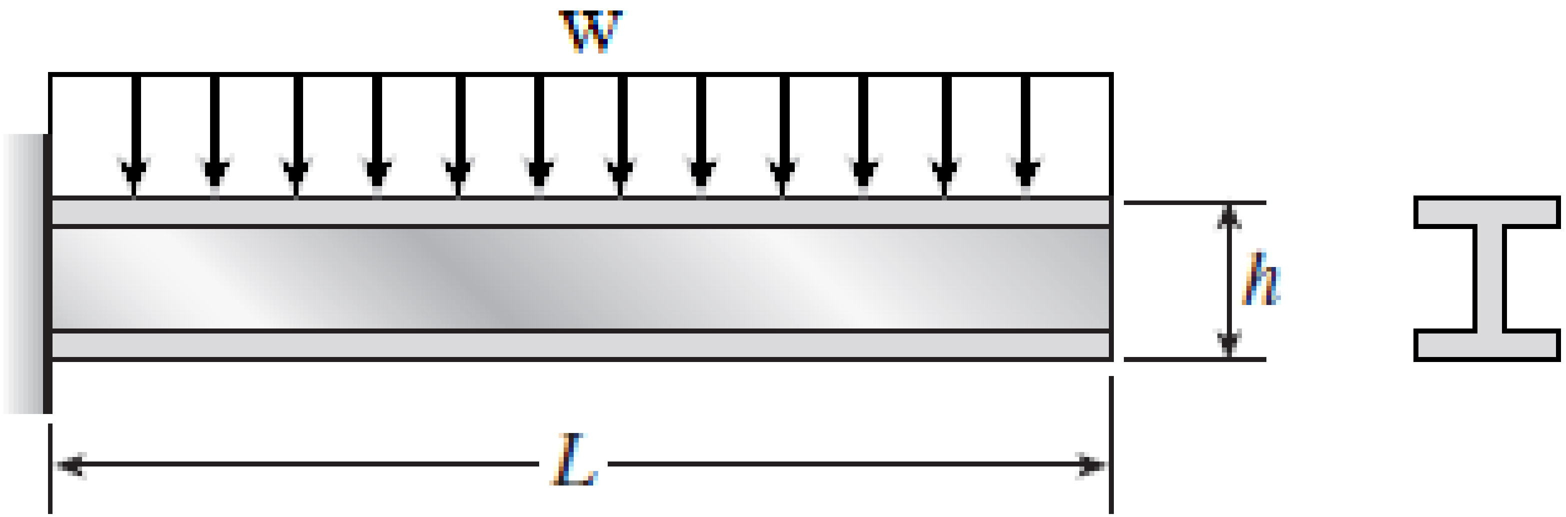
3. Determine the flexural capacity (***EI***), the moment of inertia (***I***) and the size of the cantilever beam shown in figure below, if has a rectangular cross-sectional area with the base which is a third of its depth.



GIVEN EXAMPLES

4. A cantilever beam with a UDL (see figure below) has a height h equal to 1/8 of the length L. The beam is a steel wide-flange section with the same allowable bending stress in both tension and compression.

   Calculate the ratio ${}^{\delta}/_{L}$ of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load.



$\sigma = 17{,}500$ psi

$E = 28 \times 10^6$ psi

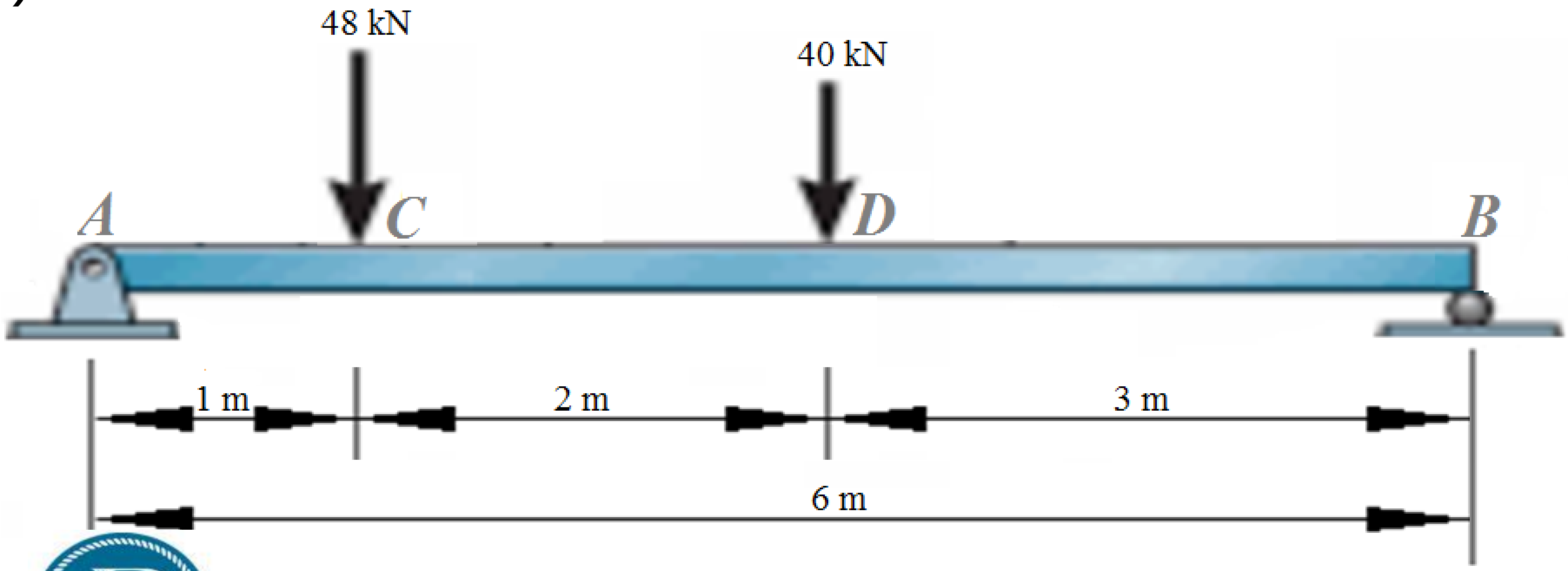
GIVEN EXAMPLES

5. Refer to the beam shown in figure below. Find

(i) Deflection under each load

(ii) The point at which the maximum deflection occurs, and

(iii) The maximum deflection.

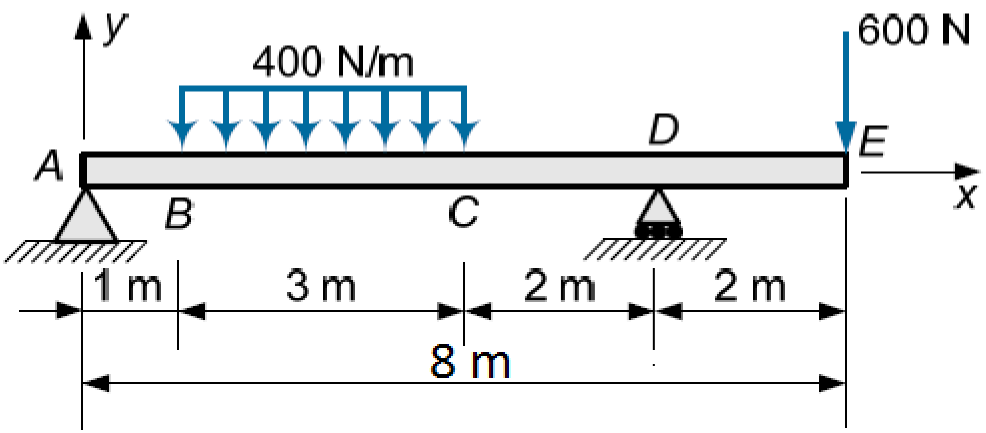


$E = 2 \times 10^5$ N/mm^2

$I = 85 \times 10^6$ mm^4

GIVEN EXAMPLES

6. Refer to the beam shown in figure below.
   Determine the maximum deflection and the deflection under the load located at the point E.

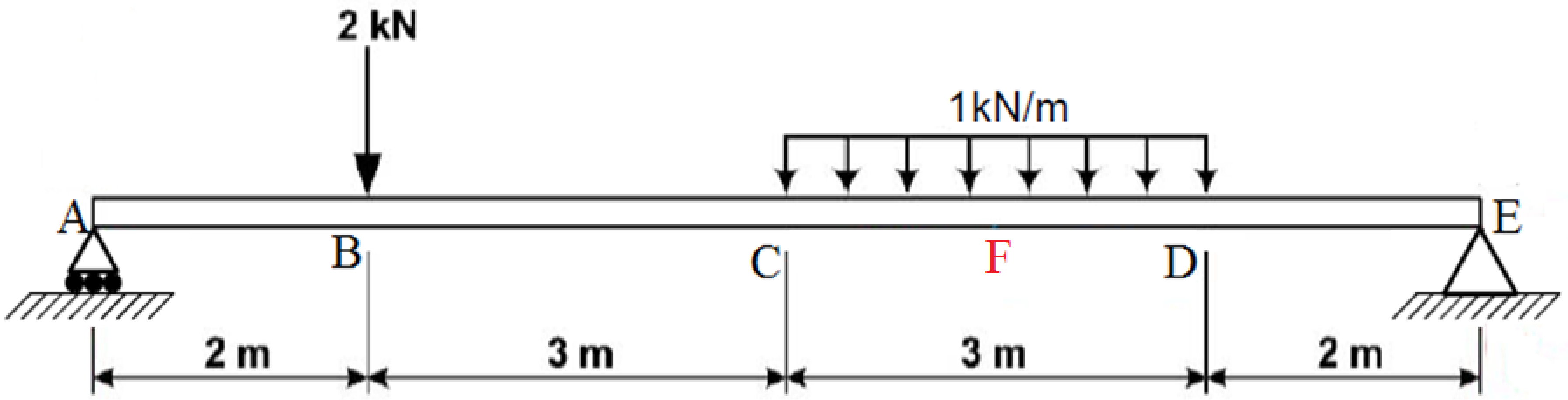


$E = 12$ GPa

$I = 8.33 \times 10^6$ mm^4

GIVEN EXAMPLES

7. Refer to the beam shown in figure below.
   Determine the deflection under the load located at the point B and the point F located under the middle of the uniform distributed load and the maximum deflection.



$E = 200\ GPa$

$I = 129 \times 10^6\ mm^4$

GIVEN EXAMPLES

1. Given data: $\mathrm{P} = 120\ lb_f$

$$\mathrm{b} = 4\ ft = 4 \times 12 = 48\ in$$

$$\mathrm{L} = 12\ ft = 12 \times 12 = 144\ in$$

$$E = 300\ psi = 300 \times 144 = 43200\ lb_f/in^2$$

First method; we apply the formula based on the elastic curve

$\delta = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$ (6 on the table of formulae)

Recall: $1lb_f = 4.448 \approx 4.45N$

$1\mathrm{psi} = 4895\ Pa = 144 lb_f/in^2$ and $\frac{1lb_f}{ft^2} = 47.88\ Pa$

GIVEN EXAMPLES

$$\delta = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$$

$$\delta = -\frac{(120 \text{ x } 48)\,x}{6 \text{ x } 144 \text{ x } 43200 \text{ x } 325}(144^2 - 48^2 - x^2)$$

$$= -\frac{5760\,x}{12130560000}(20736 - 2304 - x^2)$$

$$= -\frac{5760\,x}{12130560000}(18432 - x^2)$$

GIVEN EXAMPLES

$$= -\frac{5760\,x}{2106000} + \frac{x^3}{2106000}$$

$$\delta = -\frac{128\,x}{14625} + \frac{x^3}{2106000} \quad \textbf{(i)}$$

By using the relationship between deflection and slope, we get

$$\theta = \frac{d\delta}{dx} = -\frac{128}{14625} + \frac{3\,x^2}{2106000} \quad \boldsymbol{(ii)}$$

GIVEN EXAMPLES

We get the maximum deflection ($\boldsymbol{\delta_{max}}$) where there is zero slope ($\boldsymbol{\theta}$).

Then $\boldsymbol{ii} = 0; \Longrightarrow -\frac{128}{14625} + \frac{3\,x^2}{2106000} = 0$

$$-\frac{128}{14625} + \frac{x^2}{702000} = 0$$

$$-6144 + \ x^2 = 0$$

$$x = \sqrt{6144} = 78.384\,in.$$

GIVEN EXAMPLES

For the deflection under the load, we replace the value of $(\boldsymbol{x} = \mathbf{96}\ \boldsymbol{in}.)$ in the equation $(\boldsymbol{i})$

$$\delta_C = -\frac{128 \text{ x } (96)}{14625} + \frac{96^3}{2106000} = -0.840 + 0.420 = -0.420in.$$

For the maximum deflection, we replace the value of $(\boldsymbol{x} = 78.384\ in.)$ in the equation $(\boldsymbol{i})$

$$\delta_{max} = -\frac{128 \text{ x } (78.384)}{14625} + \frac{78.384^3}{2106000} = -0.6860 + 0.2287 = -0.457in.$$

GIVEN EXAMPLES

1. **Second method;** we apply the formula for the deflection under the load and the maximum deflection

$$\delta_C = -\frac{Pa^2b^2}{3LEI}$$ (5 on the table of formulae)

$$\delta_C = -\frac{120 \text{ x } 96^2 \text{ x } 48^2}{3 \text{ x } 144 \text{ x } 43200 \text{ x } 325} = -0.420\ in.$$

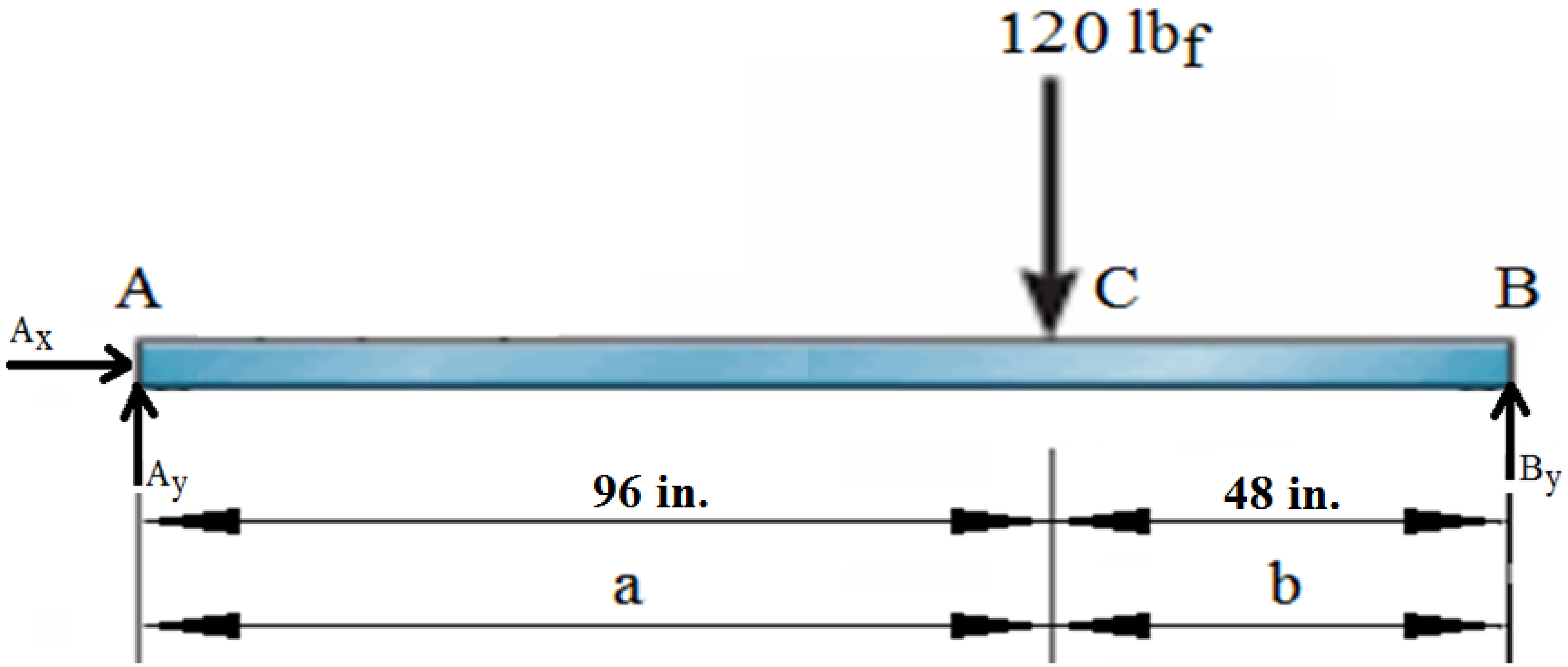
For the maximum deflection,

$$\delta_{max} = -\frac{Pb}{9\sqrt{3}\ EI\ L}(a^2 + 2ab)^{3/2}$$ (5 on the table of formulae)

$$\delta_{max} = -\frac{120 \text{ x } 48}{9 \text{ x } \sqrt{3} \text{ x } 43200 \text{ x } 325 \text{ x } 144}(96^2 + (2 \text{ x } 96 \text{ x } 48))^{3/2} = -0.457\ in.$$

GIVEN EXAMPLES

1. (i) FBD



$P = 100\ lbf$

$E = 300\ psi$

$= 300 \text{ x } 144 lbf/in.^2$

$= 43200\ lbf/in.^2$

$I = 325\ in.^4$

GIVEN EXAMPLES

(ii) Calculating the support reactions

$$\rightarrow + \sum F_x = 0 \;\; ; \quad A_x = 0$$

$$+\circlearrowleft \sum M_A = 0 \;\; ; -(120 \text{ x } 96) + \left(B_y \text{ x } 144\right) = 0$$

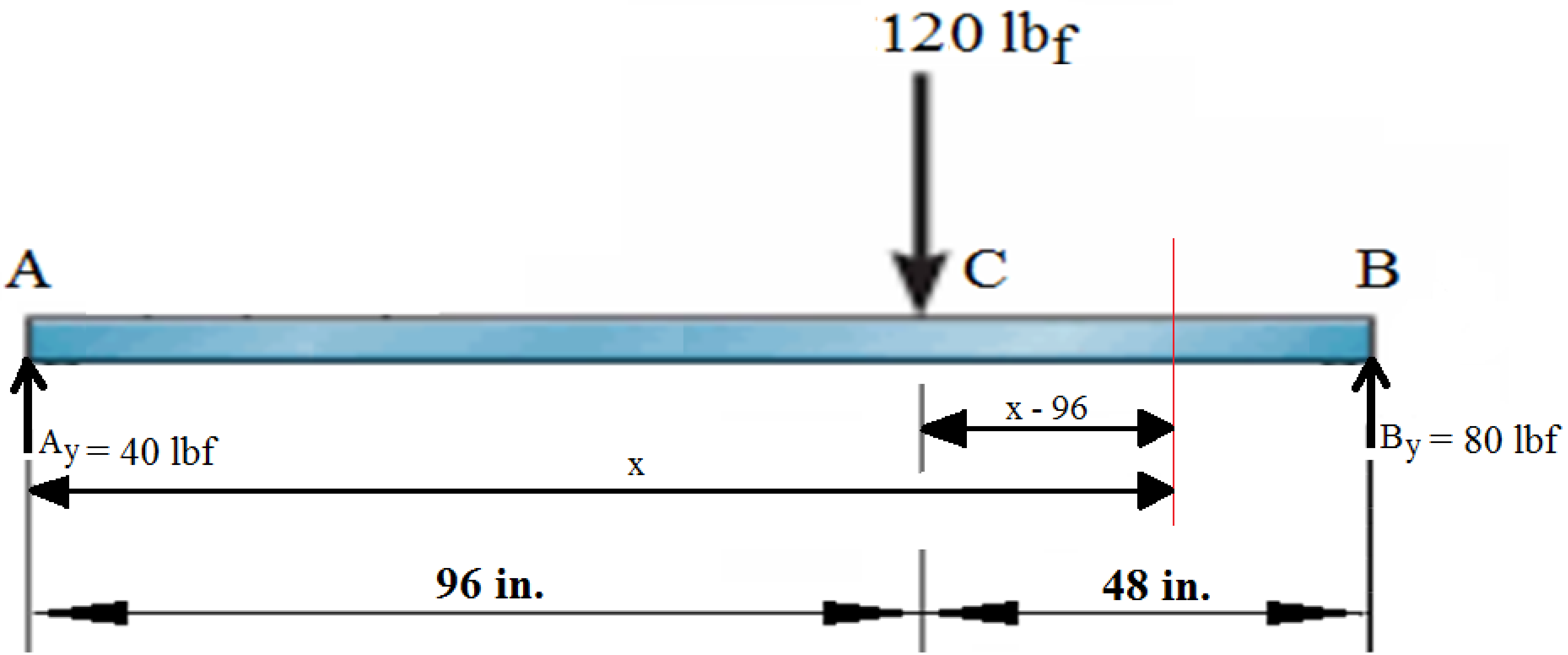
$$B_y = \frac{11520}{144} = 80\ lbf \uparrow$$

$$+\uparrow \sum F_y = 0 \;\; ; \quad A_y - 120 + B_y = 0\ ;$$

$$A_y = 120 - 80 = 40\ lbf \uparrow$$

GIVEN EXAMPLES

(iii) Making a virtual sectioning



GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$M = EI\frac{d^2y}{dx^2} = 40 < x - 0 > -120 < x - 96 > \ + 80 < x - 144 >$$

$$= 40\,x - 120 < x - 96 > (i)$$

$$\theta = EI\ \frac{dy}{dx} = 40\frac{x^2}{2} + C1 - 120\frac{< x - 96 >^2}{2}$$

$$= 20\,x^2 + C1\ - 60 < x - 96 >^2 \quad (ii)$$

$$\delta = EI\ y = 20\frac{x^3}{3} + C1\,x + C2\ - 60\frac{< x - 96 >^3}{3}$$

$$= 20\frac{x^3}{3} + C1x - C2 - 20 < x - 96 >^3 \qquad (iii)$$

GIVEN EXAMPLES

(v) Boundary conditions $\begin{cases} (1)\ x = 0\ ; \quad y = 0 \\ (2)\ x = 144\ in.;\ y = 0 \end{cases}$

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 + 0 + C2 - 0$$

$$C2 = 0$$

GIVEN EXAMPLES

Substituting the 2nd boundary condition in equation (iii), we get

$$EI(0) = \frac{20}{3}\text{ x }(144^3) + (C1\text{ x }144) - 20\text{ x }(144 - 96)^3$$

$$0 = 19906560 + 144\text{ x }C1 - 2211840$$

$$0 = 17694720 + 144\,C1$$

$$C1 = -122880$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = EI\frac{dy}{dx} = 20\,x^2 - 122880 - 60\text{ x} < x - 96 >^2 \quad (iv)$$

$$\delta = EIy = \frac{20}{3}x^3 - 122880\,x - 20\text{ x} < x - 96 >^3 \ (v)$$

(vii) Determination of the deflection under point C

Substituting $(x = 96)$ in equation (v) by considering the interval $0 < x < 144$ to find δ_C

$$\delta_C = \frac{1}{43200\text{ x }325}\text{ x }\left[\left(\frac{20}{3}\text{ x }96^{\,3}\right) - (122880\text{ x }96) - (20\text{ x }(96-96)^3)\right]$$

$$= \frac{5898240 - 11796480}{43200\text{ x }325} = -\frac{5898240}{14040000} = -0.420\ in.$$

GIVEN EXAMPLES

(viii) Determination of the point where the maximum deflection is i.e., the point where the slope is zero

$$\theta = \frac{1}{EI}[20\,x^2 - 122880 - 60\,\text{x}\,(x-96)^2]$$

$$= \frac{1}{EI}[20\,x^2 - 122880 - 60\,\text{x}\,(x^2 - 192\,x + 9216)] = 0$$

$$20\,x^2 - 122880 - 60\,x^2 + 11520\,x - 552960 = 0$$

$$-40\,x^2 + 11520\,x\ - 675840 = 0$$

$$-x^2 + 288\,x\ - 16896 = 0$$

$x_1 = 205.968\ in.$ (out of range)

$x_2 = 82.032\ in.$

GIVEN EXAMPLES

(ix) Determination of the maximum deflection

$$\delta_{max} = \frac{1}{EI} \text{ x } \left[\frac{20}{3} x^3 - 122880\, x - 20 \text{ x } (x - 96)^3\right]$$

$$= \frac{1}{43200 \text{ x } 325} \text{ x } \left[(\frac{20}{3} \text{ x } 82.032^3) - (122880 \text{ x } 82.032) - 20 \text{ x } (82.032 - 96\,)^3\right]$$

$$= \frac{1}{43200 \text{ x } 325} \text{ x } [3680091.706 - 10080092.16 + 54054.540]$$

$$\delta_{max} = \frac{-\,6345495.914}{14040000} = -\,0.452\; in.$$

xviii. DEFLECTION OF BEAMS (42 of 98)

GIVEN EXAMPLES

2. Given data: $P = 20kN$

$L = 6m$

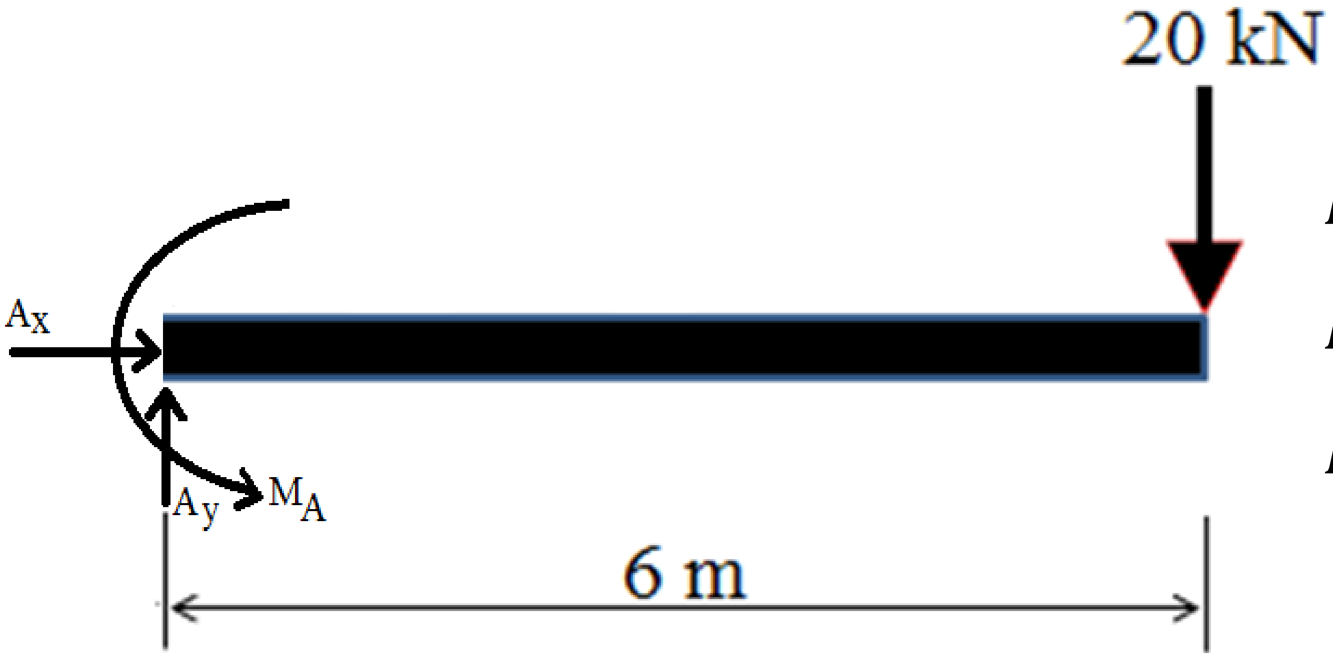
$EI = 110MNm^2 = 110 \text{ x } 10^3 kNm^2$

$$\theta_{max} = -\frac{PL^2}{2EI} = -\frac{20 \text{ x } 6^2}{2 \text{ x } 110 \text{ x } 10^3} = -\,0.0033\,rad = 0.0033\,rad$$

$$\delta_{max} = -\frac{PL^3}{3EI} = -\frac{20 \text{ x } 6^3}{3 \text{ x } 110 \text{ x } 10^3} = -0.013\,m = -13.100\,mm$$

GIVEN EXAMPLES

2. (i) FBD



$P = 20\ kN$

$L = 6\ m$

$EI = 110\ MNm^2 = 110 \text{ x } 10^3\ kNm^2$

GIVEN EXAMPLES

(ii) Support reactions

$$\rightarrow + \sum F_x = 0 \;\; ; \quad A_x = 0$$

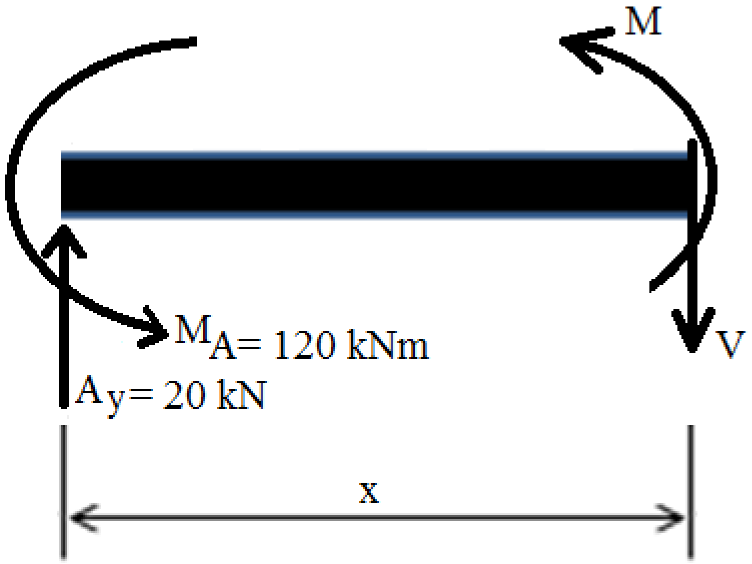
$$+\uparrow \sum F_y = 0 \;\; ; \quad A_y - 20 = 0 \; ; \quad A_y = 20\, kN \uparrow$$

$$+\circlearrowleft \sum M_A = 0 \;\; ; \quad M_A - (20 \text{ x } 6) = 0$$

$$M_A = 120\, kNm$$

GIVEN EXAMPLES

(iii) Making section of interest



$$+\uparrow \sum F_y = 0 \; ; 20 - V = 0 \; ; \mathrm{V} = 20\, kN$$

$$+\circlearrowleft \sum M_x = 0 \; ; \quad M + M_A - A_y \mathrm{x}\, x = 0$$

$$M + 120 - 20\,x = 0$$

$$M = 20\,x - 120 \quad (i)$$

GIVEN EXAMPLES

(iv) Slope and deflection equations

$$M = EI\frac{d^2y}{dx^2} = 20\,x - 120 \quad (i)$$

$$\theta = EI\frac{dy}{dx} = \frac{20\,x^2}{2} - 120\,x + C1$$

$$\theta = EI\frac{dy}{dx} = 10\,x^2 - 120\,x + C1\,(ii)$$

GIVEN EXAMPLES

(iv) Slope and deflection equations

$$\delta = EIy = \frac{10\,x^3}{3} - \frac{120\,x^2}{2} + C1\,x + C2$$

$$\delta = EIy = \frac{10\,x^3}{3} - 60\,x^2 + C1\,x + C2 \quad (iii)$$

(v) Boundary conditions $\begin{cases} (1) \quad x = 0 \;;\; y = 0 \\ (2) \quad x = 0 \;;\; \dfrac{dy}{dx} = 0 \end{cases}$

GIVEN EXAMPLES

Substituting the 2nd boundary condition in equation (ii), we get

$$EI(0) = 0 - 0 + C1$$
$$C1 = 0$$

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 - 0 + 0 + C2$$
$$C2 = 0$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = \frac{1}{EI}(10\,x^2 - 120\,x)\ (iv)$$

$$\delta = \frac{1}{EI}\left(\frac{10\,x^3}{3} - 60x^2\right)(v)$$

(v) Determination of maximum slope and maximum deflection

$$\theta = \frac{1}{110 \text{ x } 10^3}(10 \text{ x } 6^2 - 120 \text{ x } 6) = -0.0033\,rad$$

$$\delta_{max} = \frac{1}{110 \text{ x } 10^3}\left(\frac{10 \text{ x } 6^3}{3} - 60 \text{ x } 6^2\right) = -0.0131\,m = -13.100\,mm$$

GIVEN EXAMPLES

3. Given data: $P = 50kN = 50 \text{ x } 10^3$

$L = 5m = 5 \text{ x } 10^3 mm$

$\delta = 3mm \downarrow$

$E = 205GPa = 205 \text{ x } 10^3 MPa = 205 \text{ x } 10^3 N/mm^2$

$$\delta_{max} = -\frac{PL^3}{3EI}$$

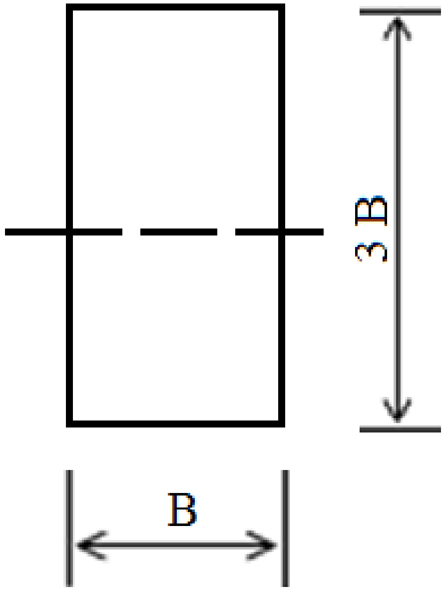
$$EI = -\frac{P \text{ x } L^3}{3\delta} = -\frac{50 \text{ x } 10^3 \text{x } (5 \text{ x } 10^3)^3}{3 \text{ x } (-3)} = 6.944 \text{ x } 10^{14} Nmm^2$$

$$= 6.944 \text{ x } 10^8 MNmm^2$$

GIVEN EXAMPLES

3.

$$I = \frac{EI}{E} = \frac{6.944 \text{ x } 10^4}{205 \text{ x } 10^3} = 3387533875mm^2$$



GIVEN EXAMPLES

3. Recall $I = \frac{bh^3}{12}$ For a rectangular section

Then in this case (h = 3B and b = B)

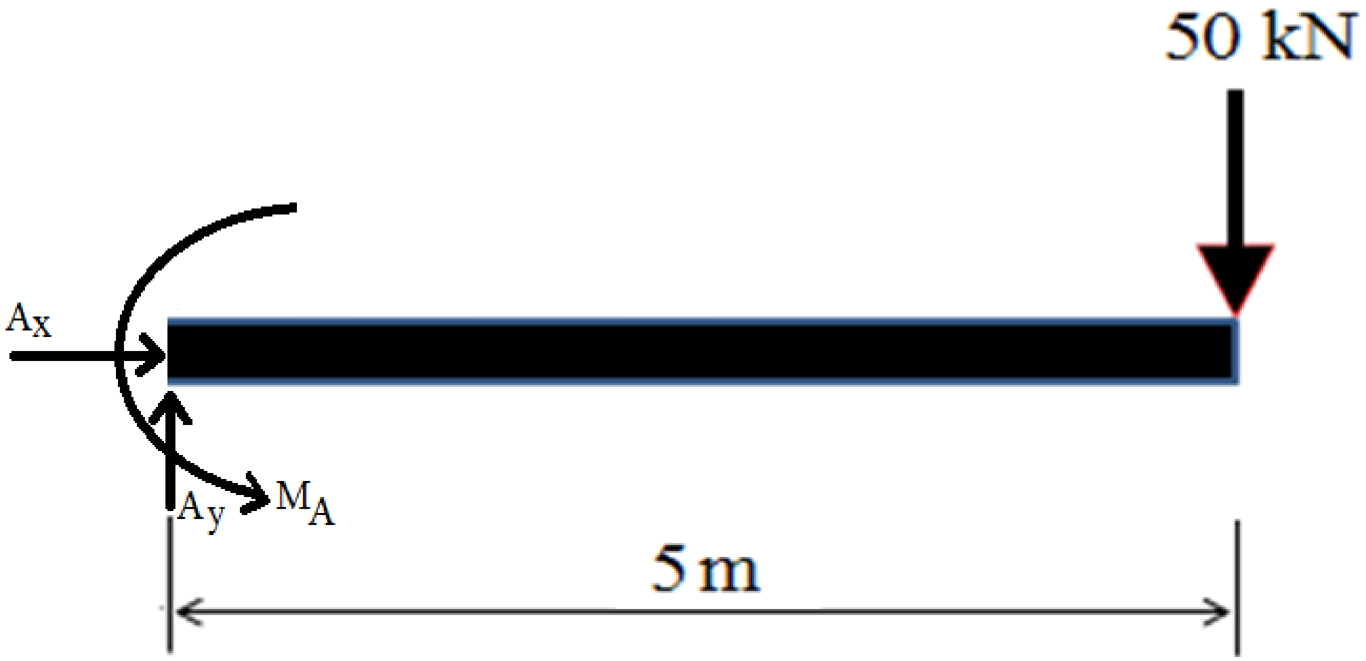
$$I = \frac{B(3B)^3}{12} = \frac{27B^4}{12} = 2.25B^4$$

Therefore, $\mathrm{B} = \sqrt[4]{\frac{I}{2.25}} = \sqrt[4]{\frac{3387533875}{2.25}} = 196.98 \approx 197mm$

$$\mathrm{h} = 3B = 3 \text{ x } 197 = 591\ mm$$

GIVEN EXAMPLES

3. (i) FBD



$E = 205\ GPa$

$L = 5\ m$

$\delta = 3\ \text{mm} \downarrow = -3 \text{ x } 10^{-3}\ \text{m}$

GIVEN EXAMPLES

(ii) Support reactions

$$\rightarrow +\sum F_x = 0 \;\; ; \quad A_x = 0$$

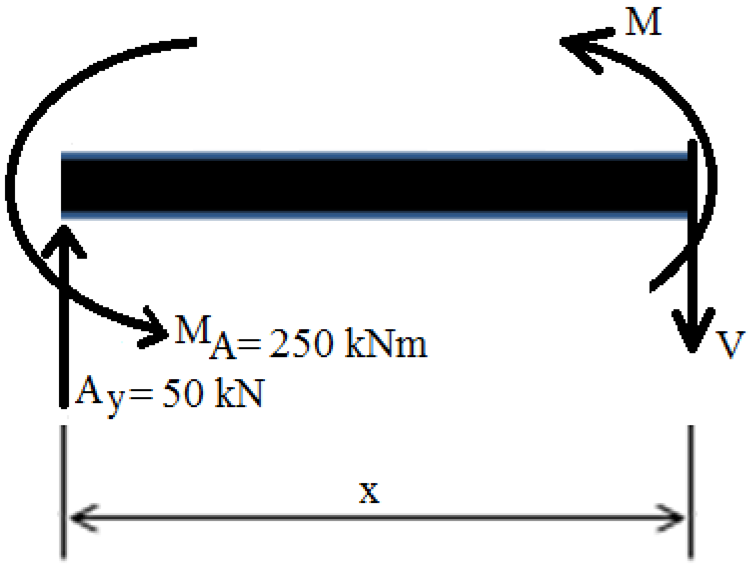
$$+\uparrow \sum F_y = 0 \;\; ; \quad A_y - 20 = 0 \; ; \quad A_y = 50\, kN \uparrow$$

$$+\circlearrowleft \sum M_A = 0 \;\; ; \quad M_A - (50 \text{ x } 5) = 0$$

$$M_A = 250\, kNm$$

GIVEN EXAMPLES

(iii) Making section of interest



$$+\uparrow \sum F_y = 0 \ ; 50 - V = 0 \ ; \mathrm{V} = 50\, kN$$

$$+\circlearrowleft \sum M_x = 0 \ ; \quad M + M_A - A_y \text{ x } x = 0$$

$$M + 250 - 50\,x = 0$$

$$M = 50\,x - 250 \quad (i)$$

GIVEN EXAMPLES

(iv) Slope and deflection equations

$$M = EI\frac{d^2y}{dx^2} = 50\,x\, - 250\quad (i)$$

$$\theta = EI\frac{dy}{dx} = \frac{50\,x^2}{2} - 250\,x + C1$$

$$\theta = EI\frac{dy}{dx} = 25x^2 - 250\,x + C1\,(ii)$$

GIVEN EXAMPLES

(iv) Slope and deflection equations

$$\delta = EIy = \frac{25\,x^3}{3} - \frac{250\,x^2}{2} + C1\,x + C2$$

$$\delta = EIy = \frac{25\,x^3}{3} - 125\,x^2 + C1\,x + C2 \quad (iii)$$

(v) Boundary conditions
$$\begin{cases} (1) \quad x = 0\ ;\ y = 0 \\ (2) \quad x = 0\ ;\ \dfrac{dy}{dx} = 0 \end{cases}$$

GIVEN EXAMPLES

Substituting the 2nd boundary condition in equation (ii), we get

$$EI(0) = 0 - 0 + C1$$
$$C1 = 0$$

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 - 0 + 0 + C2$$
$$C2 = 0$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = \frac{1}{EI}(25\,x^2 - 250\,x)\;(iv)$$

$$\delta = \frac{1}{EI}\left(\frac{25\,x^3}{3} - 125\,x^2\right)(v)$$

(v) Determination of EI at the point with maximum deflection i.e., for $L = 5m$.

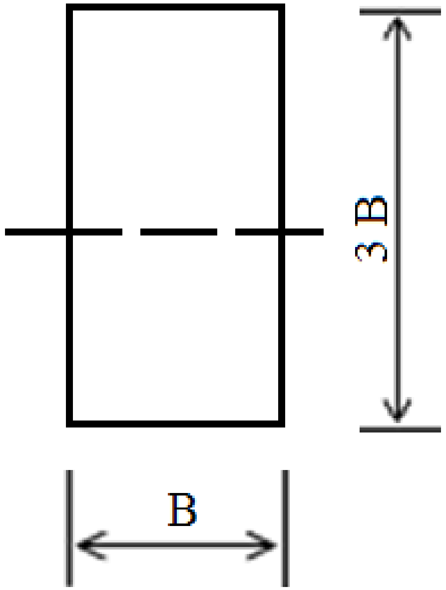
$$-3 \text{ x } 10^{-3} = \frac{1}{EI}\left(\frac{25 \text{ x } 5^3}{3} - 125 \text{ x } 5^2\right)$$

$$EI = \frac{-\frac{6250}{3}}{-3 \text{ x } 10^{-3}} = 694444.444\;kNm^2 = 694444.444 \text{ x } 10^9\;Nmm^2$$

GIVEN EXAMPLES

(vi) Determination of moment of inertia (I)

$$I = \frac{EI}{E} = \frac{6.944 \text{ x } 10^4}{205 \text{ x } 10^3} = 3387533875mm^2$$



GIVEN EXAMPLES

(vii) Determination of the cross section of the cantilever beam

Recall $I = \frac{bh^3}{12}$ For a rectangular section

Then in this case (h = 3B and b = B)

$$I = \frac{B(3B)^3}{12} = \frac{27B^4}{12} = 2.25B^4$$

Therefore, $\mathrm{B} = \sqrt[4]{\frac{I}{2.25}} = \sqrt[4]{\frac{3387533875}{2.25}} = 196.98 \approx 197mm$

$$\mathrm{h} = 3B = 3 \text{ x } 197 = 591\, mm$$

GIVEN EXAMPLES

4. Given data: $E = 28 \text{ x } 10^6 psi$

$\sigma = 17,500\ psi$

$\frac{h}{L} = \frac{1}{8}$

L and w

Find $\delta/_L$

$$\delta = \frac{wL^4}{8EI} \ldots\ldots\ldots\ldots\text{(i)}$$

$$\delta/_L = \frac{wL^3}{8EI} \ldots\ldots\ldots\ldots\text{(ii)}$$

$$M = \frac{wL^2}{2}$$

$$y = \frac{h}{2}$$

The equation of bending is: $\frac{M}{I} = \frac{\sigma}{y}$

$$\Longrightarrow \sigma = \frac{M \text{ x } y}{I} \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

$$\sigma = \frac{\frac{wL^2}{2} x \frac{h}{2}}{I} = \frac{wL^2 h}{4I}$$

GIVEN EXAMPLES

$$w = \frac{4I\sigma}{L^2 h} \quad \ldots\ldots\ldots\ldots(\mathbf{iv})$$

Replace (**iv**) in (**ii**), we get

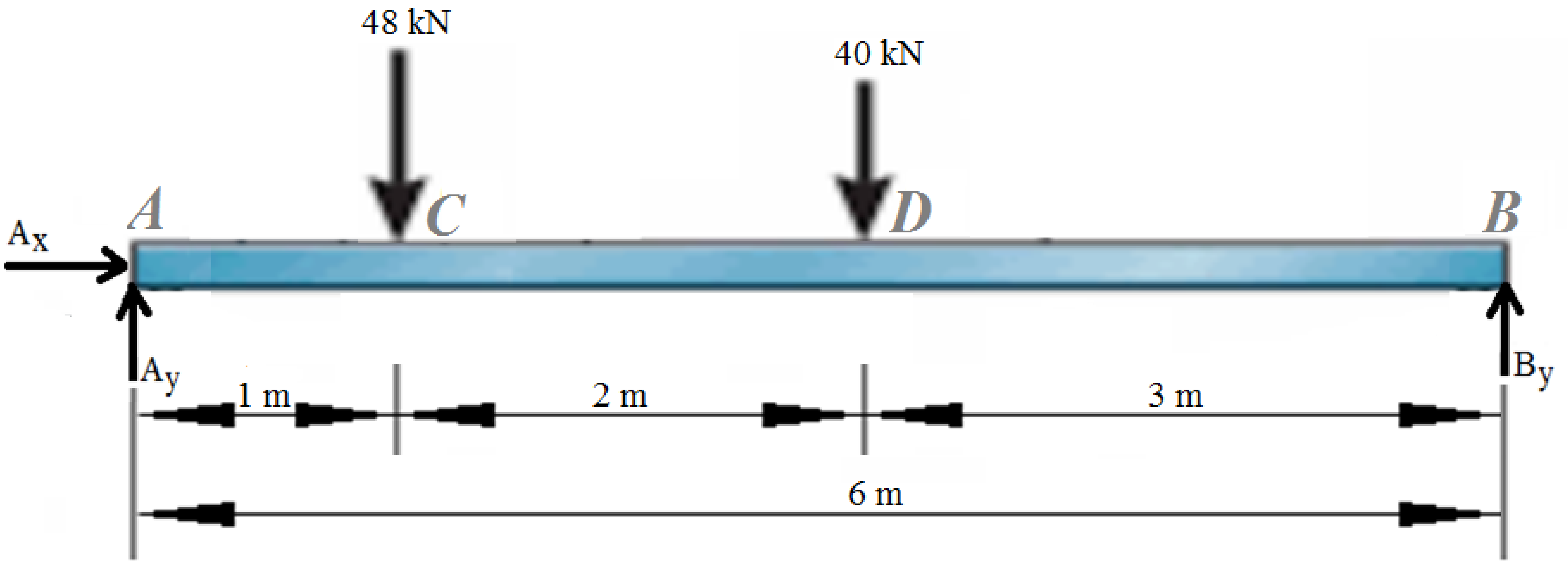
$$\delta/_L = \frac{wL^3}{8EI} = \frac{4I\sigma/_{L^2h}}{8EI} x L^3 = \frac{\sigma L}{2Eh}$$

$$= \frac{\sigma L}{2E\left(\frac{1}{8}L\right)} = \frac{4\sigma}{E}$$

$$= \frac{4 \text{ x } 17500}{28 \text{ x } 10^6} = \frac{1}{400}$$

GIVEN EXAMPLES

5. (i) FBD



$$E = 2 \text{ x } 10^5 \ N/mm^2$$

$$I = 85 \text{ x } 10^6 \ mm^4$$

GIVEN EXAMPLES

(ii) Calculating the support reactions

$$\rightarrow + \sum F_x = 0 \ ; \quad A_x = 0$$

$$+\circlearrowleft \sum M_A = 0 \ ; -(48 \text{ x } 1) - (40 \text{ x } 3) + \left(B_y \text{ x } 6\right) = 0$$

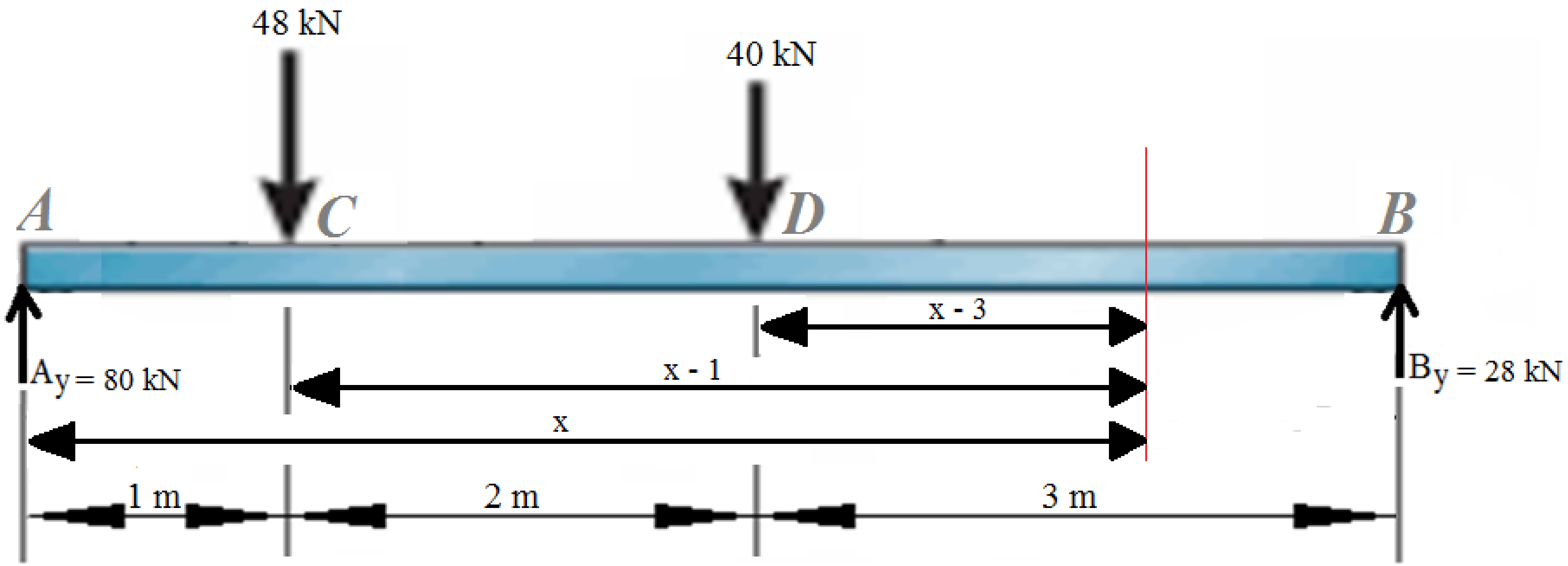
$$B_y = \frac{168}{6} = 28\, kN \uparrow$$

$$+\uparrow \sum F_y = 0 \ ; \quad A_y - 88 + B_y = 0 \ ;$$

$$A_y = 88 - 28 = 60\, kN \uparrow$$

GIVEN EXAMPLES

(iii) Making a virtual sectioning



xviii. DEFLECTION OF BEAMS (67 of 98)

GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$M = EI\frac{d^2y}{dx^2} = 60 < x - 0 > - 48 < x - 1 > - 40 < x - 3 > + 28 < x - 6 >$$

$$= 60\,x - 48 < x - 1 > - 40 < x - 3 > \qquad (i)$$

$$\theta = EI\,\frac{dy}{dx} = 60\frac{x^2}{2} + C1 - 48\frac{< x - 1 >^2}{2} - 40\frac{< x - 3 >^2}{2}$$

$$= 30\,x^2 + C1 \; - 24 < x - 1 >^2 - 20 < x - 3 >^2 \quad (ii)$$

$$\delta = EI\,y = 30\frac{x^3}{3} + C1\,x + C2\; - 24\frac{< x - 1 >^3}{3} - 20\frac{< x - 3 >^3}{3}$$

$$= 10\,x^3 + C1x - C2 - 8 < x - 1 >^3 \frac{20}{3} < x - 3 >^3 \; (iii)$$

GIVEN EXAMPLES

(v) Boundary conditions $\begin{cases} (1)\ x = 0\ ; \quad y = 0 \\ (2)\ x = 6\,m\,; \quad y = 0 \end{cases}$

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 + 0 + C2 - 0 - 0$$

$$C2 = 0$$

GIVEN EXAMPLES

Substituting the 2nd boundary condition in equation (iii), we get

$$EI(0) = 10 \text{ x } (6^3) + (C1 \text{ x } 6) + 0 - 8 \text{ x } (6 - 1)^3 - \frac{20}{3} \text{ x } (6 - 3)^3$$

$$0 = 2160 + 6\,C1 - 1000 - 180$$

$$0 = 980 + 6\,C1$$

$$C1 = -\frac{980}{6} = -163.333$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = EI\frac{dy}{dx} = 30\,x^2 - 163.333 - 24\,\mathrm{x} < x - 1 >^2 - 20\,\mathrm{x} < x - 3 >^2 \;(iv)$$

$$\delta = EIy = 10\,x^3 - 163.333\,x - 8\,\mathrm{x} < x - 1 >^3 - \frac{20}{3}\,\mathrm{x} < x - 3 >^3 \;(v)$$

(vii) Determination of the deflection under point C and D

Substituting $(x = 1)$ in equation (v) by considering the interval $0 < x < 3$ to find δ_C

$$\delta_C = \frac{1}{2\ \mathrm{x}\ 10^5\mathrm{x}\ 85\ \mathrm{x}\ 10^6}\ \mathrm{x}\ [(10\ \mathrm{x}\ 1^3) - (163.333\ \mathrm{x}\ 1) - (8\ \mathrm{x}\ (1-1)^3)]$$

GIVEN EXAMPLES

$$\delta_C = -\frac{153.333\ kNm^3}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6\ Nmm^2} = -\frac{153.33 \text{ x } 10^3 \text{x } 10^9 Nmm^3}{170 \text{ x } 10^{11} Nmm^2} = -9.020\ mm$$

Substituting $(x = 3)$ in equation (v) by considering the interval $0 < x < 6$ to find δ_D

$$\delta_D = \frac{1}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6} \text{ x } \left[(10 \text{ x } 3^{\,3}) - (163.33 \text{ x } 3) - (8 \text{ x } (3-1)^3) - (\frac{20}{3} \text{ x } (3-3)^3)\right]$$

$$= -\frac{283.999\ kNm^3}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6\ Nmm^2} = -\frac{283.999 \text{ x } 10^3 \text{x } 10^9 Nmm^3}{17000 \text{ x } 10^9 Nmm^2} = -16.706\ mm$$

GIVEN EXAMPLES

(viii) Determination of the maximum deflection

The maximum deflection is found where the slope is zero.

Consider the equation (iv) and subdivided it into intervals AC and CD.

$AC \;\; 0 < x < 1\,;\qquad 0 = 30\,x^2 - 163.333$

$$x = \sqrt{\frac{163.333}{30}} = 2.333\,m \;\; (out\ of\ range)$$

$CD \;\; 1 < x < 3\,;\qquad 0 = 30x^2 - 163.333 - 24\,\mathrm{x}\,(x-1)^2$

$$0 = 30x^2 - 163.333 - 24\,\mathrm{x}\,(x^2 - 2\,x + 1)$$

GIVEN EXAMPLES

$CD \;\; 1 < x < 3$; $\quad 0 = 30x^2 - 163.333 - 24\,x^2 + 48\,x - 24$

$$0 = 6x^2 + 48\,x - 187.333$$

$$6x^2 + 48\,x - 187.333 = 0$$

$$x_1 = 2.872\,m$$

$x_2 = -10.872\,m$ (out of range)

The point which give us the maximum deflection is $x_1 = 2.872\,m$.

GIVEN EXAMPLES

Substituting $(x = 2.872)$ in equation (v) by considering the interval $0 < x < 3$ to find δ_{max}

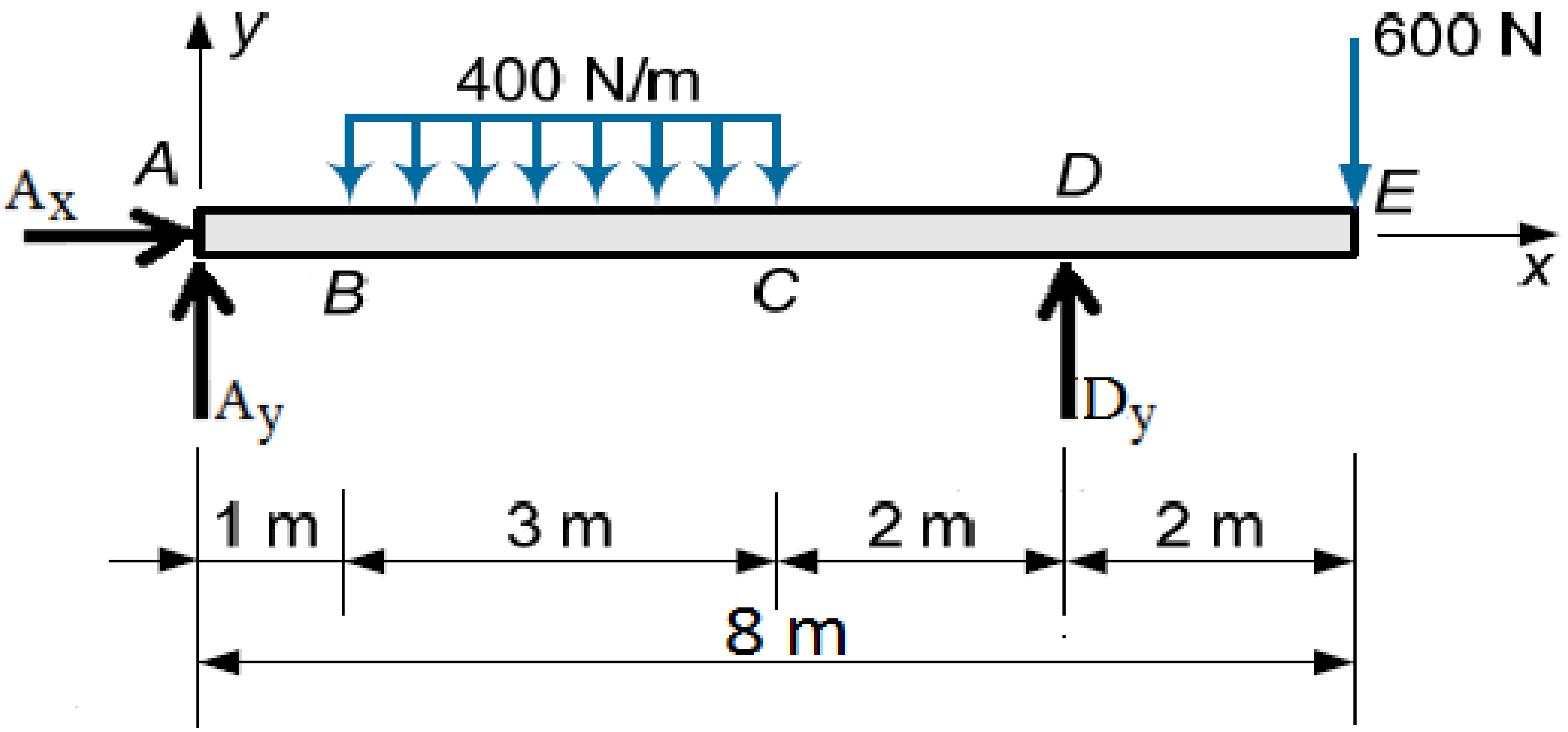
$$\delta_{max} = \frac{1}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6} \text{ x } [(10 \text{ x } 2.872^{\,3}) - (163.33 \text{ x } 2.872) - (8 \text{ x } (2.872 - 1)^3)]$$

$$= \frac{236.894 - 469.092 - 52.482 \; kNm^3}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6 \; Nmm^2}$$

$$= -\frac{284.680 \; kNm^3}{2 \text{ x } 10^5 \text{x } 85 \text{ x } 10^6 \; Nmm^2} = -\frac{284.680 \text{ x } 10^3 \text{x } 10^9 Nmm^3}{17000 \text{ x } 10^9 Nmm^2} = -16.746 \; mm$$

GIVEN EXAMPLES

6. (i) FBD



$E = 12$ GPa

$I = 8.33 \times 10^6$ mm^4

GIVEN EXAMPLES

(ii) Calculating the support reactions

$$\rightarrow + \sum F_x = 0 \;; \quad A_x = 0$$

$$+\circlearrowleft \sum M_A = 0 \;; -((400 \text{ x } 3) \text{ x } (1 + \tfrac{3}{2}) + (D_y \text{ x } 6) - (600 \text{ x } 8) = 0$$

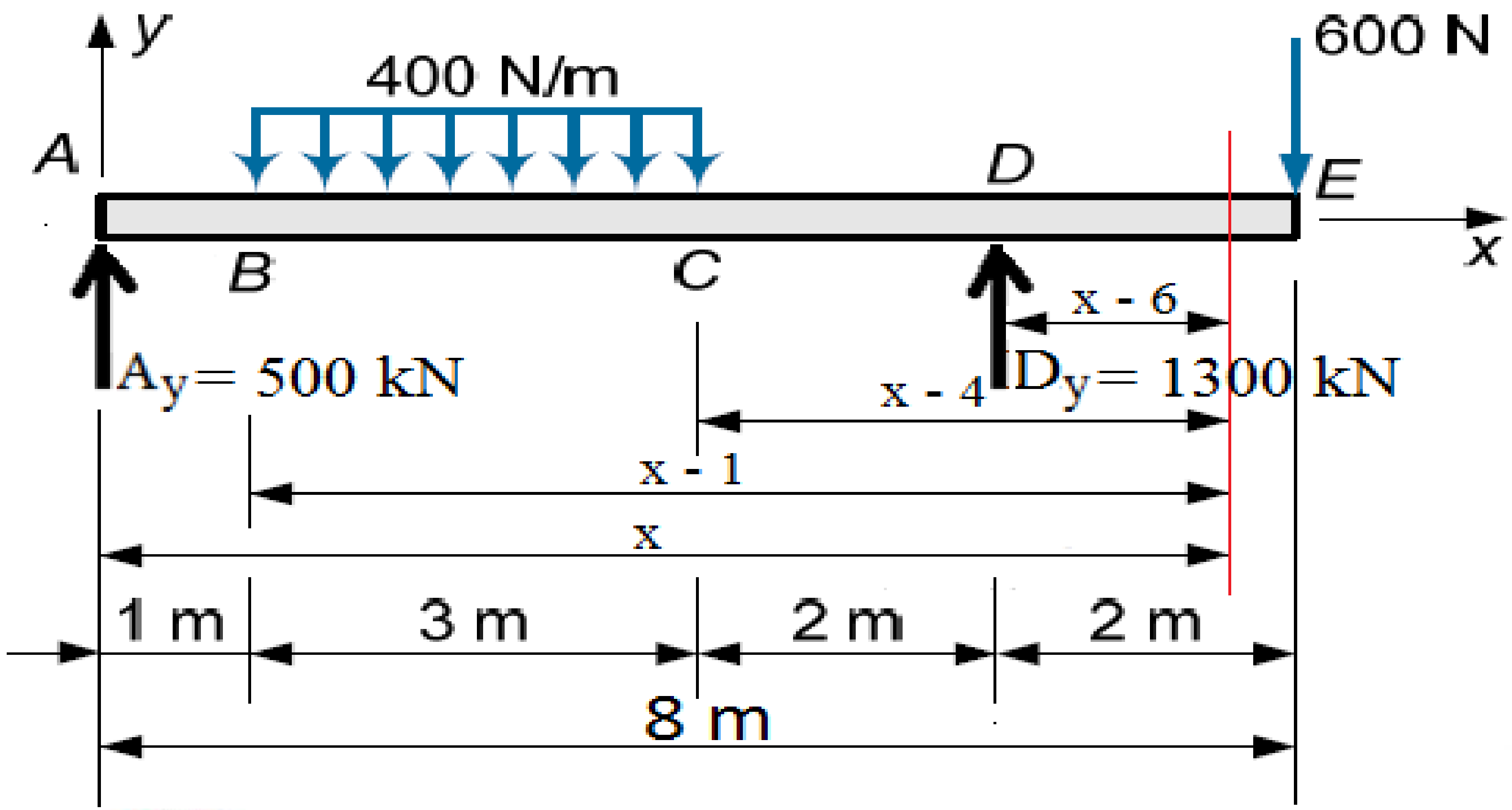
$$D_y = \frac{3000 + 4800}{6} = 1300 \; kN \uparrow$$

$$+\uparrow \sum F_y = 0 \;; \quad A_y - (400 \text{ x } 3) - 600 \; + \; D_y = 0 \;;$$

$$A_y = 1800 - 1300 = 500 \; kN \uparrow$$

GIVEN EXAMPLES

(iii) Making a virtual sectioning



$E = 12$ GPa

$I = 8.33 \times 10^6$ mm^4

GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$M = EI\frac{d^2y}{dx^2} = 500 < x - 0 > - \frac{1}{2}\text{x}\,400 < x - 1 >^2 + \frac{1}{2}\text{x}\,400 < x - 4 >^2 + 1300 < x - 3 > - 600 < x - 8 >$$

$$= 500\,x - \frac{1}{2}\text{x}\,400 < x - 1 >^2 + \frac{1}{2}\text{x}\,400 < x - 4 >^2 + 1300 < x - 6 > (i)$$

$$\theta = EI\,\frac{dy}{dx} = 500\frac{x^2}{2} + C1 - \frac{400}{2}\,\frac{< x - 1 >^3}{3} + \frac{400}{2}\frac{< x - 4 >^3}{3} + 1300\frac{< x - 6 >^2}{2}$$

$$= 250\,x^2 + C1 - \frac{200}{3} < x - 1 >^3 + \frac{200}{3} < x - 4 >^3 + 650 < x - 6 >^2\ (ii)$$

GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$\delta = EI\,y = 250\frac{x^3}{3} + C1\,x + C2\; - \frac{200}{3}\frac{< x-1 >^4}{4} + \frac{200}{3}\frac{< x-4 >^4}{4} + 650\frac{< x-6 >^3}{3}$$

$$= \frac{250}{3}x^3 + C1x - C2 - \frac{50}{3} < x-1 >^4 + \frac{50}{3} < x-4 >^4 + \frac{650}{3} < x-6 >^3 \;(iii)$$

(v) Boundary conditions $\begin{cases} (1)\; x = 0\;; \quad y = 0 \\ (2)\; x = 6\,m\;; \quad y = 0 \end{cases}$

GIVEN EXAMPLES

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 + 0 + C2 - 0 + 0 + 0$$

$$C2 = 0$$

Substituting the 2nd boundary condition in equation (iii), we get

$$EI(0) = (\frac{250}{3} \text{ x } 6^3) + (C1 \text{ x } 6) + 0 - (\frac{50}{3} \text{ x } (6-1)^4) + (\frac{50}{3} \text{ x } (6-4)^4) + (\frac{650}{3} \text{ x } (6-6)^3)$$

$$0 = 18000 + 6\,C1 \; - 10416.667 + 266.667$$

$$C1 = -\frac{7850}{6} = -1308.333$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = EI\frac{dy}{dx} = 250\,x^2 - 1308.333 - \frac{200}{3} < x-1 >^3 + \frac{200}{3} < x-4 >^3 + 650 < x-6 >^2 \;(iv)$$

$$\delta = EIy = \frac{250}{3}x^3 - 1308.333\,x - \frac{50}{3} < x-1 >^4 + \frac{50}{3} < x-4 >^4 + \frac{650}{3} < x-6 >^3 \;(v)$$

(vii) Determination of the deflection under point E.

Substituting $(x = 8)$ in equation (v) by considering the interval $0 < x < 8$ to find δ_E

$$\delta_E = \frac{1}{12 \text{ x } 10^3 \text{x } 8.33 \text{ x } 10^6} \text{ x } \left[\left(\frac{250}{3}\text{ x } 8^{\,3}\right) - (1308.333 \text{ x } 8) - \left(\frac{50}{3}\text{x } (8-1)^4\right) + \left(\frac{50}{3}\text{x } (8-4)^4\right) + \left(\frac{650}{3}\text{x } (8-6)^3\right)\right]$$

$$= \frac{(42666.667 - 10466.664 - 40016.667 + 4266.667 + 1733.333) \text{ x } 10^9 Nmm^3}{99.960 \text{ x } 10^9 Nmm^2} = -18.174\,mm$$

GIVEN EXAMPLES

(viii) Determination of the maximum deflection

The maximum deflection is found where the slope is zero.

Consider the equation (iv) and subdivided it into intervals AB, BC and CD.

$AB \;\; 0 < x < 1$; $\qquad 0 = 250\,x^2 - 1308.333$

$$x = \sqrt{\frac{1308.333}{250}} = 2.288\,m \quad \text{(out of range)}$$

$BC \;\; 1 < x < 4$; $\qquad 0 = 250\,x^2 - 1308.333 - \frac{200}{3}\,\text{x}\,(x-1)^3$

$$0 = 250\,x^2 - 163.333 - \frac{200}{3}\text{x}\,(x^3 - 3x^2 + 3\,x - 1)$$

GIVEN EXAMPLES

$BC \ \ 1 < x < 4 \ ;$ $\quad 0 = 250\, x^2 - 1308.333 - \frac{200}{3} x^3 + 200\, x^2 \ - 200\, x + \frac{200}{3}$

$$0 = 450\, x^2 - 1241.666 - 66.667 x^3 - 200\, x$$

$$-66.667 x^3 + 450\, x^2 - 200\, x - 1241.666 = 0$$

$x_1 = -1.343\, m$ (out of range)

$x_2 = 5.629\, m$ (out of range)

$x_3 = 2.464\, m$

The point which give us the maximum deflection is $x_2 = 2.464\, m$.

GIVEN EXAMPLES

$$CD \;\; 4 < x < 6\;; \quad 0 = 250\,x^2 - 1308.333 - \frac{200}{3}\,\mathrm{x}\,(x-1)^3 + \frac{200}{3}\,\mathrm{x}\,(x-4)^3$$

$$0 = 250\,x^2 - 163.333 - \frac{200}{3}\mathrm{x}\,(x^3 - 3\,x^2 + 3\,x - 1) + \frac{200}{3}\mathrm{x}\,(x^3 - 12\,x^2 + 48\,x - 64)$$

$$0 = 250\,x^2 - 1308.333 - \frac{200}{3}x^3 + 200\,x^2 \; - 200\,x + \frac{200}{3} + \frac{200}{3}x^3 - 1200\,x^2 + 3200\,x - \frac{12800}{3}$$

$$0 = -750\,x^2 - 5508.333 + 3000\,x$$

$$-\,750\,x^2 + 3000\,x - 5508.33 = 0$$

$x_1 = 2.000 + 1.829i\,m$ (complex root)

$x_2 = 2.000 - 1.829i\,m$ (complex root)

GIVEN EXAMPLES

Substituting $(x = 2.464)$ in equation (v) by considering the interval $0 < x < 4$ to find δ_{max}

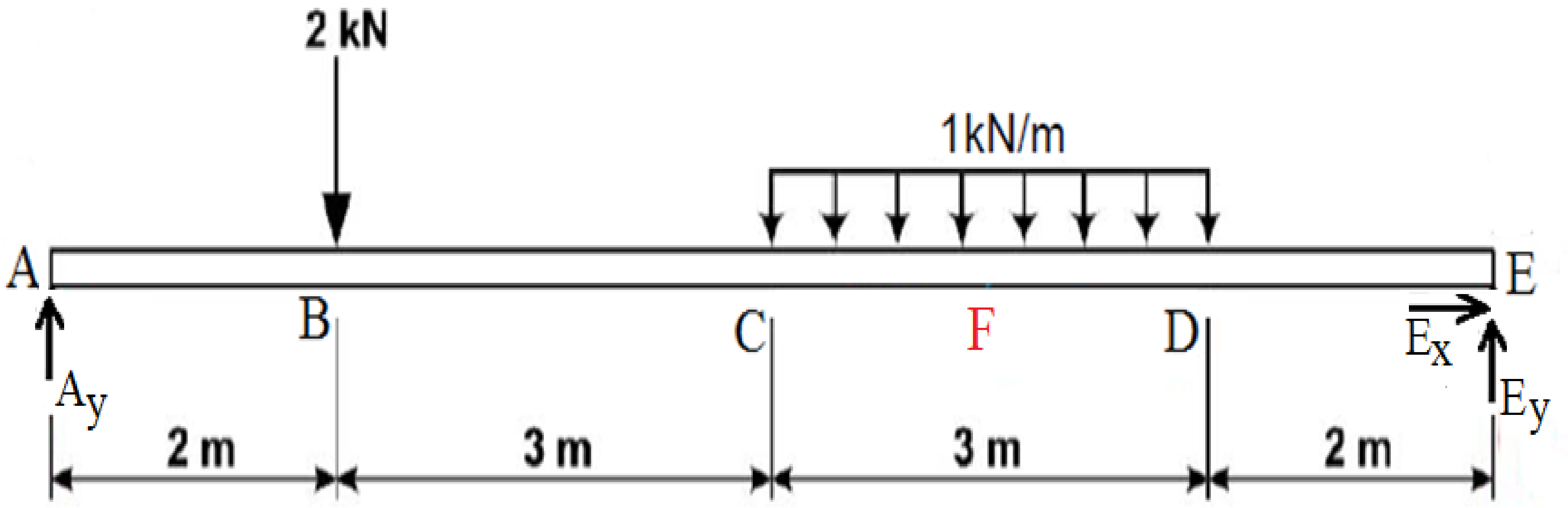
$$\delta_{max} = \frac{1}{12 \text{ x } 10^3 \text{x } 8.33 \text{ x } 10^6} \text{ x } \left[\left(\frac{250}{3} \text{x } 2.464^{\,3}\right) - (1308.333 \text{ x } 2.464) - \left(\frac{50}{3} \text{x } (2.464 - 1)^4\right)\right]$$

$$= \frac{(1246.639 - 3223.733 - 76.562)Nm^3}{99.960 \text{ x } 10^{10}}$$

$$= \frac{(-2053.656) \text{ x } 10^9 Nmm^3}{99.960 \text{ x } 10^9 Nmm^2} = -20.545\ mm$$

GIVEN EXAMPLES

7. (i) FBD



$E = 200\ GPa$

$I = 129 \times 10^6\ mm^4$

GIVEN EXAMPLES

(ii) Calculating the support reactions

$$\rightarrow + \sum F_x = 0 \; ; \quad E_x = 0$$

$$+\circlearrowleft \sum M_A = 0 \; ; -(2 \text{ x } 2) - [(3 \text{ x } 1)\text{x } (5 + \tfrac{3}{2})] + \left(E_y \text{ x } 10\right) = 0$$

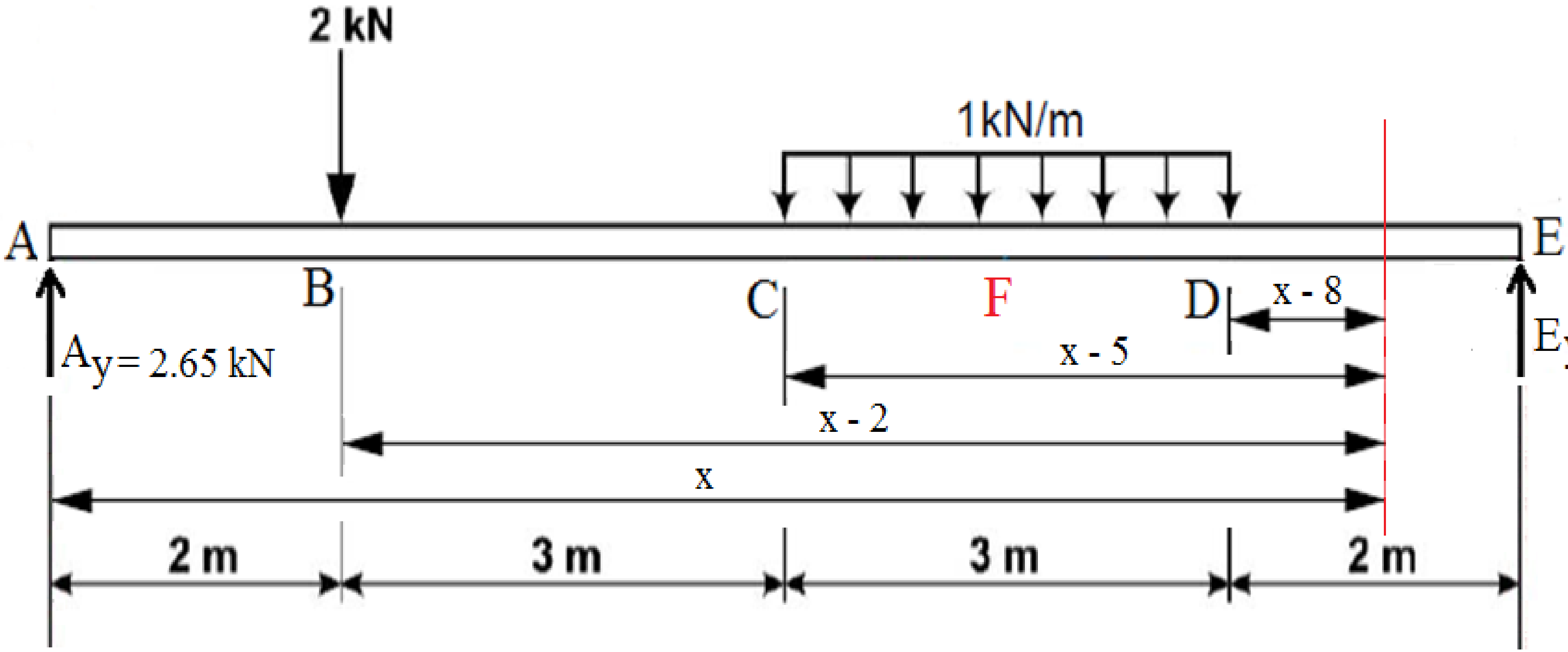
$$E_y = \frac{4 + 19.5}{10} = 2.35 \, kN \uparrow$$

$$+\uparrow \sum F_y = 0 \; ; \quad A_y - 2 - (3 \text{ x } 1) + E_y = 0 \; ;$$

$$A_y = 5 - 2.35 = 2.65 \, kN \uparrow$$

GIVEN EXAMPLES

(iii) Making a virtual sectioning



$E = 200\ GPa$

$I = 129 \times 10^6\ mm^4$

GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$M = EI\frac{d^2y}{dx^2} = 2.65 < x-0 > -2 < x-2 > -\frac{1}{2}\text{x}\,1 < x-5 >^2 + \frac{1}{2}\text{x}\,1 < x-8 >^2 + 2.65 < x-10 >$$

$$= 2.65\,x - \ 2 < x-2 > -\frac{1}{2} < x-5 >^2 + \frac{1}{2} < x-8 >^2 \ (i)$$

$$\theta = EI\ \frac{dy}{dx} = 2.65\frac{x^2}{2} + C1 - 2\ \frac{< x-2 >^2}{2} - \frac{1}{2}\frac{< x-5 >^3}{3} + \frac{1}{2}\frac{< x-8 >^3}{3}$$

$$= \ \frac{2.65}{2}x^2 + C1 - 1 < x-2 >^2 - \frac{1}{6} < x-5 >^3 + \frac{1}{6} < x-8 >^2 \ (ii)$$

GIVEN EXAMPLES

(iv) Applying discontinuity functions method

$$\delta = EI\,y = \frac{2.65}{2}\frac{x^3}{3} + C1\,x + C2\; - 1\,\frac{< x-2 >^3}{3} - \frac{1}{6}\frac{< x-5 >^4}{4} + \frac{1}{6}\frac{< x-8 >^4}{4}$$

$$= \frac{2.65}{6}x^3 + C1\,x - C2 - \frac{1}{3} < x-2 >^3 - \frac{1}{24} < x-5 >^4 + \frac{1}{24} < x-8 >^4 \;(iii)$$

(v) Boundary conditions
$$\begin{cases} (1)\; x = 0\;; \quad y = 0 \\ (2)\; x = 10\,m\;; \quad y = 0 \end{cases}$$

GIVEN EXAMPLES

Substituting the 1st boundary condition in equation (iii), we get

$$EI(0) = 0 + 0 + C2 - 0 - 0 + 0$$

$$C2 = 0$$

Substituting the 2nd boundary condition in equation (iii), we get

$$EI(0) = (\frac{2.65}{3} \text{ x } 6^3) + (C1 \text{ x } 10) - (\frac{1}{3} \text{ x } (10-2)^3) - (\frac{1}{24} \text{ x } (10-5)^4) + (\frac{1}{24} \text{ x } (10-8)^4)$$

$$0 = 441.667 + 10\,C1 - 170.667 - 26.042 + 0.667$$

$$C1 = -\frac{245.625}{10} = -24.563$$

GIVEN EXAMPLES

(vi) Re-writing slope and deflection equations

$$\theta = EI\frac{dy}{dx} = \frac{2.65}{2}x^2 - 24.563 - 1 < x - 2 >^2 - \frac{1}{6} < x - 5 >^3 + \frac{1}{6} < x - 8 >^3 \ (iv)$$

$$\delta = EIy = \frac{2.65}{6}x^3 - 24.563\, x - \frac{1}{3} < x - 1 >^3 - \frac{1}{24} < x - 5 >^4 + \frac{1}{24} < x - 8 >^4 \ (v)$$

(vii) Determination of the deflection under point B and F.

Substituting $(x = 2)$ in equation (v) by considering the interval $0 < x < 2$ to find δ_B

$$\delta_B = \frac{1}{200 \text{ x } 10^3 \text{ x } 129 \text{ x } 10^6} \text{ x } \left[\left(\frac{2.65}{6} \text{ x } 2^{\ 3}\right) - (24.563 \text{ x } 2)\right] = \frac{(3.533 - 490126)\ kNm^3}{25.800 \text{ x } 10^{12}\ Nmm^2}$$

$$= -\frac{45.593 \text{ x } 10^{12} Nmm^3}{25.800 \text{ x } 10^{12} Nmm^2} = -1.767\ mm$$

GIVEN EXAMPLES

Substituting $(x = 6.5)$ in equation (v) by considering the interval $0 < x < 8$ to find δ_F

$$\delta_F = \frac{1}{200 \text{ x } 10^3 \text{x } 129 \text{ x } 10^6} \text{ x } \left[\left(\frac{2.65}{6} \text{x } 6.5^{\ 3}\right) - (24.563 \text{ x } 6.5) - (\frac{1}{3}\text{x } (6.5-2)^3) - (\frac{1}{24} \text{ x } (6.5-5)^4)\right]$$

$$= -\frac{(121.293 - 159.660 - 30.375 - 0.211)kNm^3}{200 \text{ x } 10^3 \text{x } 129 \text{ x } 10^6\ Nmm^2}$$

$$\delta_F = -\frac{68.953\ kNm^3}{200 \text{ x } 10^3 \text{x } 129 \text{ x } 10^6\ Nmm^2} = -\frac{68.953 \text{ x } 10^{12} Nmm^3}{25.800 \text{ x } 10^{12} Nmm^2} = -2.673\ mm$$

GIVEN EXAMPLES

(viii) Determination of the maximum deflection

The maximum deflection is found where the slope is zero.

Consider the equation (iv) and subdivided it into intervals AB, BC, CD and DE.

AB $0 < x < 2$;

$$0 = \frac{2.65}{2} x^2 - 24.563$$

$$x = \sqrt{\frac{24.563 \text{ x } 2}{2.65}} = 4.306\, m$$ (out of range)

GIVEN EXAMPLES

$BC \;\; 2 < x < 5 \; ; \qquad 0 = \frac{2.65}{2}\, x^2 - 24.563 - 1\,(x-2)^2$

$$0 = \frac{2.65}{2}\, x^2 - 24.563 - \;(x^2 - 4\,x + 4)$$

$$0 = \frac{2.65}{2}\, x^2 - x^2 - 24.563 + 4\,x - 4$$

$$0 = 0.325\,x^2 + 4\,x - 28.563$$

$$0.325\,x^2 + 4\,x - 28.563 = 0$$

$x_1 = 5.060\,m$ (out of range)

$x_2 = -17.368\,m$ (out of range)

GIVEN EXAMPLES

$$CD \;\; 5 < x < 8\,; \qquad 0 = \frac{2.65}{2}\,x^2 - 24.563 - 1\,(x-2)^2 - \frac{1}{6}\,(x-5)^3$$

$$0 = \frac{2.65}{2}\,x^2 - 24.563 - \,(x^2 - 4\,x + 4) - \frac{1}{6}(x^3 - 15x^2 + 75\,x - 125)$$

$$0 = -\frac{x^3}{2} + 2.825\,x^2 - 8.5\,x - 7.730$$

$$-x^3 + 16.950\,x^2 - 51\,x - 46.378 = 0$$

$x_1 = 12.616\,m$ (out of range)

$x_2 = 5.060\,m$

$x_3 = -0.726\,m$ (out of range)

The point which give us the maximum deflection is $x_2 = 5.060\,m$.

GIVEN EXAMPLES

$DE \;\; 8 < x < 10 \; ; \qquad 0 = \frac{2.65}{2}\, x^2 - 24.563 - 1\,(x-2)^2 - \frac{1}{6}\,(x-5)^3 + \frac{1}{6}\,(x-8)^3$

$$0 = \frac{2.65}{2}\, x^2 - 24.563 - \;(x^2 - 4\,x + 4) - \frac{1}{6}(x^3 - 15\,x^2 + 75\,x - 125) + \frac{1}{6}(x^3 - 24\,x^2 + 192\,x - 512)$$

$$0 = -1.175\,x^2 + 23.5\,x - 93.063$$

$$-1.175\,x^2 + 23.5\,x - 93.063 = 0$$

$x_1 = 14.560\,m$ (out of range)

$x_2 = 5.440\,m$ (out of range)

GIVEN EXAMPLES

Substituting $(x = 5.060)$ in equation (v) by considering the interval $0 < x < 8$ to find δ_{max}

$$\delta_{max} = \frac{1}{200 \text{ x } 10^3 \text{x } 129 \text{ x } 10^6} \text{ x } \left[\left(\frac{2.65}{6} \text{x } 5.06^{\,3}\right) - (24.563 \text{ x } 5.06) - \left(\frac{1}{3} \text{x } (5.06-2)^3\right) - \left(\frac{1}{24} \text{x } (5.06-5)^4\right)\right]$$

$$= \frac{(57.220 - 124.289 - 9.551 - 0.0000054)\ kNm^3}{200 \text{ x } 10^3 \text{x } 129 \text{ x } 10^6\ Nmm^2}$$

$$\delta_{max} = -\frac{76.620 \text{ x } 10^{12} Nmm^3}{25.800 \text{ x } 10^{12} Nmm^2} = -2.970\ mm$$