Chapter 16

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Kinematics is important for the design of the mechanism used on this dump truck.

Planar Kinematics of a Rigid Body

CHAPTER OBJECTIVES

- To classify the various types of rigid-body planar motion.
- To investigate rigid-body translation and angular motion about a fixed axis.
- To study planar motion using an absolute motion analysis.
- To provide a relative motion analysis of velocity and acceleration using a translating frame of reference.
- To show how to find the instantaneous center of zero velocity and determine the velocity of a point on a body using this method.
- To provide a relative-motion analysis of velocity and acceleration using a rotating frame of reference.

16.1 Planar Rigid-Body Motion

In this chapter, the planar kinematics of a rigid body will be discussed. This study is important for the design of gears, cams, and mechanisms used for many mechanical operations. Once the kinematics is thoroughly understood, then we can apply the equations of motion, which relate the forces on the body to the body's motion.

The *planar motion* of a body occurs when all the particles of a rigid body move along paths which are equidistant from a fixed plane. There are three types of rigid-body planar motion. In order of increasing complexity, they are

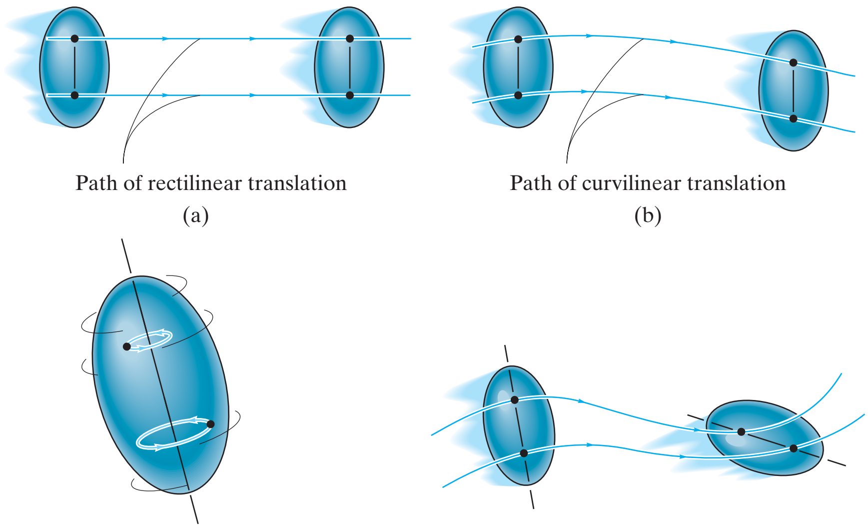

Fig. 16–1

- *Translation.* This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion. When the paths of motion for any two points on the body are parallel lines, the motion is called *rectilinear translation*, Fig. 16–1*a*. If the paths of motion are along curved lines, the motion is called *curvilinear translation*, Fig. 16–1*b*.
- *Rotation about a fixed axis.* When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along circular paths, Fig. 16–1*c*.
- *General plane motion.* When a body is subjected to general plane motion, it undergoes a combination of translation *and* rotation, Fig. 16–1*d*. The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.

In the following sections we will consider each of these motions in detail. Examples of bodies undergoing these motions are shown in Fig. 16–2.

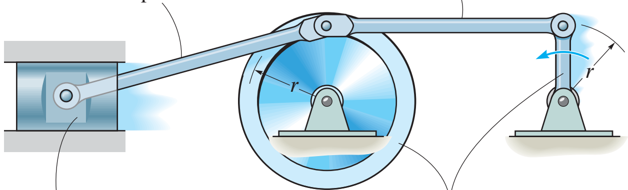

Fig. 16–2

16.2 Translation

Consider a rigid body which is subjected to either rectilinear or curvilinear translation in the *x*–*y* plane, Fig. 16–3.

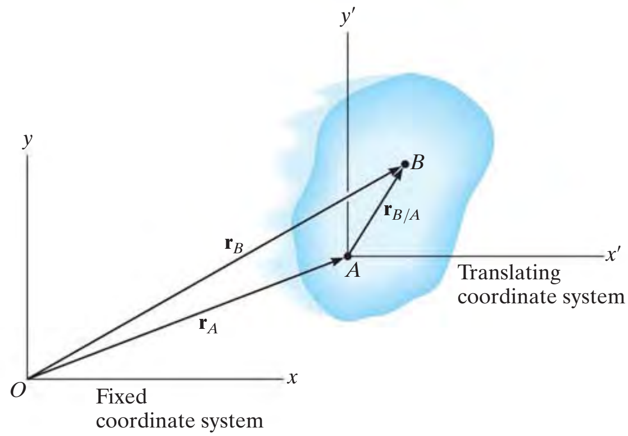

Fig. 16–3

Position. The locations of points A and B on the body are defined with respect to fixed *x, y* reference frame using *position vectors* $\mathbf{r}_A$ and $\mathbf{r}_B$. The translating x', y' coordinate system is *fixed in the body* and has its origin at A, hereafter referred to as the *base point*. The position of B with respect to A is denoted by the *relative-position vector* $\mathbf{r}_{B/A}$ ("**r** of B with respect to A"). By vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Velocity. A relation between the instantaneous velocities of A and B is obtained by taking the time derivative of this equation, which yields $\mathbf{v}_B = \mathbf{v}_A + d\mathbf{r}_{B/A}/dt$. Here $\mathbf{v}_A$ and $\mathbf{v}_B$ denote *absolute velocities* since these vectors are measured with respect to the *x, y* axes. The term $d\mathbf{r}_{B/A}/dt = \mathbf{0}$, since the *magnitude* of $\mathbf{r}_{B/A}$ is *constant* by definition of a rigid body, and because the body is translating the *direction* of $\mathbf{r}_{B/A}$ is also *constant*. Therefore,

$$\mathbf{v}_B = \mathbf{v}_A$$

Acceleration. Taking the time derivative of the velocity equation yields a similar relationship between the instantaneous accelerations of A and B:

$$\mathbf{a}_B = \mathbf{a}_A$$

The above two equations indicate that *all points in a rigid body subjected to either rectilinear or curvilinear translation move with the same velocity and acceleration*. As a result, the kinematics of particle motion, discussed in Chapter 12, can also be used to specify the kinematics of points located in a translating rigid body.

Riders on this Ferris wheel are subjected to curvilinear translation, since the gondolas move in a circular path, yet it always remains in the upright position. (© R.C. Hibbeler)

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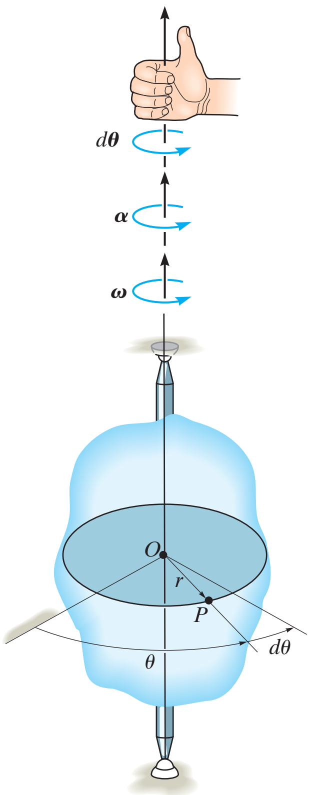


(a)

16.3 Rotation about a Fixed Axis

When a body rotates about a fixed axis, any point P located in the body travels along a *circular path*. To study this motion it is first necessary to discuss the angular motion of the body about the axis.

Angular Motion. Since a point is without dimension, it cannot have angular motion. *Only lines or bodies undergo angular motion*. For example, consider the body shown in Fig. 16–4*a* and the angular motion of a radial line r located within the shaded plane.

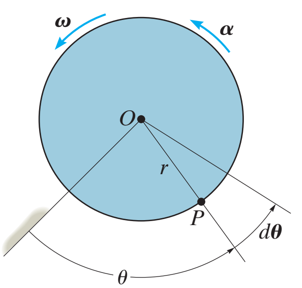
Angular Position. At the instant shown, the *angular position* of r is defined by the angle θ, measured from a *fixed* reference line to r.

Angular Displacement. The change in the angular position, which can be measured as a differential $d\boldsymbol{\theta}$, is called the *angular displacement*.* This vector has a *magnitude* of $d\theta$, measured in degrees, radians, or revolutions, where 1 rev $= 2\pi$ rad. Since motion is about a *fixed axis*, the direction of $d\boldsymbol{\theta}$ is *always* along this axis. Specifically, the *direction* is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or $d\boldsymbol{\theta}$, points upward, Fig. 16–4*a*. In two dimensions, as shown by the top view of the shaded plane, Fig. 16–4*b*, both θ and $d\theta$ are counterclockwise, and so the thumb points outward from the page.

Angular Velocity. The time rate of change in the angular position is called the *angular velocity* $\boldsymbol{\omega}$ (omega). Since $d\boldsymbol{\theta}$ occurs during an instant of time dt, then,

$$(\circlearrowleft +) \qquad \omega = \frac{d\theta}{dt} \qquad (16\text{–}1)$$

This vector has a *magnitude* which is often measured in rad/s. It is expressed here in scalar form since its *direction* is also along the axis of rotation, Fig. 16–4*a*. When indicating the angular motion in the shaded plane, Fig. 16–4*b*, we can refer to the sense of rotation as clockwise or counterclockwise. Here we have *arbitrarily* chosen counterclockwise rotations as *positive* and indicated this by the curl shown in parentheses next to Eq. 16–1. Realize, however, that the directional sense of $\boldsymbol{\omega}$ is actually outward from the page.



(b)

Fig. 16–4

*It is shown in Sec. 20.1 that finite rotations or finite angular displacements are *not* vector quantities, although differential rotations $d\boldsymbol{\theta}$ are vectors.

Angular Acceleration. The *angular acceleration* $\boldsymbol{\alpha}$ (alpha) measures the time rate of change of the angular velocity. The *magnitude* of this vector is

$$(\circlearrowleft+) \qquad \boxed{\alpha = \frac{d\omega}{dt}} \qquad (16\text{–}2)$$

Using Eq. 16–1, it is also possible to express α as

$$(\circlearrowleft+) \qquad \alpha = \frac{d^2\theta}{dt^2} \qquad (16\text{–}3)$$

The line of action of $\boldsymbol{\alpha}$ is the same as that for $\boldsymbol{\omega}$, Fig. 16–4*a*; however, its sense of *direction* depends on whether $\boldsymbol{\omega}$ is increasing or decreasing. If $\boldsymbol{\omega}$ is decreasing, then $\boldsymbol{\alpha}$ is called an *angular deceleration* and therefore has a sense of direction which is opposite to $\boldsymbol{\omega}$.

By eliminating dt from Eqs. 16–1 and 16–2, we obtain a differential relation between the angular acceleration, angular velocity, and angular displacement, namely,

$$(\circlearrowleft+) \qquad \boxed{\alpha\, d\theta = \omega\, d\omega} \qquad (16\text{–}4)$$

The similarity between the differential relations for angular motion and those developed for rectilinear motion of a particle ($v = ds/dt$, $a = dv/dt$, and $a\,ds = v\,dv$) should be apparent.

Constant Angular Acceleration. If the angular acceleration of the body is *constant*, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_c$, then Eqs. 16–1, 16–2, and 16–4, when integrated, yield a set of formulas which relate the body's angular velocity, angular position, and time. These equations are similar to Eqs. 12–4 to 12–6 used for rectilinear motion. The results are

$$(\circlearrowleft+) \qquad \omega = \omega_0 + \alpha_c t \qquad (16\text{–}5)$$

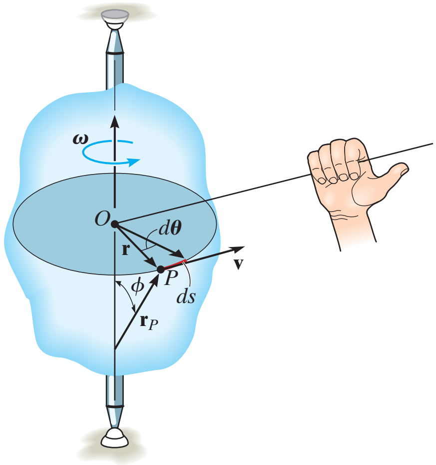
$$(\circlearrowleft+) \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2 \qquad (16\text{–}6)$$

$$(\circlearrowleft+) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0) \qquad (16\text{–}7)$$

Constant Angular Acceleration

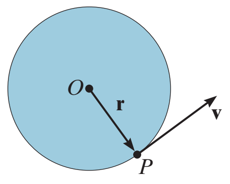
Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity, respectively.

The gears used in the operation of a crane all rotate about fixed axes. Engineers must be able to relate their angular motions in order to properly design this gear system. (© R.C. Hibbeler)



(c)

Fig. 16–4 (cont.)



(d)

Motion of Point *P*. As the rigid body in Fig. 16–4*c* rotates, point P travels along a *circular path* of radius r with center at point O. This path is contained within the shaded plane shown in top view, Fig. 16–4*d*.

Position and Displacement. The position of P is defined by the position vector $\mathbf{r}$, which extends from O to P. If the body rotates $d\theta$ then P will displace $ds = r\,d\theta$.

Velocity. The velocity of P has a magnitude which can be found by dividing $ds = r\,d\theta$ by dt so that

$$v = \omega r \tag{16–8}$$

As shown in Figs. 16–4*c* and 16–4*d*, the *direction* of $\mathbf{v}$ is *tangent* to the circular path.

Both the magnitude and direction of $\mathbf{v}$ can also be accounted for by using the cross product of $\boldsymbol{\omega}$ and $\mathbf{r}_P$ (see Appendix B). Here, $\mathbf{r}_P$ is directed from *any point* on the axis of rotation to point P, Fig. 16–4*c*. We have

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P \tag{16–9}$$

The order of the vectors in this formulation is important, since the cross product is not commutative, i.e., $\boldsymbol{\omega} \times \mathbf{r}_P \neq \mathbf{r}_P \times \boldsymbol{\omega}$. Notice in Fig. 16–4*c* how the correct direction of $\mathbf{v}$ is established by the right-hand rule. The fingers of the right hand are curled from $\boldsymbol{\omega}$ toward $\mathbf{r}_P$ ($\boldsymbol{\omega}$ "cross" $\mathbf{r}_P$). The thumb indicates the correct direction of $\mathbf{v}$, which is tangent to the path in the direction of motion. From Eq. B–8, the magnitude of $\mathbf{v}$ in Eq. 16–9 is $v = \omega r_P \sin\phi$, and since $r = r_P \sin\phi$, Fig. 16–4*c*, then $v = \omega r$, which agrees with Eq. 16–8. As a special case, the position vector $\mathbf{r}$ can be chosen for $\mathbf{r}_P$. Here $\mathbf{r}$ lies in the plane of motion and again the velocity of point P is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{16–10}$$

Acceleration. The acceleration of P can be expressed in terms of its normal and tangential components. Applying Eq. 12–19 and Eq. 12–20, $a_t = dv/dt$ and $a_n = v^2/\rho$, where $\rho = r$, $v = \omega r$, and $\alpha = d\omega/dt$, we get

$$\boxed{a_t = \alpha r} \qquad (16\text{–}11)$$

$$\boxed{a_n = \omega^2 r} \qquad (16\text{–}12)$$

The *tangential component of acceleration*, Figs. 16–4*e* and 16–4*f*, represents the time rate of change in the velocity's magnitude. If the speed of P is increasing, then $\mathbf{a}_t$ acts in the same direction as $\mathbf{v}$; if the speed is decreasing, $\mathbf{a}_t$ acts in the opposite direction of $\mathbf{v}$; and finally, if the speed is constant, $\mathbf{a}_t$ is zero.

The *normal component of acceleration* represents the time rate of change in the velocity's direction. The *direction* of $\mathbf{a}_n$ is always toward O, the center of the circular path, Figs. 16–4*e* and 16–4*f*.

Like the velocity, the acceleration of point P can be expressed in terms of the vector cross product. Taking the time derivative of Eq. 16–9 we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

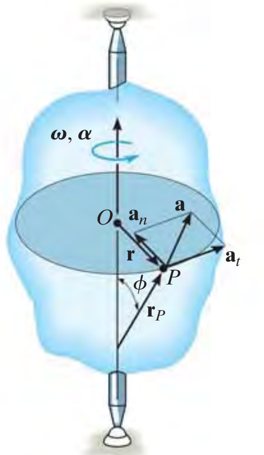
Recalling that $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$, and using Eq. 16–9 ($d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$), yields

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) \qquad (16\text{–}13)$$

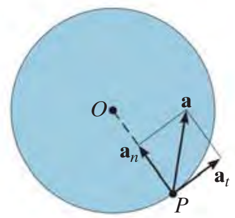
From the definition of the cross product, the first term on the right has a magnitude $a_t = \alpha r_P \sin\phi = \alpha r$, and by the right-hand rule, $\boldsymbol{\alpha} \times \mathbf{r}_P$ is in the direction of $\mathbf{a}_t$, Fig. 16–4*e*. Likewise, the second term has a magnitude $a_n = \omega^2 r_P \sin\phi = \omega^2 r$, and applying the right-hand rule twice, first to determine the result $\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_P$ then $\boldsymbol{\omega} \times \mathbf{v}_P$, it can be seen that this result is in the same direction as $\mathbf{a}_n$, shown in Fig. 16–4*e*. Noting that this is also the *same* direction as $-\mathbf{r}$, which lies in the plane of motion, we can express $\mathbf{a}_n$ in a much simpler form as $\mathbf{a}_n = -\omega^2\mathbf{r}$. Hence, Eq. 16–13 can be identified by its two components as

$$\boxed{\begin{aligned}\mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}\end{aligned}} \qquad (16\text{–}14)$$

Since $\mathbf{a}_t$ and $\mathbf{a}_n$ are perpendicular to one another, if needed the magnitude of acceleration can be determined from the Pythagorean theorem; namely, $a = \sqrt{a_n^2 + a_t^2}$, Fig. 16–4*f*.

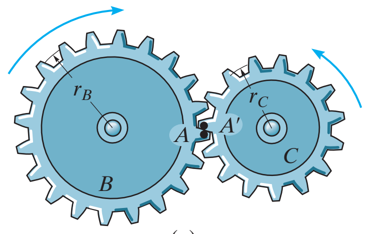


(e)



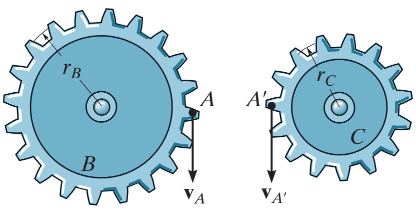
(f)

Fig. 16–4 (cont.)



(a)

If two rotating bodies contact one another, then the *points in contact* move along *different circular paths*, and the velocity and *tangential components* of acceleration of the points will be the *same*: however, the *normal components* of acceleration will *not* be the same. For example, consider the two meshed gears in Fig. 16–5*a*. Point *A* is located on gear *B* and a coincident point *A*′ is located on gear *C*. Due to the rotational motion, $\mathbf{v}_A = \mathbf{v}_{A'}$, Fig. 16–5*b*, and as a result, $\omega_B r_B = \omega_C r_C$ or $\omega_B = \omega_C(r_C/r_B)$. Also, from Fig. 16–5*c*, $(\mathbf{a}_A)_t = (\mathbf{a}_{A'})_t$, so that $\alpha_B = \alpha_C(r_C/r_B)$; however, since both points follow different circular paths, $(\mathbf{a}_A)_n \neq (\mathbf{a}_{A'})_n$ and therefore, as shown, $\mathbf{a}_A \neq \mathbf{a}_{A'}$.



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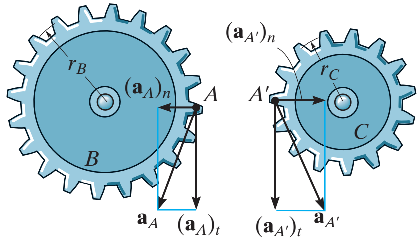


(c)

Fig. 16–5

Important Points

- A body can undergo two types of translation. During rectilinear translation all points follow parallel straight-line paths, and during curvilinear translation the points follow curved paths that are the same shape.
- All the points on a translating body move with the same velocity and acceleration.
- Points located on a body that rotates about a fixed axis follow circular paths.
- The relation $\alpha\, d\theta = \omega\, d\omega$ is derived from $\alpha = d\omega/dt$ and $\omega = d\theta/dt$ by eliminating *dt*.
- Once angular motions ω and α are known, the velocity and acceleration of any point on the body can be determined.
- The velocity always acts tangent to the path of motion.
- The acceleration has two components. The tangential acceleration measures the rate of change in the magnitude of the velocity and can be determined from $a_t = \alpha r$. The normal acceleration measures the rate of change in the direction of the velocity and can be determined from $a_n = \omega^2 r$.

Procedure for Analysis

The velocity and acceleration of a point located on a rigid body that is rotating about a fixed axis can be determined using the following procedure.

Angular Motion.

- Establish the positive sense of rotation about the axis of rotation and show it alongside each kinematic equation as it is applied.
- If a relation is known between any *two* of the four variables α, ω, θ, and t, then a third variable can be obtained by using one of the following kinematic equations which relates all three variables.

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha\, d\theta = \omega\, d\omega$$

- If the body's angular acceleration is *constant*, then the following equations can be used:

$$\omega = \omega_0 + \alpha_c t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

- Once the solution is obtained, the sense of θ, ω, and α is determined from the algebraic signs of their numerical quantities.

Motion of Point *P*.

- In most cases the velocity of P and its two components of acceleration can be determined from the scalar equations

$$v = \omega r$$
$$a_t = \alpha r$$
$$a_n = \omega^2 r$$

- If the geometry of the problem is difficult to visualize, the following vector equations should be used:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}_P = \boldsymbol{\alpha} \times \mathbf{r}$$
$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) = -\omega^2 \mathbf{r}$$

- Here $\mathbf{r}_P$ is directed from any point on the axis of rotation to point P, whereas $\mathbf{r}$ lies in the plane of motion of P. Either of these vectors, along with $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, should be expressed in terms of its $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components, and, if necessary, the cross products determined using a determinant expansion (see Eq. B–12).

EXAMPLE 16.1

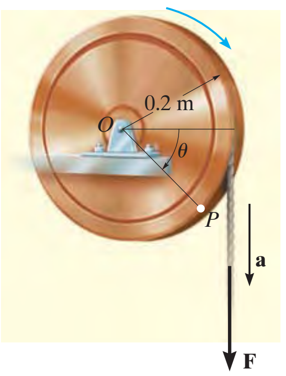


Fig. 16–6

A cord is wrapped around a wheel in Fig. 16–6, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \text{m/s}^2$, where t is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line OP in radians.

SOLUTION

Part (a). The wheel is subjected to rotation about a fixed axis passing through point O. Thus, point P on the wheel has motion about a circular path, and the acceleration of this point has *both* tangential and normal components. The tangential component is $(a_P)_t = (4t)\ \text{m/s}^2$, since the cord is wrapped around the wheel and moves *tangent* to it. Hence the angular acceleration of the wheel is

$$(\circlearrowright +) \qquad (a_P)_t = \alpha r$$

$$(4t)\ \text{m/s}^2 = \alpha(0.2\ \text{m})$$

$$\alpha = (20t)\ \text{rad/s}^2 \circlearrowright$$

Using this result, the wheel's angular velocity ω can now be determined from $\alpha = d\omega/dt$, since this equation relates α, t, and ω. Integrating, with the initial condition that $\omega = 0$ when $t = 0$, yields

$$(\circlearrowright +) \qquad \alpha = \frac{d\omega}{dt} = (20t)\ \text{rad/s}^2$$

$$\int_0^\omega d\omega = \int_0^t 20t\ dt$$

$$\omega = 10t^2\ \text{rad/s} \circlearrowright \qquad \textit{Ans.}$$

Part (b). Using this result, the angular position θ of OP can be found from $\omega = d\theta/dt$, since this equation relates θ, ω, and t. Integrating, with the initial condition $\theta = 0$ when $t = 0$, we have

$$(\circlearrowright +) \qquad \frac{d\theta}{dt} = \omega = (10t^2)\ \text{rad/s}$$

$$\int_0^\theta d\theta = \int_0^t 10t^2\ dt$$

$$\theta = 3.33t^3\ \text{rad} \qquad \textit{Ans.}$$

NOTE: We cannot use the equation of constant angular acceleration, since α is a function of time.

The motor shown in the photo is used to turn a wheel and attached blower contained within the housing. The details are shown in Fig. 16–7*a*. If the pulley *A* connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$, determine the magnitudes of the velocity and acceleration of point *P* on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.

(© R.C. Hibbeler)

SOLUTION

Angular Motion. First we will convert the two revolutions to radians. Since there are 2π rad in one revolution, then

$$\theta_A = 2\text{ rev}\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right) = 12.57\text{ rad}$$

Since α_A is constant, the angular velocity of pulley *A* is therefore

$$(\circlearrowright +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$\omega_A^2 = 0 + 2(2\ \text{rad/s}^2)(12.57\text{ rad} - 0)$$

$$\omega_A = 7.090\ \text{rad/s}$$

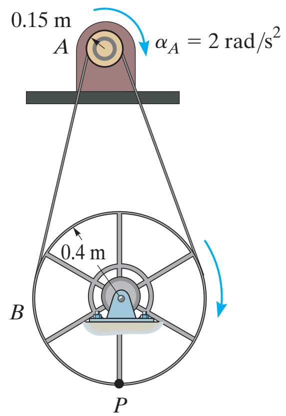
The belt has the same speed and tangential component of acceleration as it passes over the pulley and wheel. Thus,

$$v = \omega_A r_A = \omega_B r_B; \quad 7.090\ \text{rad/s}\ (0.15\text{ m}) = \omega_B(0.4\text{ m})$$

$$\omega_B = 2.659\ \text{rad/s}$$

$$a_t = \alpha_A r_A = \alpha_B r_B; \quad 2\ \text{rad/s}^2\ (0.15\text{ m}) = \alpha_B(0.4\text{ m})$$

$$\alpha_B = 0.750\ \text{rad/s}^2$$



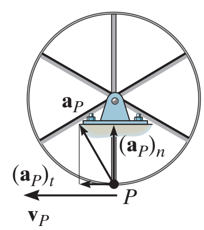
(a)

Motion of *P*. As shown on the kinematic diagram in Fig. 16–7*b*, we have

$$v_P = \omega_B r_B = 2.659\ \text{rad/s}\ (0.4\text{ m}) = 1.06\ \text{m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha_B r_B = 0.750\ \text{rad/s}^2\ (0.4\text{ m}) = 0.3\ \text{m/s}^2$$

$$(a_P)_n = \omega_B^2 r_B = (2.659\ \text{rad/s})^2(0.4\text{ m}) = 2.827\ \text{m/s}^2$$



(b)

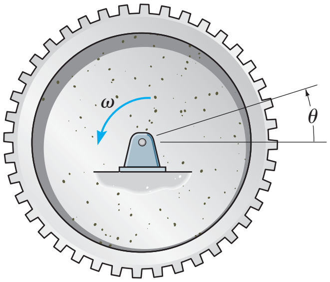
Thus

$$a_P = \sqrt{(0.3\ \text{m/s}^2)^2 + (2.827\ \text{m/s}^2)^2} = 2.84\ \text{m/s}^2 \qquad \textit{Ans.}$$

Fig. 16–7

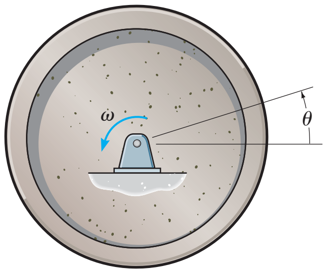
FUNDAMENTAL PROBLEMS

F16–1. When the gear rotates 20 revolutions, it achieves an angular velocity of $\omega = 30$ rad/s, starting from rest. Determine its constant angular acceleration and the time required.



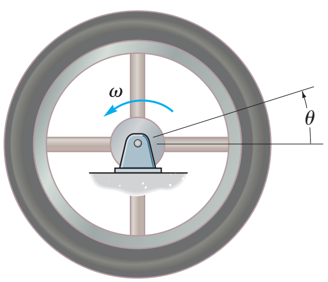
Prob. F16–1

F16–2. The flywheel rotates with an angular velocity of $\omega = (0.005\theta^2)$ rad/s, where θ is in radians. Determine the angular acceleration when it has rotated 20 revolutions.



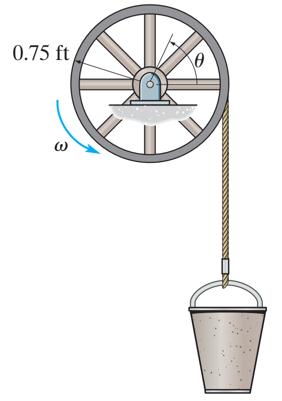
Prob. F16–2

F16–3. The flywheel rotates with an angular velocity of $\omega = (4\,\theta^{1/2})$ rad/s, where θ is in radians. Determine the time it takes to achieve an angular velocity of $\omega = 150$ rad/s. When $t = 0$, $\theta = 1$ rad.



Prob. F16–3

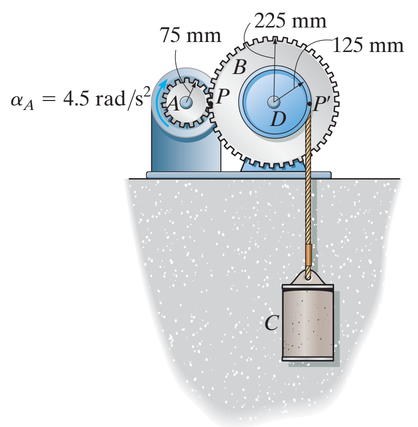
F16–4. The bucket is hoisted by the rope that wraps around a drum wheel. If the angular displacement of the wheel is $\theta = (0.5t^3 + 15t)$ rad, where t is in seconds, determine the velocity and acceleration of the bucket when $t = 3$ s.



Prob. F16–4

F16–5. A wheel has an angular acceleration of $\alpha = (0.5\,\theta)$ rad/s^2, where θ is in radians. Determine the magnitude of the velocity and acceleration of a point P located on its rim after the wheel has rotated 2 revolutions. The wheel has a radius of 0.2 m and starts at $\omega_0 = 2$ rad/s.

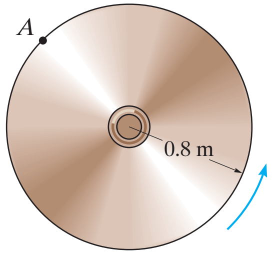
F16–6. For a short period of time, the motor turns gear A with a constant angular acceleration of $\alpha_A = 4.5$ rad/s^2, starting from rest. Determine the velocity of the cylinder and the distance it travels in three seconds. The cord is wrapped around pulley D which is rigidly attached to gear B.



Prob. F16–6

PROBLEMS

16–1. The angular velocity of the disk is defined by $\omega = (5t^2 + 2)$ rad/s, where t is in seconds. Determine the magnitudes of the velocity and acceleration of point A on the disk when $t = 0.5$ s.

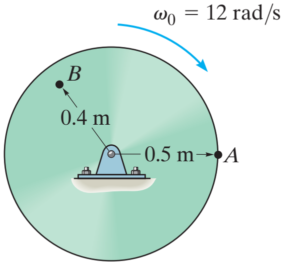


Prob. 16–1

16–2. The angular acceleration of the disk is defined by $\alpha = 3t^2 + 12$ rad/s, where t is in seconds. If the disk is originally rotating at $\omega_0 = 12$ rad/s, determine the magnitude of the velocity and the n and t components of acceleration of point A on the disk when $t = 2$ s.

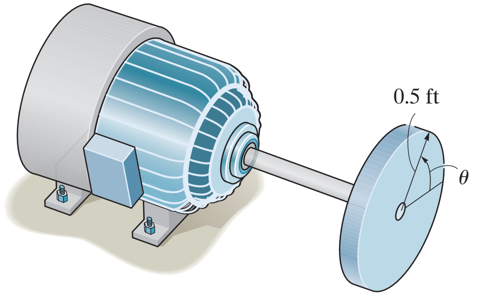
16–3. The disk is originally rotating at $\omega_0 = 12$ rad/s. If it is subjected to a constant angular acceleration of $\alpha = 20\ \text{rad/s}^2$, determine the magnitudes of the velocity and the n and t components of acceleration of point A at the instant $t = 2$ s.

***16–4.** The disk is originally rotating at $\omega_0 = 12$ rad/s. If it is subjected to a constant angular acceleration of $\alpha = 20\ \text{rad/s}^2$, determine the magnitudes of the velocity and the n and t components of acceleration of point B when the disk undergoes 2 revolutions.



Probs. 16–2/3/4

16–5. The disk is driven by a motor such that the angular position of the disk is defined by $\theta = (20t + 4t^2)$ rad, where t is in seconds. Determine the number of revolutions, the angular velocity, and angular acceleration of the disk when $t = 90$ s.

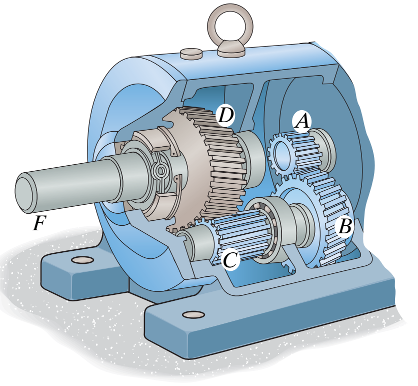


Prob. 16–5

16–6. A wheel has an initial clockwise angular velocity of 10 rad/s and a constant angular acceleration of $3\ \text{rad/s}^2$. Determine the number of revolutions it must undergo to acquire a clockwise angular velocity of 15 rad/s. What time is required?

16–7. If gear A rotates with a constant angular acceleration of $\alpha_A = 90\ \text{rad/s}^2$, starting from rest, determine the time required for gear D to attain an angular velocity of 600 rpm. Also, find the number of revolutions of gear D to attain this angular velocity. Gears A, B, C, and D have radii of 15 mm, 50 mm, 25 mm, and 75 mm, respectively.

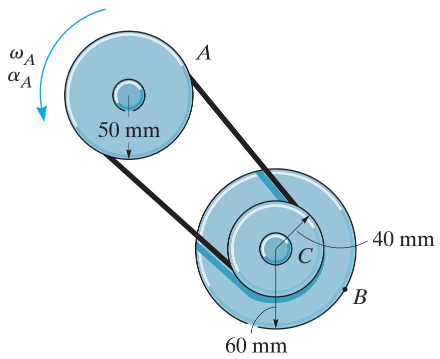
***16–8.** If gear A rotates with an angular velocity of $\omega_A = (\theta_A + 1)$ rad/s, where θ_A is the angular displacement of gear A, measured in radians, determine the angular acceleration of gear D when $\theta_A = 3$ rad, starting from rest. Gears A, B, C, and D have radii of 15 mm, 50 mm, 25 mm, and 75 mm, respectively.



Probs. 16–7/8

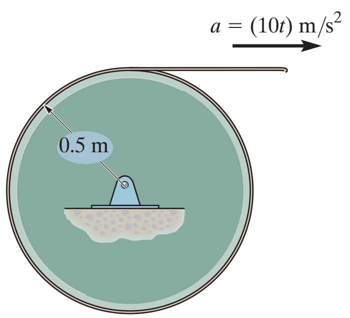
16–9. At the instant $\omega_A = 5$ rad/s, pulley A is given an angular acceleration $\alpha = (0.8\theta)$ rad/s^2, where θ is in radians. Determine the magnitude of acceleration of point B on pulley C when A rotates 3 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it.

16–10. At the instant $\omega_A = 5$ rad/s, pulley A is given a constant angular acceleration $\alpha_A = 6$ rad/s^2. Determine the magnitude of acceleration of point B on pulley C when A rotates 2 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it.



Probs. 16–9/10

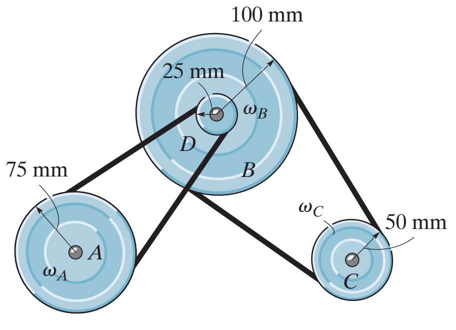
16–11. The cord, which is wrapped around the disk, is given an acceleration of $a = (10t)$ m/s^2, where t is in seconds. Starting from rest, determine the angular displacement, angular velocity, and angular acceleration of the disk when $t = 3$ s.



Prob. 16–11

***16–12.** The power of a bus engine is transmitted using the belt-and-pulley arrangement shown. If the engine turns pulley A at $\omega_A = (20t + 40)$ rad/s, where t is in seconds, determine the angular velocities of the generator pulley B and the air-conditioning pulley C when $t = 3$ s.

16–13. The power of a bus engine is transmitted using the belt-and-pulley arrangement shown. If the engine turns pulley A at $\omega_A = 60$ rad/s, determine the angular velocities of the generator pulley B and the air-conditioning pulley C. The hub at D is rigidly *connected* to B and turns with it.

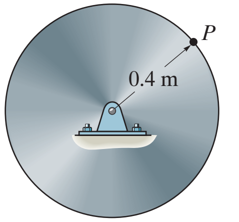


Probs. 16–12/13

16–14. The disk starts from rest and is given an angular acceleration $\alpha = (2t^2)$ rad/s^2, where t is in seconds. Determine the angular velocity of the disk and its angular displacement when $t = 4$ s.

16–15. The disk starts from rest and is given an angular acceleration $\alpha = (5t^{1/2})$ rad/s^2, where t is in seconds. Determine the magnitudes of the normal and tangential components of acceleration of a point P on the rim of the disk when $t = 2$ s.

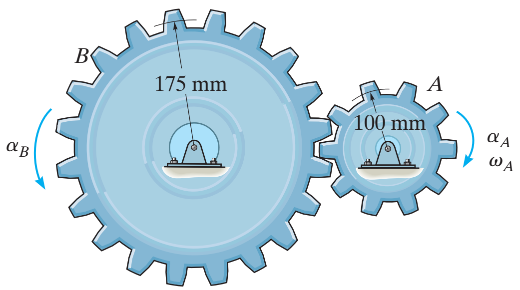
***16–16.** The disk starts at $\omega_0 = 1$ rad/s when $\theta = 0$, and is given an angular acceleration $\alpha = (0.3\theta)$ rad/s^2, where θ is in radians. Determine the magnitudes of the normal and tangential components of acceleration of a point P on the rim of the disk when $\theta = 1$ rev.



Probs. 16–14/15/16

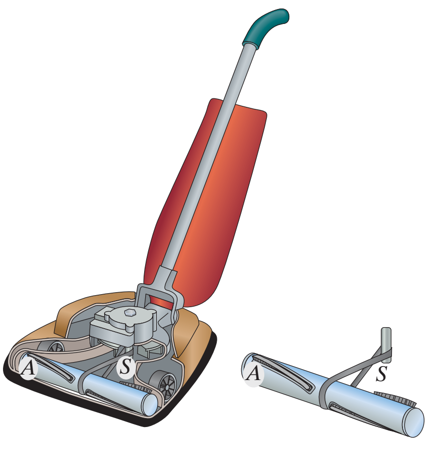
16–17. A motor gives gear A an angular acceleration of $\alpha_A = (2 + 0.006\ \theta^2)\ \text{rad/s}^2$, where θ is in radians. If this gear is initially turning at $\omega_A = 15\ \text{rad/s}$, determine the angular velocity of gear B after A undergoes an angular displacement of 10 rev.

16–18. A motor gives gear A an angular acceleration of $\alpha_A = (2t^3)\ \text{rad/s}^2$, where t is in seconds. If this gear is initially turning at $\omega_A = 15\ \text{rad/s}$, determine the angular velocity of gear B when $t = 3$ s.



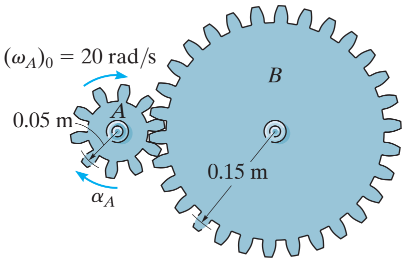
Probs. 16–17/18

16–19. The vacuum cleaner's armature shaft S rotates with an angular acceleration of $\alpha = 4\omega^{3/4}\ \text{rad/s}^2$, where ω is in rad/s. Determine the brush's angular velocity when $t = 4$ s, starting from $\omega_0 = 1$ rad/s, at $\theta = 0$. The radii of the shaft and the brush are 0.25 in. and 1 in., respectively. Neglect the thickness of the drive belt.



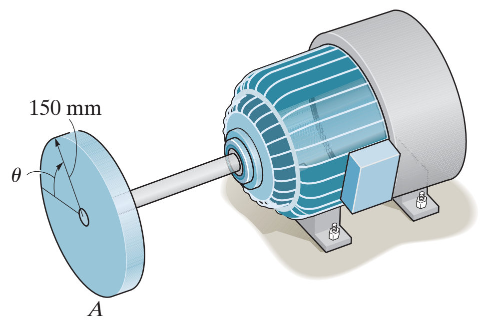
Prob. 16–19

***16–20.** A motor gives gear A an angular acceleration of $\alpha_A = (4t^3)\ \text{rad/s}^2$, where t is in seconds. If this gear is initially turning at $(\omega_A)_0 = 20\ \text{rad/s}$, determine the angular velocity of gear B when $t = 2$ s.



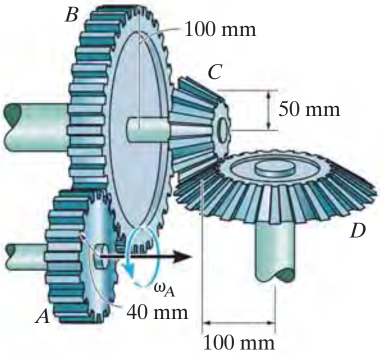
Prob. 16–20

16–21. The motor turns the disk with an angular velocity of $\omega = (5t^2 + 3t)\ \text{rad/s}$, where t is in seconds. Determine the magnitudes of the velocity and the n and t components of acceleration of the point A on the disk when $t = 3$ s.



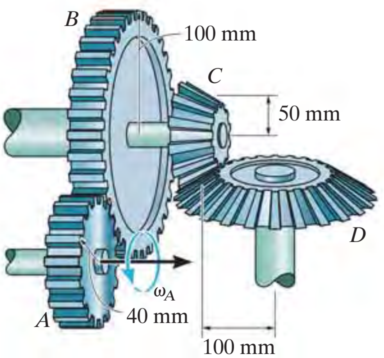
Prob. 16–21

16–22. If the motor turns gear A with an angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$ when the angular velocity is $\omega_A = 20\ \text{rad/s}$, determine the angular acceleration and angular velocity of gear D.



Prob. 16–22

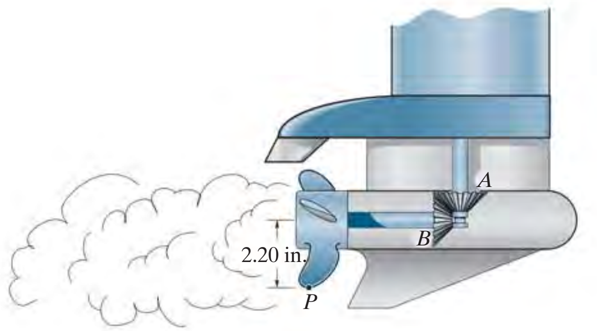
16–23. If the motor turns gear A with an angular acceleration of $\alpha_A = 3\ \text{rad/s}^2$ when the angular velocity is $\omega_A = 60\ \text{rad/s}$, determine the angular acceleration and angular velocity of gear D.



Prob. 16–23

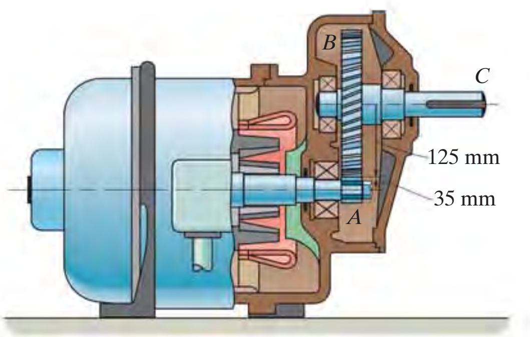
***16–24.** The gear A on the drive shaft of the outboard motor has a radius $r_A = 0.5$ in. and the meshed pinion gear B on the propeller shaft has a radius $r_B = 1.2$ in. Determine the angular velocity of the propeller in $t = 1.5$ s, if the drive shaft rotates with an angular acceleration $\alpha = (400t^3)\ \text{rad/s}^2$, where t is in seconds. The propeller is originally at rest and the motor frame does not move.

16–25. For the outboard motor in Prob. 16–24, determine the magnitude of the velocity and acceleration of point P located on the tip of the propeller at the instant $t = 0.75$ s.



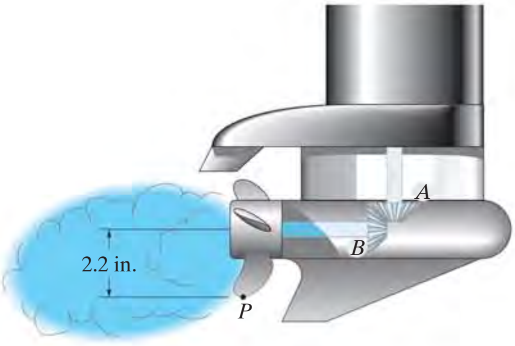
Probs. 16–24/25

16–26. The pinion gear A on the motor shaft is given a constant angular acceleration $\alpha = 3\ \text{rad/s}^2$. If the gears A and B have the dimensions shown, determine the angular velocity and angular displacement of the output shaft C, when $t = 2$ s starting from rest. The shaft is fixed to B and turns with it.



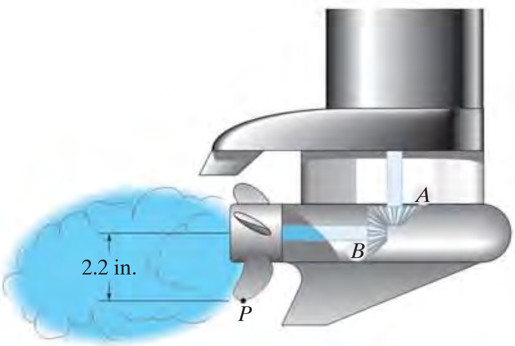
Prob. 16–26

16–27. The gear A on the drive shaft of the outboard motor has a radius $r_A = 0.7$ in. and the meshed pinion gear B on the propeller shaft has a radius $r_B = 1.4$ in. Determine the angular velocity of the propeller in $t = 1.3$ s if the drive shaft rotates with an angular acceleration $\alpha = (300\sqrt{t})$ rad/s^2, where t is in seconds. The propeller is originally at rest and the motor frame does not move.



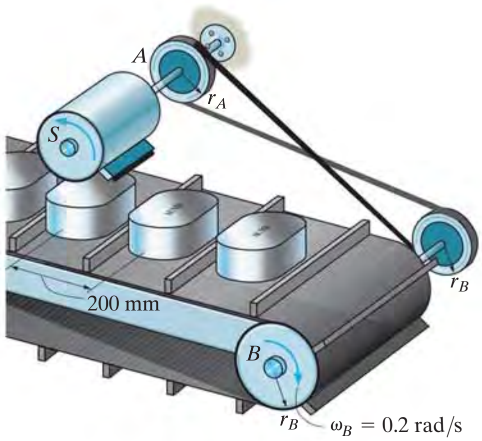
Prob. 16–27

***16–28.** The gear A on the drive shaft of the outboard motor has a radius $r_A = 0.7$ in. and the meshed pinion gear B on the propeller shaft has a radius $r_B = 1.4$ in. Determine the magnitudes of the velocity and acceleration of a point P located on the tip of the propeller at the instant $t = 0.75$ s. The drive shaft rotates with an angular acceleration $\alpha = (300\sqrt{t})$ rad/s^2, where t is in seconds. The propeller is originally at rest and the motor frame does not move.



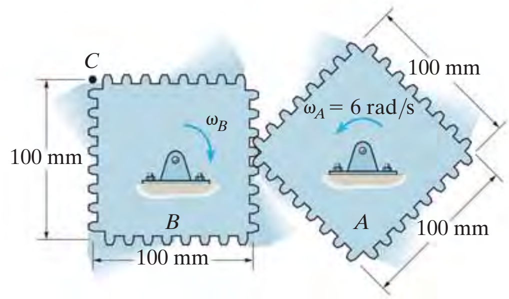
Prob. 16–28

16–29. A stamp S, located on the revolving drum, is used to label canisters. If the canisters are centered 200 mm apart on the conveyor, determine the radius r_A of the driving wheel A and the radius r_B of the conveyor belt drum so that for each revolution of the stamp it marks the top of a canister. How many canisters are marked per minute if the drum at B is rotating at $\omega_B = 0.2$ rad/s? Note that the driving belt is twisted as it passes between the wheels.



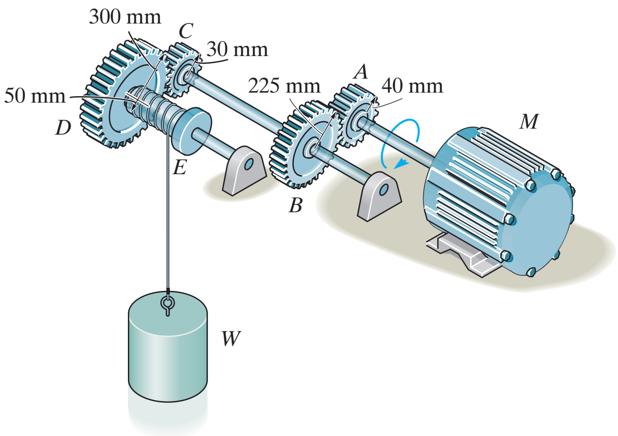
Prob. 16–29

16–30. At the instant shown, gear A is rotating with a constant angular velocity of $\omega_A = 6$ rad/s. Determine the largest angular velocity of gear B and the maximum speed of point C.



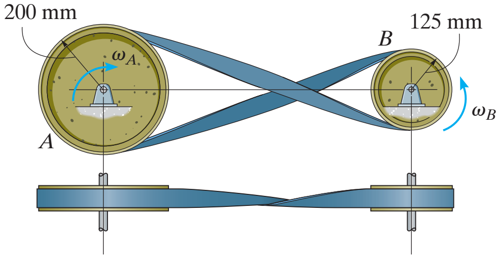
Prob. 16–30

16–31. Determine the distance the load W is lifted in $t = 5$ s using the hoist. The shaft of the motor M turns with an angular velocity $\omega = 100(4 + t)$ rad/s, where t is in seconds.



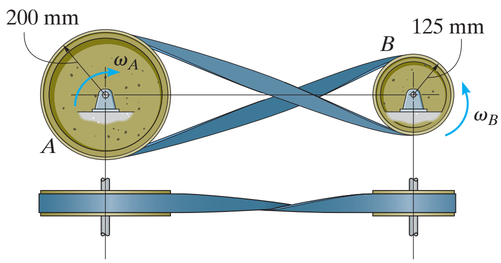
Prob. 16–31

***16–32.** The driving belt is twisted so that pulley B rotates in the opposite direction to that of drive wheel A. If A has a constant angular acceleration of $\alpha_A = 30$ rad/s^2, determine the tangential and normal components of acceleration of a point located at the rim of B when $t = 3$ s, starting from rest.



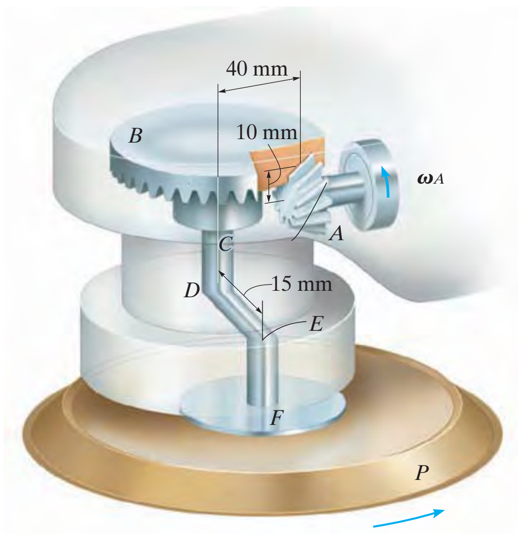
Prob. 16–32

16–33. The driving belt is twisted so that pulley B rotates in the opposite direction to that of drive wheel A. If the angular displacement of A is $\theta_A = (5t^3 + 10t^2)$ rad, where t is in seconds, determine the angular velocity and angular acceleration of B when $t = 3$ s.



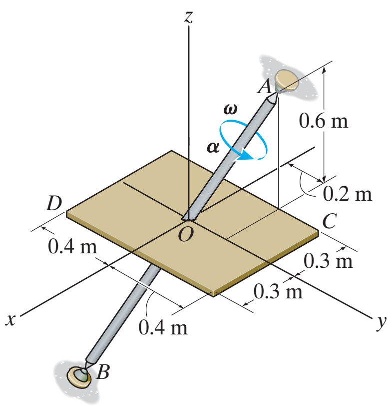
Prob. 16–33

16–34. For a short time a motor of the random-orbit sander drives the gear A with an angular velocity of $\omega_A = 40(t^3 + 6t)$ rad/s, where t is in seconds. This gear is connected to gear B, which is fixed connected to the shaft CD. The end of this shaft is connected to the eccentric spindle EF and pad P, which causes the pad to orbit around shaft CD at a radius of 15 mm. Determine the magnitudes of the velocity and the tangential and normal components of acceleration of the spindle EF when $t = 2$ s after starting from rest.



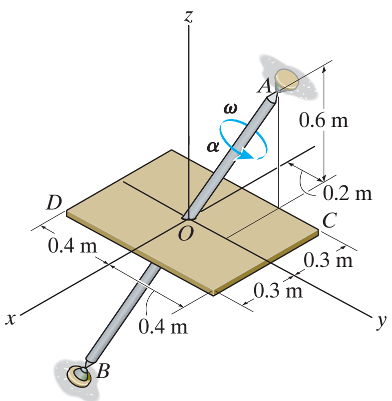
Prob. 16–34

16–35. If the shaft and plate rotates with a constant angular velocity of $\omega = 14$ rad/s, determine the velocity and acceleration of point C located on the corner of the plate at the instant shown. Express the result in Cartesian vector form.



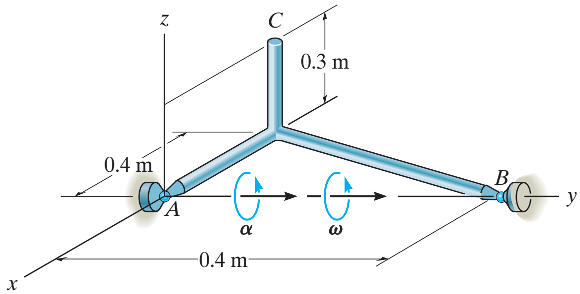
Prob. 16–35

***16–36.** At the instant shown, the shaft and plate rotates with an angular velocity of $\omega = 14$ rad/s and angular acceleration of $\alpha = 7$ rad/s^2. Determine the velocity and acceleration of point D located on the corner of the plate at this instant. Express the result in Cartesian vector form.



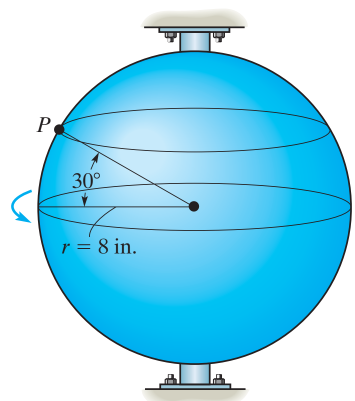
Prob. 16–36

16–37. The rod assembly is supported by ball-and-socket joints at A and B. At the instant shown it is rotating about the y axis with an angular velocity $\omega = 5$ rad/s and has an angular acceleration $\alpha = 8$ rad/s^2. Determine the magnitudes of the velocity and acceleration of point C at this instant. Solve the problem using Cartesian vectors and Eqs. 16–9 and 16–13.



Prob. 16–37

16–38. The sphere starts from rest at $\theta = 0°$ and rotates with an angular acceleration of $\alpha = (4\theta + 1)$ rad/s^2, where θ is in radians. Determine the magnitudes of the velocity and acceleration of point P on the sphere at the instant $\theta = 6$ rad.



Prob. 16–38

16.4 Absolute Motion Analysis

The dumping bin on the truck rotates about a fixed axis passing through the pin at A. It is operated by the extension of the hydraulic cylinder BC. The angular position of the bin can be specified using the angular position coordinate θ, and the position of point C on the bin is specified using the rectilinear position coordinate s. Since a and b are fixed lengths, then the two coordinates can be related by the cosine law, $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$. The time derivative of this equation relates the speed at which the hydraulic cylinder extends to the angular velocity of the bin. (© R.C. Hibbeler)

A body subjected to *general plane motion* undergoes a *simultaneous* translation and rotation. If the body is represented by a thin slab, the slab translates in the plane of the slab and rotates about an axis perpendicular to this plane. The motion can be completely specified by knowing *both* the angular rotation of a line fixed in the body and the motion of a point on the body. One way to relate these motions is to use a rectilinear position coordinate s to locate the point along its path and an angular position coordinate θ to specify the orientation of the line. The two coordinates are then related using the geometry of the problem. By *direct application* of the time-differential equations $v = ds/dt$, $a = dv/dt$, $\omega = d\theta/dt$, and $\alpha = d\omega/dt$, the *motion* of the point and the *angular motion* of the line can then be related. This procedure is similar to that used to solve dependent motion problems involving pulleys, Sec. 12.9. In some cases, this same procedure may be used to relate the motion of one body, undergoing either rotation about a fixed axis or translation, to that of a connected body undergoing general plane motion.

Procedure for Analysis

The velocity and acceleration of a point P undergoing rectilinear motion can be related to the angular velocity and angular acceleration of a line contained within a body using the following procedure.

Position Coordinate Equation.

- Locate point P on the body using a position coordinate s, which is measured from a *fixed origin* and is *directed along the straight-line path of motion* of point P.
- Measure from a fixed reference line the angular position θ of a line lying in the body.
- From the dimensions of the body, relate s to θ, $s = f(\theta)$, using geometry and/or trigonometry.

Time Derivatives.

- Take the first derivative of $s = f(\theta)$ with respect to time to get a relation between v and ω.
- Take the second time derivative to get a relation between a and α.
- In each case the chain rule of calculus must be used when taking the time derivatives of the position coordinate equation. See Appendix C.

EXAMPLE 16.3

The end of rod R shown in Fig. 16–8 maintains contact with the cam by means of a spring. If the cam rotates about an axis passing through point O with an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, determine the velocity and acceleration of the rod when the cam is in the arbitrary position θ.

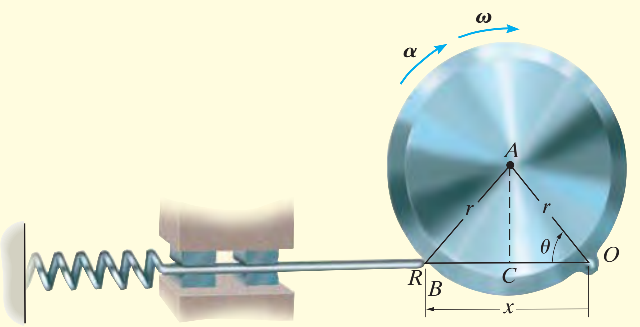


Fig. 16–8

SOLUTION

Position Coordinate Equation. Coordinates θ and x are chosen in order to relate the *rotational motion* of the line segment OA on the cam to the *rectilinear translation* of the rod. These coordinates are measured from the *fixed point O* and can be related to each other using trigonometry. Since $OC = CB = r\cos\theta$, Fig. 16–8, then

$$x = 2r\cos\theta$$

Time Derivatives. Using the chain rule of calculus, we have

$$\frac{dx}{dt} = -2r(\sin\theta)\frac{d\theta}{dt}$$

$$v = -2r\omega\sin\theta \qquad \textit{Ans.}$$

$$\frac{dv}{dt} = -2r\left(\frac{d\omega}{dt}\right)\sin\theta - 2r\omega(\cos\theta)\frac{d\theta}{dt}$$

$$a = -2r(\alpha\sin\theta + \omega^2\cos\theta) \qquad \textit{Ans.}$$

NOTE: The negative signs indicate that v and a are opposite to the direction of positive x. This seems reasonable when you visualize the motion.

EXAMPLE 16.4

At a given instant, the cylinder of radius r, shown in Fig. 16–9, has an angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Determine the velocity and acceleration of its center G if the cylinder rolls without slipping.

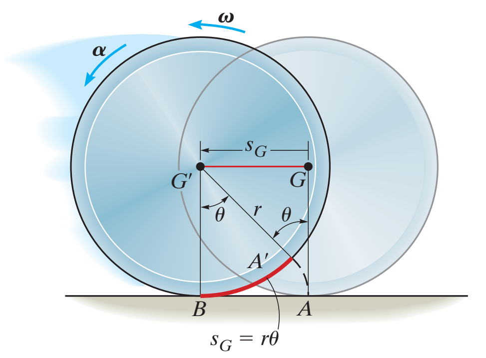


Fig. 16–9

SOLUTION

Position Coordinate Equation. The cylinder undergoes general plane motion since it simultaneously translates and rotates. By inspection, point G moves in a *straight line* to the left, from G to G', as the cylinder rolls, Fig. 16–9. Consequently its new position G' will be specified by the *horizontal* position coordinate s_G, which is measured from G to G'. Also, as the cylinder rolls (without slipping), the arc length $A'B$ on the rim which was in contact with the ground from A to B, is equivalent to s_G. Consequently, the motion requires the radial line GA to rotate θ to the position $G'A'$. Since the arc $A'B = r\theta$, then G travels a distance

$$s_G = r\theta$$

Time Derivatives. Taking successive time derivatives of this equation, realizing that r is constant, $\omega = d\theta/dt$, and $\alpha = d\omega/dt$, gives the necessary relationships:

$$s_G = r\theta$$

$$v_G = r\omega \qquad \textit{Ans.}$$

$$a_G = r\alpha \qquad \textit{Ans.}$$

NOTE: Remember that these relationships are valid only if the cylinder (disk, wheel, ball, etc.) rolls *without* slipping.

EXAMPLE 16.5

The large window in Fig. 16–10 is opened using a hydraulic cylinder AB. If the cylinder extends at a constant rate of 0.5 m/s, determine the angular velocity and angular acceleration of the window at the instant $\theta = 30^\circ$.

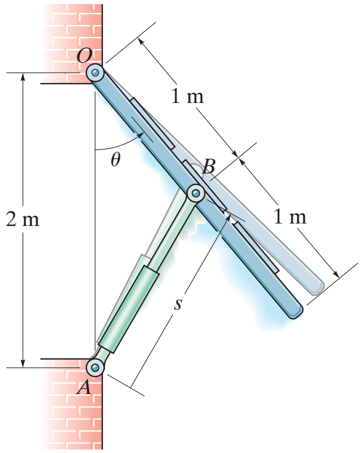


Fig. 16–10

SOLUTION

Position Coordinate Equation. The angular motion of the window can be obtained using the coordinate θ, whereas the extension or motion *along the hydraulic cylinder* is defined using a coordinate s, which measures its length from the fixed point A to the moving point B. These coordinates can be related using the law of cosines, namely,

$$s^2 = (2\text{ m})^2 + (1\text{ m})^2 - 2(2\text{ m})(1\text{ m})\cos\theta$$

$$s^2 = 5 - 4\cos\theta \qquad (1)$$

When $\theta = 30^\circ$,

$$s = 1.239\text{ m}$$

Time Derivatives. Taking the time derivatives of Eq. 1, we have

$$2s\frac{ds}{dt} = 0 - 4(-\sin\theta)\frac{d\theta}{dt}$$

$$s(v_s) = 2(\sin\theta)\omega \qquad (2)$$

Since $v_s = 0.5$ m/s, then at $\theta = 30^\circ$,

$$(1.239\text{ m})(0.5\text{ m/s}) = 2\sin 30^\circ\omega$$

$$\omega = 0.6197\text{ rad/s} = 0.620\text{ rad/s} \qquad \textit{Ans.}$$

Taking the time derivative of Eq. 2 yields

$$\frac{ds}{dt}v_s + s\frac{dv_s}{dt} = 2(\cos\theta)\frac{d\theta}{dt}\omega + 2(\sin\theta)\frac{d\omega}{dt}$$

$$v_s^2 + sa_s = 2(\cos\theta)\omega^2 + 2(\sin\theta)\alpha$$

Since $a_s = dv_s/dt = 0$, then

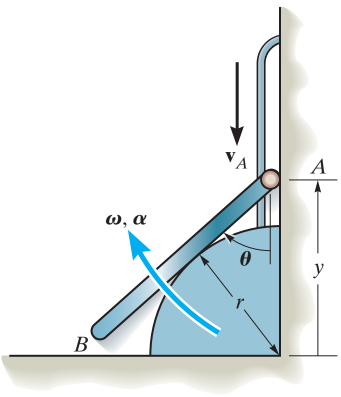
$$(0.5\text{ m/s})^2 + 0 = 2\cos 30^\circ(0.6197\text{ rad/s})^2 + 2\sin 30^\circ\alpha$$

$$\alpha = -0.415\text{ rad/s}^2 \qquad \textit{Ans.}$$

Because the result is negative, it indicates the window has an angular deceleration.

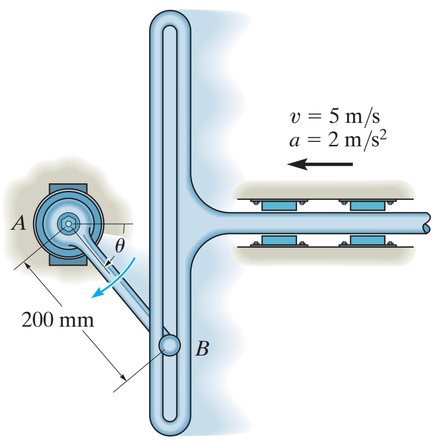
PROBLEMS

16–39. The end A of the bar is moving downward along the slotted guide with a constant velocity $\mathbf{v}_A$. Determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the bar as a function of its position y.



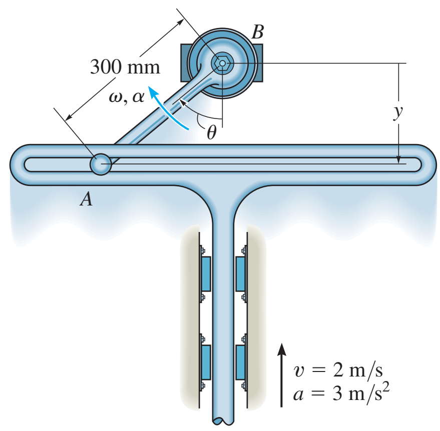
Prob. 16–39

***16–40.** At the instant $\theta = 60°$, the slotted guide rod is moving to the left with an acceleration of $2\ \mathrm{m/s^2}$ and a velocity of $5\ \mathrm{m/s}$. Determine the angular acceleration and angular velocity of link AB at this instant.



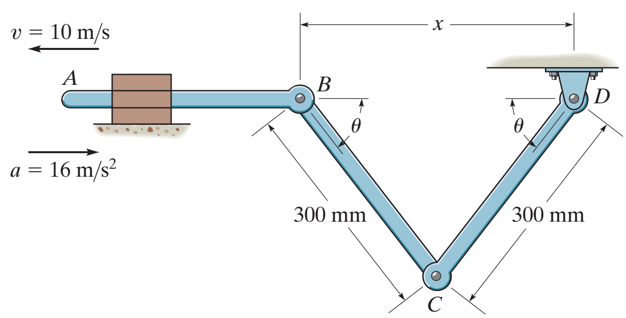
Prob. 16–40

16–41. At the instant $\theta = 50°$, the slotted guide is moving upward with an acceleration of $3\ \mathrm{m/s^2}$ and a velocity of $2\ \mathrm{m/s}$. Determine the angular acceleration and angular velocity of link AB at this instant. *Note:* The upward motion of the guide is in the negative y direction.



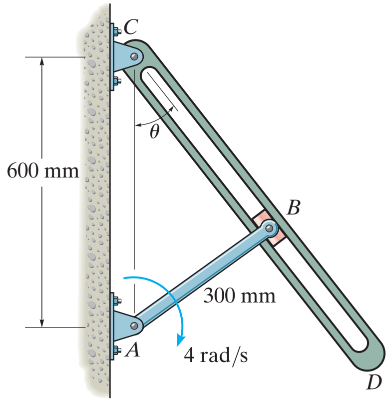
Prob. 16–41

16–42. At the instant shown, $\theta = 60°$, and rod AB is subjected to a deceleration of $16\ \mathrm{m/s^2}$ when the velocity is $10\ \mathrm{m/s}$. Determine the angular velocity and angular acceleration of link CD at this instant.



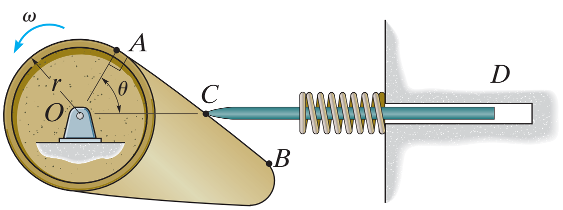
Prob. 16–42

16–43. The crank AB is rotating with a constant angular velocity of 4 rad/s. Determine the angular velocity of the connecting rod CD at the instant $\theta = 30°$.



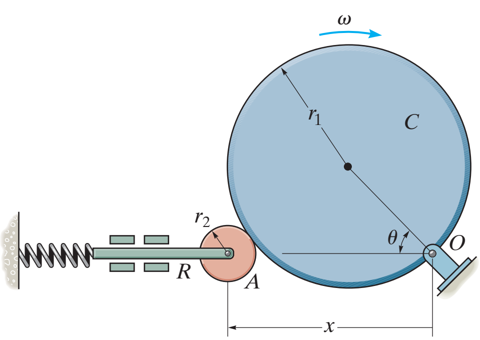
Prob. 16–43

***16–44.** Determine the velocity and acceleration of the follower rod CD as a function of θ when the contact between the cam and follower is along the straight region AB on the face of the cam. The cam rotates with a constant counterclockwise angular velocity $\boldsymbol{\omega}$.



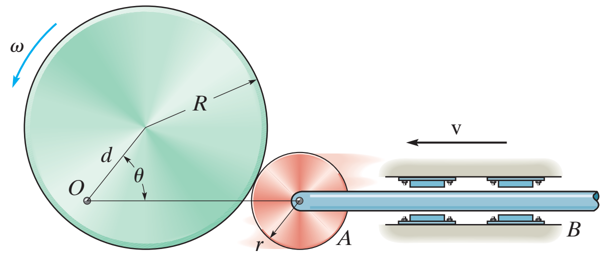
Prob. 16–44

16–45. Determine the velocity of rod R for any angle θ of the cam C if the cam rotates with a constant angular velocity $\boldsymbol{\omega}$. The pin connection at O does not cause an interference with the motion of A on C.



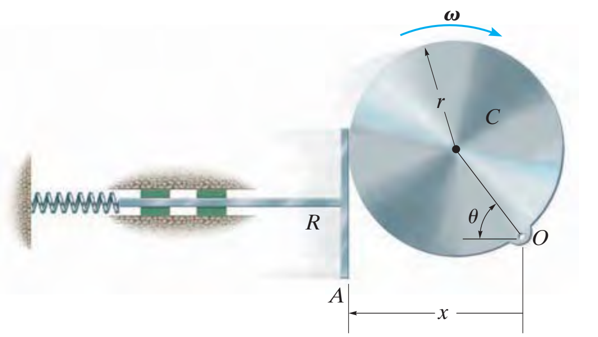
Prob. 16–45

16–46. The circular cam rotates about the fixed point O with a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity v of the follower rod AB as a function of θ.



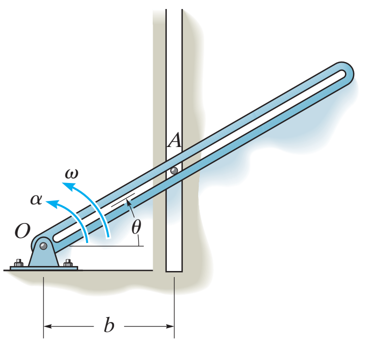
Prob. 16–46

16–47. Determine the velocity of the rod R for any angle θ of cam C as the cam rotates with a constant angular velocity $\boldsymbol{\omega}$. The pin connection at O does not cause an interference with the motion of plate A on C.



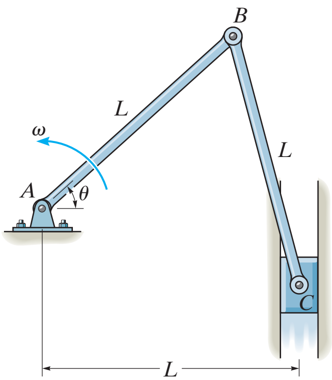
Prob. 16–47

***16–48.** Determine the velocity and acceleration of the peg A which is confined between the vertical guide and the rotating slotted rod.



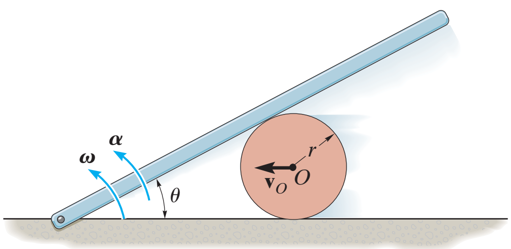
Prob. 16–48

16–49. Bar AB rotates uniformly about the fixed pin A with a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity and acceleration of block C, at the instant $\theta = 60°$.



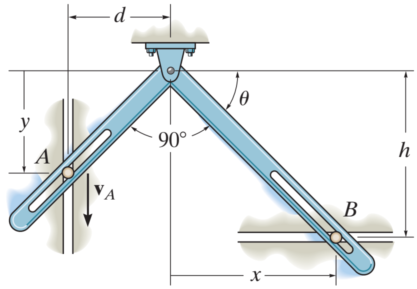
Prob. 16–49

16–50. The center of the cylinder is moving to the left with a constant velocity $\mathbf{v}_0$. Determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the bar. Neglect the thickness of the bar.



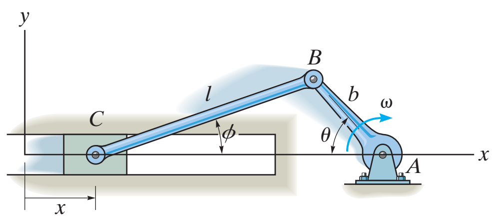
Prob. 16–50

16–51. The pins at A and B are confined to move in the vertical and horizontal tracks. If the slotted arm is causing A to move downward at $\mathbf{v}_A$, determine the velocity of B at the instant shown.



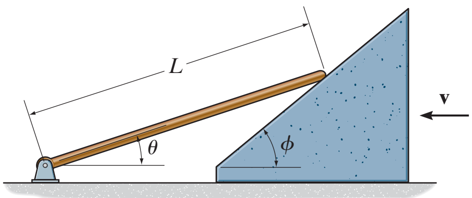
Prob. 16–51

***16–52.** The crank AB has a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity and acceleration of the slider at C as a function of θ. *Suggestion:* Use the x coordinate to express the motion of C and the ϕ coordinate for CB. $x = 0$ when $\phi = 0°$.



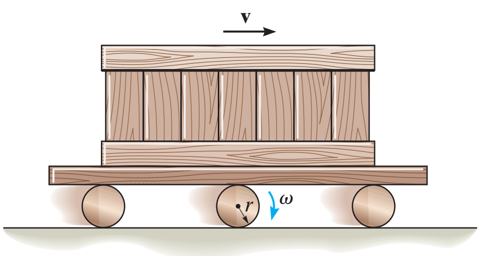
Prob. 16–52

16–53. If the wedge moves to the left with a constant velocity **v**, determine the angular velocity of the rod as a function of θ.



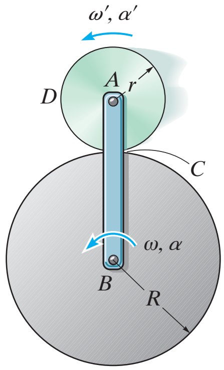
Prob. 16–53

16–54. The crate is transported on a platform which rests on rollers, each having a radius r. If the rollers do not slip, determine their angular velocity if the platform moves forward with a velocity **v**.



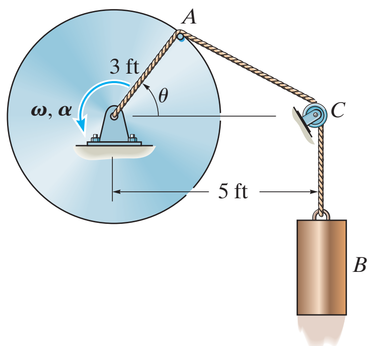
Prob. 16–54

16–55. Arm AB has an angular velocity of $\boldsymbol{\omega}$ and an angular acceleration of $\boldsymbol{\alpha}$. If no slipping occurs between the disk D and the fixed curved surface, determine the angular velocity and angular acceleration of the disk.

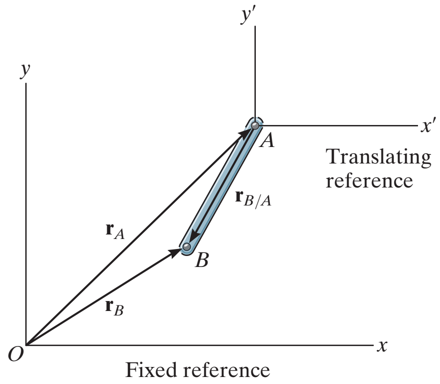


Prob. 16–55

***16–56.** At the instant shown, the disk is rotating with an angular velocity of $\boldsymbol{\omega}$ and has an angular acceleration of $\boldsymbol{\alpha}$. Determine the velocity and acceleration of cylinder B at this instant. Neglect the size of the pulley at C.



Prob. 16–56



(a)

Fig. 16–11

16.5 Relative-Motion Analysis: Velocity

The general plane motion of a rigid body can be described as a *combination* of translation and rotation. To view these "component" motions *separately* we will use a *relative-motion analysis* involving two sets of coordinate axes. The x, y coordinate system is fixed and measures the *absolute* position of two points A and B on the body, here represented as a bar, Fig. 16–11*a*. The origin of the x', y' coordinate system will be attached to the selected "base point" A, which generally has a *known* motion. The axes of this coordinate system *translate* with respect to the fixed frame but do not rotate with the bar.

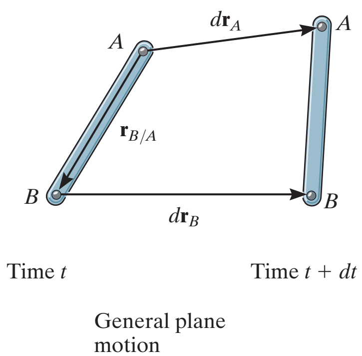
Position. The position vector $\mathbf{r}_A$ in Fig. 16–11*a* specifies the location of the "base point" A, and the relative-position vector $\mathbf{r}_{B/A}$ locates point B with respect to point A. By vector addition, the *position* of B is then

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

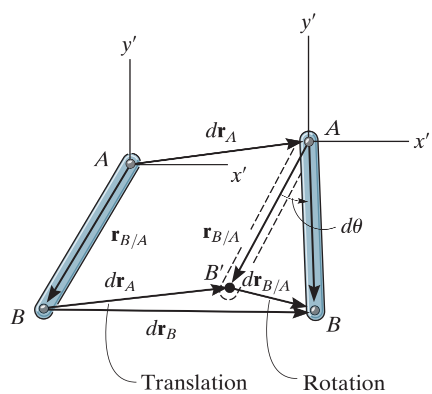
Displacement. During an instant of time dt, points A and B undergo displacements $d\mathbf{r}_A$ and $d\mathbf{r}_B$ as shown in Fig. 16–11*b*. If we consider the general plane motion by its component parts then the *entire bar* first *translates* by an amount $d\mathbf{r}_A$ so that A, the base point, moves to its *final position* and point B moves to B', Fig. 16–11*c*. The bar is then *rotated* about A by an amount $d\theta$ so that B' undergoes a *relative displacement* $d\mathbf{r}_{B/A}$ and thus moves to its final position B. Due to the rotation about A, $dr_{B/A} = r_{B/A}\, d\theta$, and the displacement of B is

$$d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}_{B/A}$$

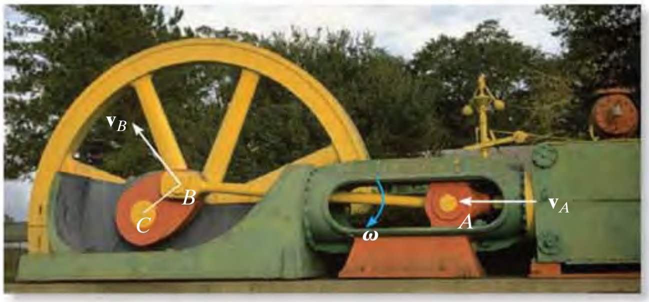
($d\mathbf{r}_{B/A}$) due to rotation about A
($d\mathbf{r}_A$) due to translation of A
($d\mathbf{r}_B$) due to translation and rotation



(b)



(c)



As slider block A moves horizontally to the left with a velocity $\mathbf{v}_A$, it causes crank CB to rotate counterclockwise, such that $\mathbf{v}_B$ is directed tangent to its circular path, i.e., upward to the left. The connecting rod AB is subjected to general plane motion, and at the instant shown it has an angular velocity $\boldsymbol{\omega}$. (© R.C. Hibbeler)

Velocity. To determine the relation between the velocities of points A and B, it is necessary to take the time derivative of the position equation, or simply divide the displacement equation by dt. This yields

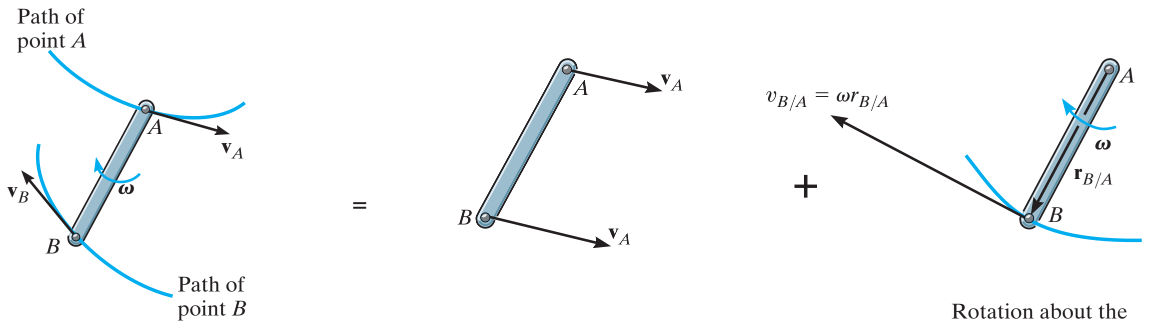
$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}$$

The terms $d\mathbf{r}_B/dt = \mathbf{v}_B$ and $d\mathbf{r}_A/dt = \mathbf{v}_A$ are measured with respect to the fixed x, y axes and represent the *absolute velocities* of points A and B, respectively. Since the relative displacement is caused by a rotation, the magnitude of the third term is $dr_{B/A}/dt = r_{B/A}\,d\theta/dt = r_{B/A}\dot{\theta} = r_{B/A}\omega$, where ω is the angular velocity of the body at the instant considered. We will denote this term as the *relative velocity* $\mathbf{v}_{B/A}$, since it represents the velocity of B with respect to A as measured by an observer fixed to the translating x', y' axes. In other words, *the bar appears to move as if it were rotating with an angular velocity* $\boldsymbol{\omega}$ *about the* z' *axis passing through* A. Consequently, $\mathbf{v}_{B/A}$ has a magnitude of $v_{B/A} = \omega r_{B/A}$ and a *direction* which is perpendicular to $\mathbf{r}_{B/A}$. We therefore have

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}} \qquad (16\text{–}15)$$

where

$\mathbf{v}_B$ = velocity of point B
$\mathbf{v}_A$ = velocity of the base point A
$\mathbf{v}_{B/A}$ = velocity of B with respect to A



General plane motion

(d)

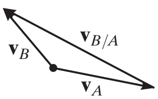
Translation

(e)

Rotation about the base point A

(f)

Fig. 16–11 (cont.)



(g)

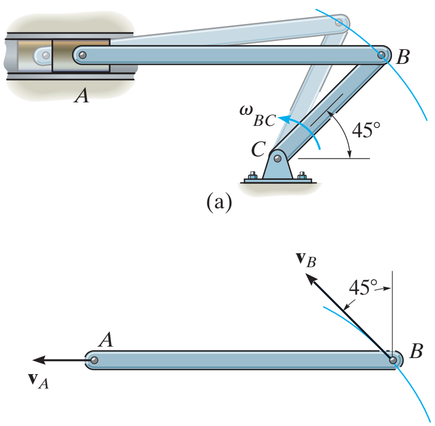
What the equation $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ states is that the velocity of B, Fig. 16–11*d*, is determined by considering the entire bar to translate with a velocity of $\mathbf{v}_A$, Fig. 16–11*e*, and rotate about A with an angular velocity $\boldsymbol{\omega}$, Fig. 16–11*f*. Vector addition of these two effects, applied to B, yields $\mathbf{v}_B$, as shown in Fig. 16–11*g*.

Since the relative velocity $\mathbf{v}_{B/A}$ represents the effect of *circular motion*, about A, this term can be expressed by the cross product $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 16–9. Hence, for application using Cartesian vector analysis, we can also write Eq. 16–15 as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{16–16}$$

where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the base point A
$\boldsymbol{\omega}$ = angular velocity of the body
$\mathbf{r}_{B/A}$ = position vector directed from A to B



(b)

Fig. 16–12

The velocity equation 16–15 or 16–16 may be used in a practical manner to study the general plane motion of a rigid body which is either pin connected to or in contact with other moving bodies. When applying this equation, points A and B should generally be selected as points on the body which are pin-connected to other bodies, or as points in contact with adjacent bodies which have a *known motion*. For example, point A on link AB in Fig. 16–12*a* must move along a horizontal path, whereas point B moves on a circular path. The *directions* of $\mathbf{v}_A$ and $\mathbf{v}_B$ can therefore be established since they are always *tangent* to their paths of motion, Fig. 16–12*b*. In the case of the wheel in Fig. 16–13, which rolls *without slipping*, point A on the wheel can be selected at the ground. Here A (momentarily) has zero velocity since the ground does not move. Furthermore, the center of the wheel, B, moves along a horizontal path so that $\mathbf{v}_B$ is horizontal.

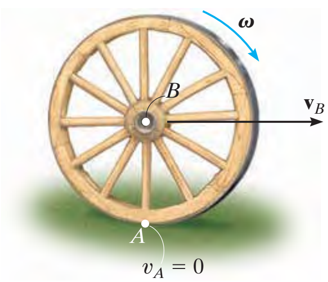


Fig. 16–13

Procedure for Analysis

The relative velocity equation can be applied either by using Cartesian vector analysis, or by writing the x and y scalar component equations directly. For application, it is suggested that the following procedure be used.

Vector Analysis

Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw a kinematic diagram of the body. Indicate on it the velocities $\mathbf{v}_A$, $\mathbf{v}_B$ of points A and B, the angular velocity $\boldsymbol{\omega}$, and the relative-position vector $\mathbf{r}_{B/A}$.
- If the magnitudes of $\mathbf{v}_A$, $\mathbf{v}_B$, or $\boldsymbol{\omega}$ are unknown, the sense of direction of these vectors can be assumed.

Velocity Equation.

- To apply $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective $\mathbf{i}$ and $\mathbf{j}$ components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates the sense of direction of the vector is opposite to that shown on the kinematic diagram.

Scalar Analysis

Kinematic Diagram.

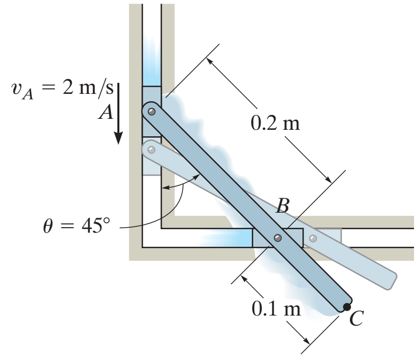
- If the velocity equation is to be applied in scalar form, then the magnitude and direction of the relative velocity $\mathbf{v}_{B/A}$ must be established. Draw a kinematic diagram such as shown in Fig. 16–11g, which shows the relative motion. Since the body is considered to be "pinned" momentarily at the base point A, the magnitude of $\mathbf{v}_{B/A}$ is $v_{B/A} = \omega r_{B/A}$. The sense of direction of $\mathbf{v}_{B/A}$ is always perpendicular to $\mathbf{r}_{B/A}$ in accordance with the rotational motion $\boldsymbol{\omega}$ of the body.*

Velocity Equation.

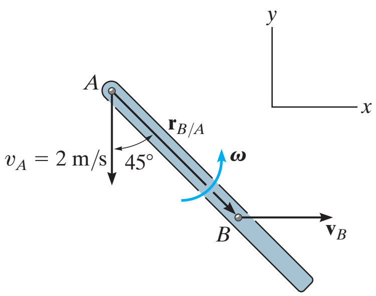
- Write Eq. 16–15 in symbolic form, $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$, and underneath each of the terms represent the vectors graphically by showing their magnitudes and directions. The scalar equations are determined from the x and y components of these vectors.

*The notation $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})}$ may be helpful in recalling that A is "pinned."

EXAMPLE 16.6



(a)



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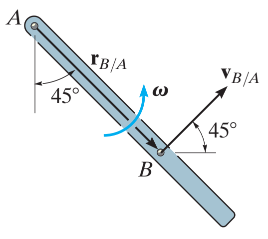


Relative motion

(c)

Fig. 16–14

The link shown in Fig. 16–14*a* is guided by two blocks at A and B, which move in the fixed slots. If the velocity of A is 2 m/s downward, determine the velocity of B at the instant $\theta = 45°$.

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since points A and B are restricted to move along the fixed slots and $\mathbf{v}_A$ is directed downward, then velocity $\mathbf{v}_B$ must be directed horizontally to the right, Fig. 16–14*b*. This motion causes the link to rotate counterclockwise; that is, by the right-hand rule the angular velocity $\boldsymbol{\omega}$ is directed outward, perpendicular to the plane of motion.

Velocity Equation. Expressing each of the vectors in Fig. 16–14*b* in terms of their $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components and applying Eq. 16–16 to A, the base point, and B, we have

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$
$$v_B\mathbf{i} = -2\mathbf{j} + [\omega\mathbf{k} \times (0.2 \sin 45°\mathbf{i} - 0.2 \cos 45°\mathbf{j})]$$
$$v_B\mathbf{i} = -2\mathbf{j} + 0.2\omega \sin 45°\mathbf{j} + 0.2\omega \cos 45°\mathbf{i}$$

Equating the $\mathbf{i}$ and $\mathbf{j}$ components gives

$$v_B = 0.2\omega \cos 45° \qquad 0 = -2 + 0.2\omega \sin 45°$$

Thus,

$$\omega = 14.1 \text{ rad/s} \circlearrowleft \qquad v_B = 2 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

The kinematic diagram of the relative "circular motion" which produces $\mathbf{v}_{B/A}$ is shown in Fig. 16–14*c*. Here $v_{B/A} = \omega(0.2 \text{ m})$.

Thus,

$$v_B = v_A + v_{B/A}$$

$$\begin{bmatrix} v_B \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 2 \text{ m/s} \\ \downarrow \end{bmatrix} + \begin{bmatrix} \omega(0.2 \text{ m}) \\ \angle 45° \end{bmatrix}$$

$(\overset{+}{\rightarrow})$ $$v_B = 0 + \omega(0.2) \cos 45°$$

$(+\uparrow)$ $$0 = -2 + \omega(0.2) \sin 45°$$

The solution produces the above results.

It should be emphasized that these results are *valid only* at the instant $\theta = 45°$. A recalculation for $\theta = 44°$ yields $v_B = 2.07$ m/s and $\omega = 14.4$ rad/s; whereas when $\theta = 46°$, $v_B = 1.93$ m/s and $\omega = 13.9$ rad/s, etc.

NOTE: Since v_A and ω are *known*, the velocity of any other point on the link can be determined. As an exercise, see if you can apply Eq. 16–16 to points A and C or to points B and C and show that when $\theta = 45°$, $v_C = 3.16$ m/s, directed at an angle of 18.4° up from the horizontal.

EXAMPLE 16.7

The cylinder shown in Fig. 16–15*a* rolls without slipping on the surface of a conveyor belt which is moving at 2 ft/s. Determine the velocity of point *A*. The cylinder has a clockwise angular velocity $\omega = 15$ rad/s at the instant shown.

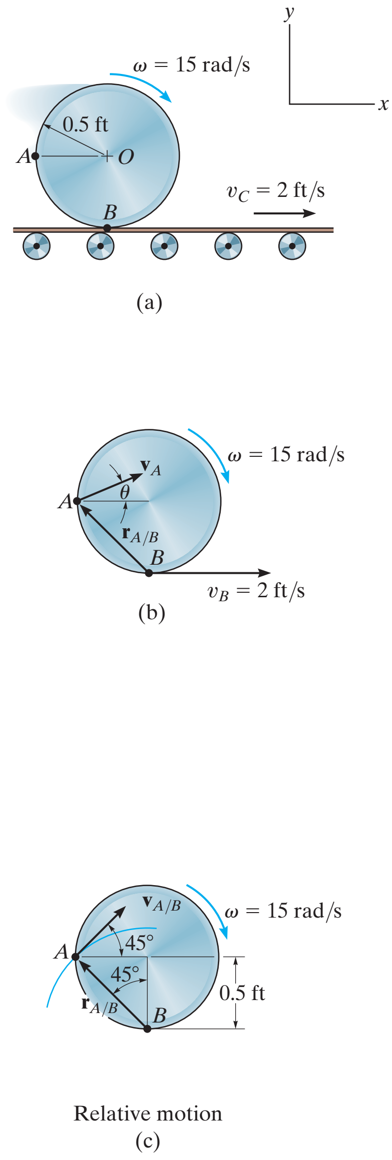


Fig. 16–15

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs, point *B* on the cylinder has the same velocity as the conveyor, Fig. 16–15*b*. Also, the angular velocity of the cylinder is known, so we can apply the velocity equation to *B*, the base point, and *A* to determine $\mathbf{v}_A$.

Velocity Equation

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + (-15\mathbf{k}) \times (-0.5\mathbf{i} + 0.5\mathbf{j})$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + 7.50\mathbf{j} + 7.50\mathbf{i}$$

so that

$$(v_A)_x = 2 + 7.50 = 9.50 \text{ ft/s} \qquad (1)$$

$$(v_A)_y = 7.50 \text{ ft/s} \qquad (2)$$

Thus,

$$v_A = \sqrt{(9.50)^2 + (7.50)^2} = 12.1 \text{ ft/s} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{7.50}{9.50} = 38.3^\circ \; \measuredangle \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

As an alternative procedure, the scalar components of $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ can be obtained directly. From the kinematic diagram showing the relative "circular" motion which produces $\mathbf{v}_{A/B}$, Fig. 16–15*c*, we have

$$v_{A/B} = \omega r_{A/B} = (15 \text{ rad/s})\left(\frac{0.5 \text{ ft}}{\cos 45^\circ}\right) = 10.6 \text{ ft/s}$$

Thus,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

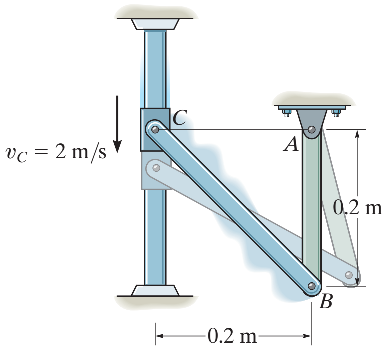
$$\begin{bmatrix}(v_A)_x\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(v_A)_y\\ \uparrow\end{bmatrix} = \begin{bmatrix}2 \text{ ft/s}\\ \rightarrow\end{bmatrix} + \begin{bmatrix}10.6 \text{ ft/s}\\ \measuredangle 45^\circ\end{bmatrix}$$

Equating the *x* and *y* components gives the same results as before, namely,

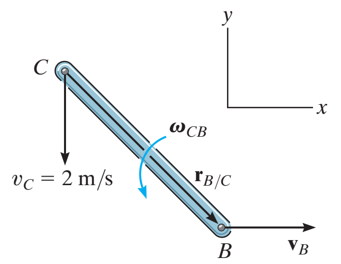
$$(\overset{+}{\rightarrow}) \qquad (v_A)_x = 2 + 10.6 \cos 45^\circ = 9.50 \text{ ft/s}$$

$$(+\uparrow) \qquad (v_A)_y = 0 + 10.6 \sin 45^\circ = 7.50 \text{ ft/s}$$

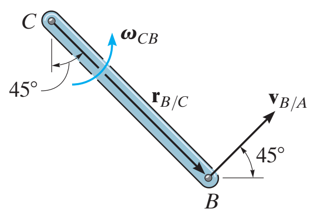
EXAMPLE 16.8



(a)

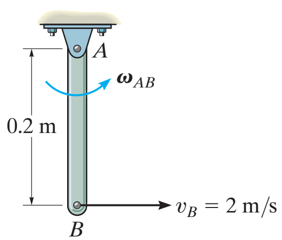


(b)



Relative motion

(c)



(d)

Fig. 16–16

The collar C in Fig. 16–16a is moving downward with a velocity of 2 m/s. Determine the angular velocity of CB at this instant.

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. The downward motion of C causes B to move to the right along a curved path. Also, CB and AB rotate counterclockwise.

Velocity Equation. Link CB (general plane motion): See Fig. 16–16b.

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{CB} \times \mathbf{r}_{B/C}$$

$$v_B\mathbf{i} = -2\mathbf{j} + \omega_{CB}\mathbf{k} \times (0.2\mathbf{i} - 0.2\mathbf{j})$$

$$v_B\mathbf{i} = -2\mathbf{j} + 0.2\omega_{CB}\mathbf{j} + 0.2\omega_{CB}\mathbf{i}$$

$$v_B = 0.2\omega_{CB} \qquad (1)$$

$$0 = -2 + 0.2\omega_{CB} \qquad (2)$$

$$\omega_{CB} = 10 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

$$v_B = 2 \text{ m/s} \rightarrow$$

SOLUTION II (SCALAR ANALYSIS)

The scalar component equations of $\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C}$ can be obtained directly. The kinematic diagram in Fig. 16–16c shows the relative "circular" motion which produces $\mathbf{v}_{B/C}$. We have

$$\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C}$$

$$\begin{bmatrix} v_B \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 2 \text{ m/s} \\ \downarrow \end{bmatrix} + \begin{bmatrix} \omega_{CB}(0.2\sqrt{2} \text{ m}) \\ \measuredangle 45^\circ \end{bmatrix}$$

Resolving these vectors in the x and y directions yields

$$(\overset{+}{\rightarrow}) \qquad v_B = 0 + \omega_{CB}(0.2\sqrt{2}\cos 45^\circ)$$

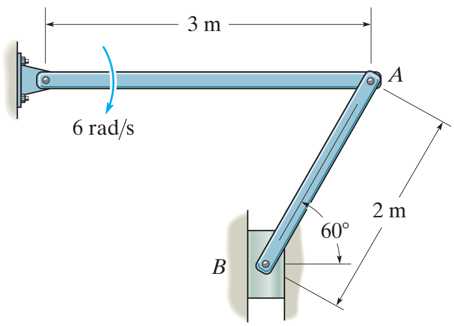
$$(+\uparrow) \qquad 0 = -2 + \omega_{CB}(0.2\sqrt{2}\sin 45^\circ)$$

which is the same as Eqs. 1 and 2.

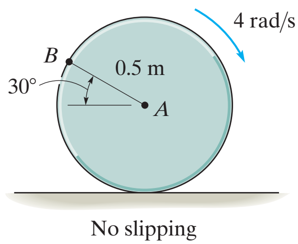
NOTE: Since link AB rotates about a fixed axis and v_B is known, Fig. 16–16d, its angular velocity is found from $v_B = \omega_{AB} r_{AB}$ or 2 m/s $= \omega_{AB}$ (0.2 m), $\omega_{AB} = 10$ rad/s.

PRELIMINARY PROBLEM

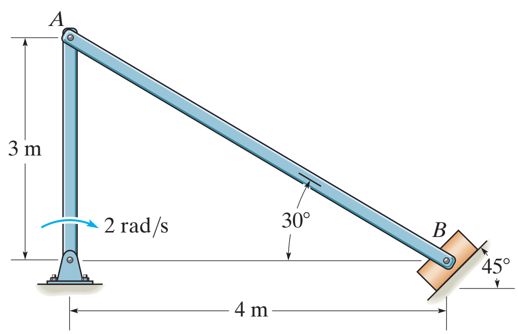
P16–1. Set up the relative velocity equation between points A and B.



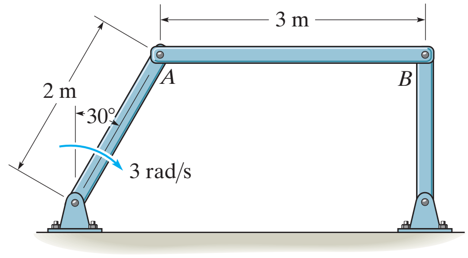
(a)



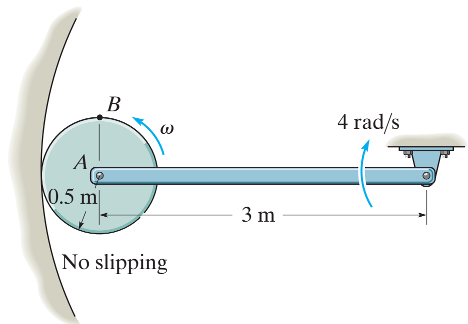
(b)



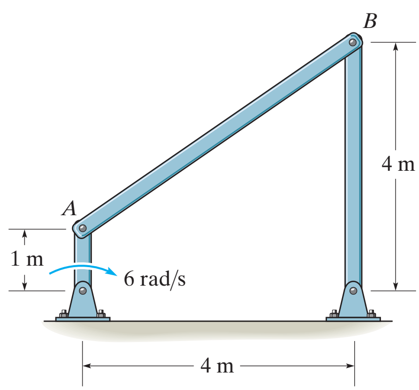
(c)



(d)



(e)

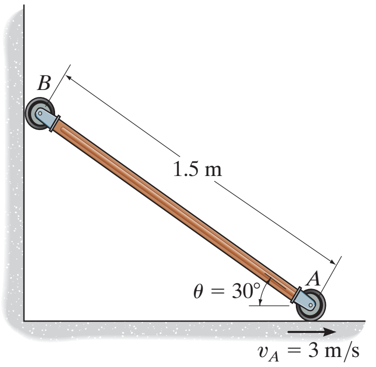


(f)

Prob. P16–1

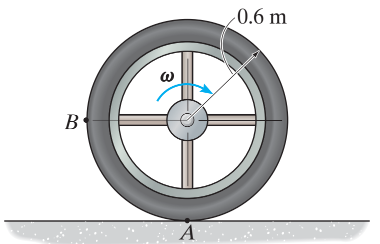
FUNDAMENTAL PROBLEMS

F16–7. If roller A moves to the right with a constant velocity of $v_A = 3$ m/s, determine the angular velocity of the link and the velocity of roller B at the instant $\theta = 30°$.



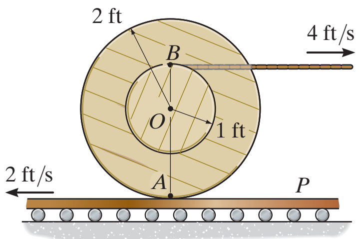
Prob. F16–7

F16–8. The wheel rolls without slipping with an angular velocity of $\omega = 10$ rad/s. Determine the magnitude of the velocity of point B at the instant shown.



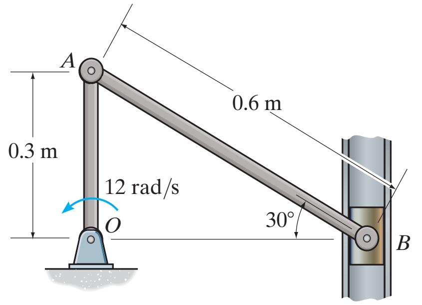
Prob. F16–8

F16–9. Determine the angular velocity of the spool. The cable wraps around the inner core, and the spool does not slip on the platform P.



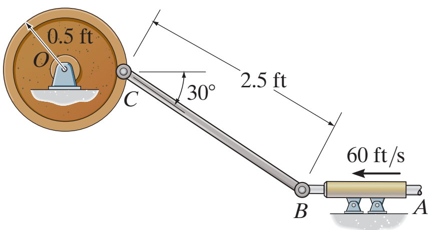
Prob. F16–9

F16–10. If crank OA rotates with an angular velocity of $\omega = 12$ rad/s, determine the velocity of piston B and the angular velocity of rod AB at the instant shown.



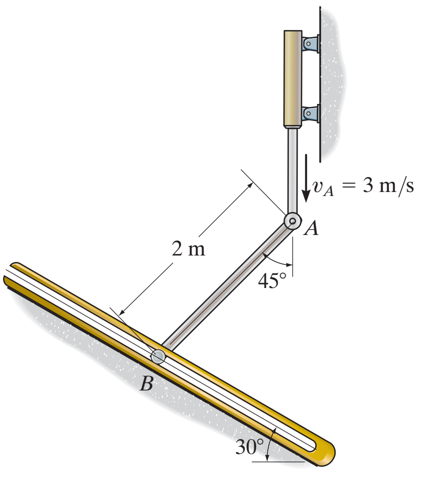
Prob. F16–10

F16–11. If rod AB slides along the horizontal slot with a velocity of 60 ft/s, determine the angular velocity of link BC at the instant shown.



Prob. F16–11

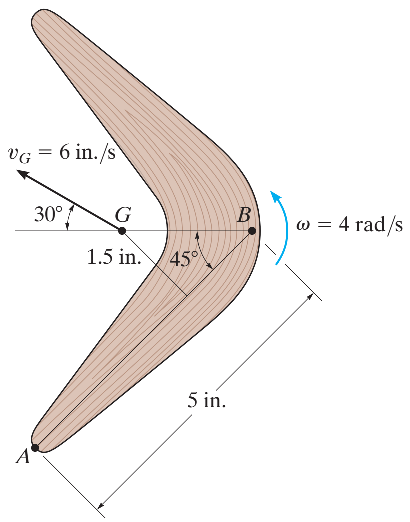
F16–12. End A of the link has a velocity of $v_A = 3$ m/s. Determine the velocity of the peg at B at this instant. The peg is constrained to move along the slot.



Prob. F16–12

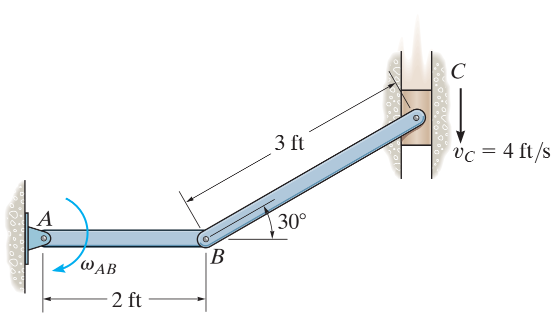
PROBLEMS

16–57. At the instant shown the boomerang has an angular velocity $\omega = 4$ rad/s, and its mass center G has a velocity $v_G = 6$ in./s. Determine the velocity of point B at this instant.



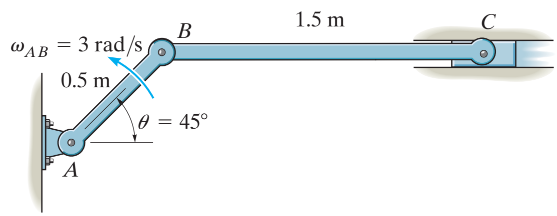
Prob. 16–57

16–58. If the block at C is moving downward at 4 ft/s, determine the angular velocity of bar AB at the instant shown.



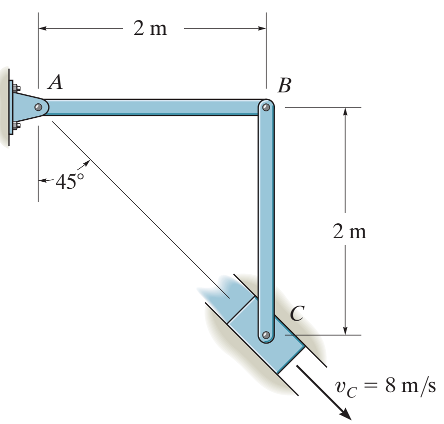
Prob. 16–58

16–59. The link AB has an angular velocity of 3 rad/s. Determine the velocity of block C and the angular velocity of link BC at the instant $\theta = 45°$. Also, sketch the position of link BC when $\theta = 60°$, 45°, and 30° to show its general plane motion.



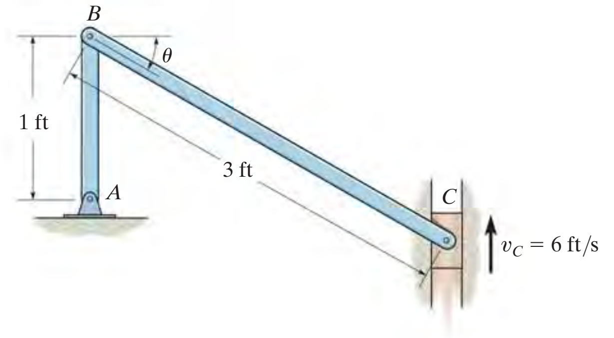
Prob. 16–59

***16–60.** The slider block C moves at 8 m/s down the inclined groove. Determine the angular velocities of links AB and BC, at the instant shown.



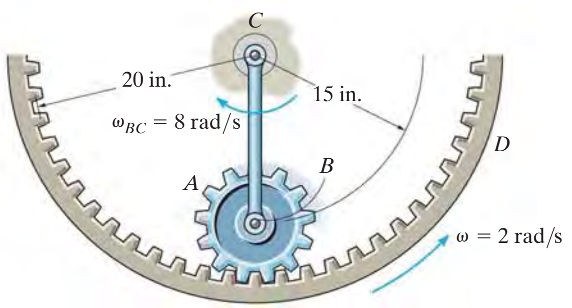
Prob. 16–60

16–61. Determine the angular velocity of links AB and BC at the instant $\theta = 30°$. Also, sketch the position of link BC when $\theta = 55°$, $45°$, and $30°$ to show its general plane motion.



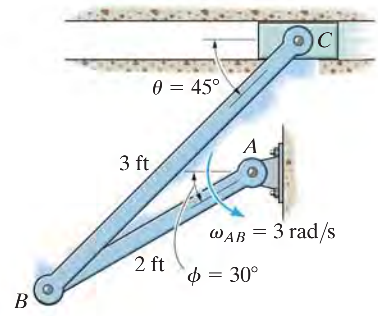
Prob. 16–61

16–62. The planetary gear A is pinned at B. Link BC rotates clockwise with an angular velocity of 8 rad/s, while the outer gear rack rotates counterclockwise with an angular velocity of 2 rad/s. Determine the angular velocity of gear A.



Prob. 16–62

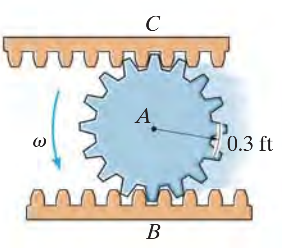
16–63. If the angular velocity of link AB is $\omega_{AB} = 3$ rad/s, determine the velocity of the block at C and the angular velocity of the connecting link CB at the instant $\theta = 45°$ and $\phi = 30°$.



Prob. 16–63

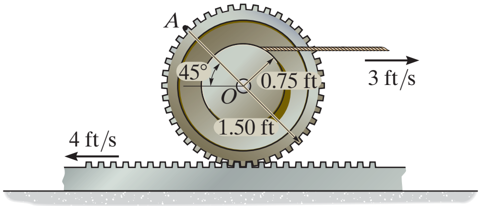
***16–64.** The pinion gear A rolls on the fixed gear rack B with an angular velocity $\omega = 4$ rad/s. Determine the velocity of the gear rack C.

16–65. The pinion gear rolls on the gear racks. If B is moving to the right at 8 ft/s and C is moving to the left at 4 ft/s, determine the angular velocity of the pinion gear and the velocity of its center A.



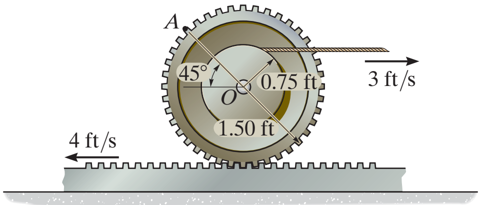
Probs. 16–64/65

16–66. Determine the angular velocity of the gear and the velocity of its center O at the instant shown.



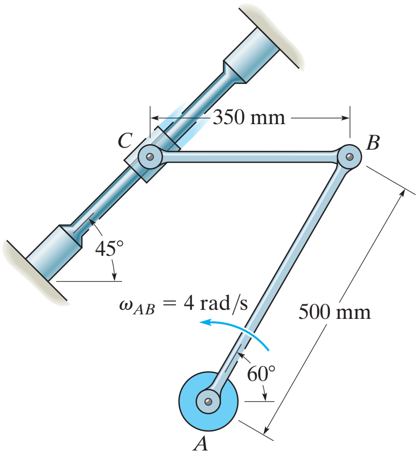
Prob. 16–66

16–67. Determine the velocity of point A on the rim of the gear at the instant shown.



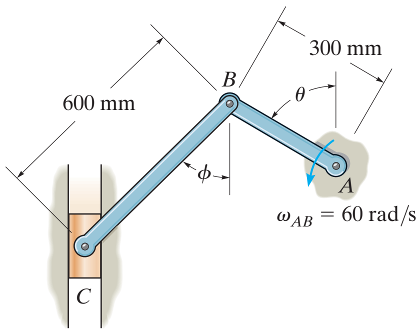
Prob. 16–67

***16–68.** Knowing that angular velocity of link AB is $\omega_{AB} = 4$ rad/s, determine the velocity of the collar at C and the angular velocity of link CB at the instant shown. Link CB is horizontal at this instant.



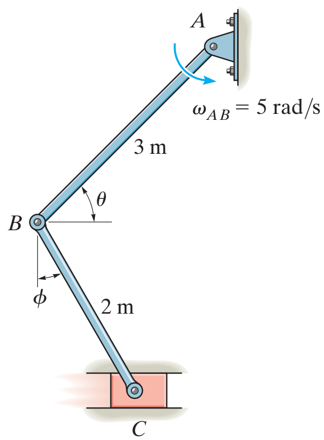
Prob. 16–68

16–69. Rod AB is rotating with an angular velocity of $\omega_{AB} = 60$ rad/s. Determine the velocity of the slider C at the instant $\theta = 60^\circ$ and $\phi = 45^\circ$. Also, sketch the position of bar BC when $\theta = 30^\circ$, 60° and 90° to show its general plane motion.



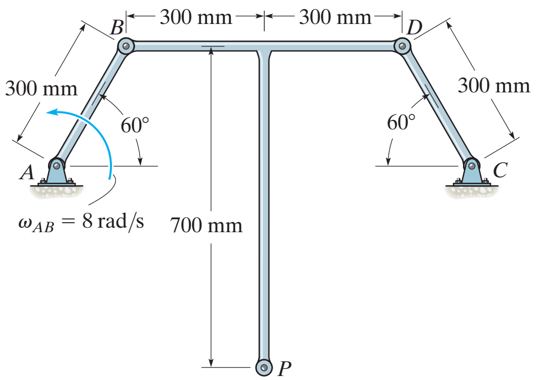
Prob. 16–69

16–70. The angular velocity of link AB is $\omega_{AB} = 5$ rad/s. Determine the velocity of block C and the angular velocity of link BC at the instant $\theta = 45^\circ$ and $\phi = 30^\circ$. Also, sketch the position of link CB when $\theta = 45^\circ$, 60°, and 75° to show its general plane motion.



Prob. 16–70

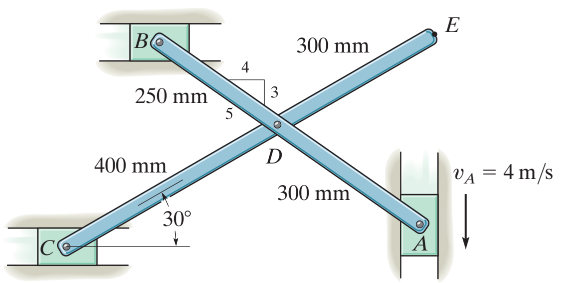
16–71. The similar links AB and CD rotate about the fixed pins at A and C. If AB has an angular velocity $\omega_{AB} = 8$ rad/s, determine the angular velocity of BDP and the velocity of point P.



Prob. 16–71

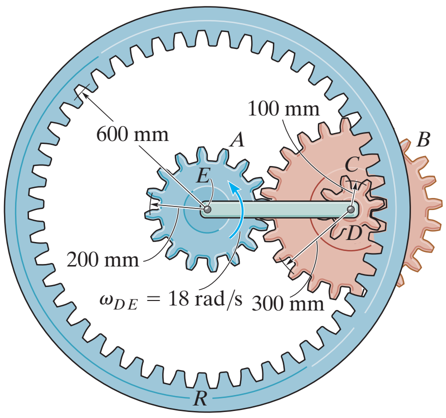
***16–72.** If the slider block A is moving downward at $v_A = 4$ m/s, determine the velocities of blocks B and C at the instant shown.

16–73. If the slider block A is moving downward at $v_A = 4$ m/s, determine the velocity of point E at the instant shown.



Probs. 16–72/73

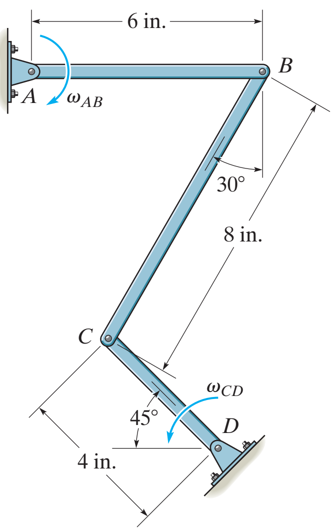
16–74. The epicyclic gear train consists of the sun gear A which is in mesh with the planet gear B. This gear has an inner hub C which is fixed to B and in mesh with the fixed ring gear R. If the connecting link DE pinned to B and C is rotating at $\omega_{DE} = 18$ rad/s about the pin at E, determine the angular velocities of the planet and sun gears.



Prob. 16–74

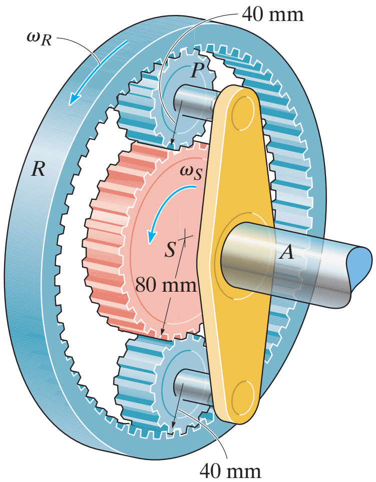
16–75. If link AB is rotating at $\omega_{AB} = 3$ rad/s, determine the angular velocity of link CD at the instant shown.

***16–76.** If link CD is rotating at $\omega_{CD} = 5$ rad/s, determine the angular velocity of link AB at the instant shown.



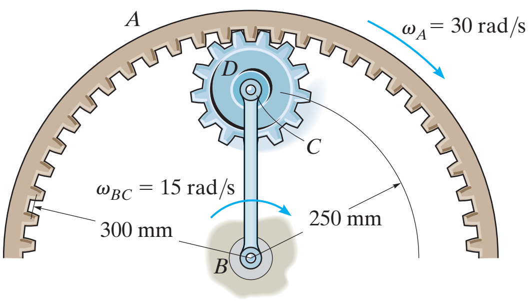
Probs. 16–75/76

16–77. The planetary gear system is used in an automatic transmission for an automobile. By locking or releasing certain gears, it has the advantage of operating the car at different speeds. Consider the case where the ring gear R is held fixed, $\omega_R = 0$, and the sun gear S is rotating at $\omega_S = 5$ rad/s. Determine the angular velocity of each of the planet gears P and shaft A.



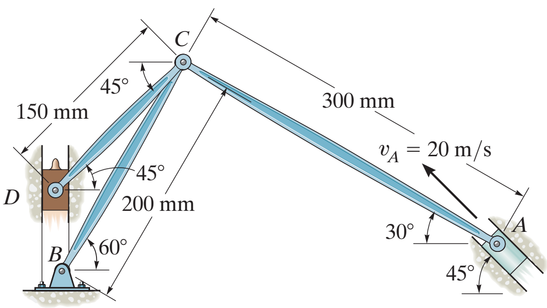
Prob. 16–77

16–78. If the ring gear A rotates clockwise with an angular velocity of $\omega_A = 30$ rad/s, while link BC rotates clockwise with an angular velocity of $\omega_{BC} = 15$ rad/s, determine the angular velocity of gear D.



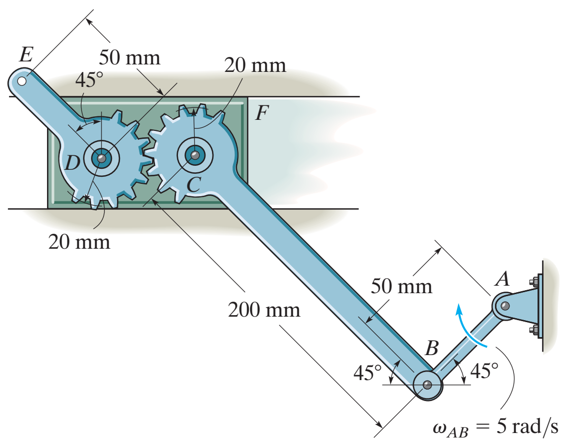
Prob. 16–78

16–79. The mechanism shown is used in a riveting machine. It consists of a driving piston A, three links, and a riveter which is attached to the slider block D. Determine the velocity of D at the instant shown, when the piston at A is traveling at $v_A = 20$ m/s.



Prob. 16–79

***16–80.** The mechanism is used on a machine for the manufacturing of a wire product. Because of the rotational motion of link AB and the sliding of block F, the segmental gear lever DE undergoes general plane motion. If AB is rotating at $\omega_{AB} = 5$ rad/s, determine the velocity of point E at the instant shown.

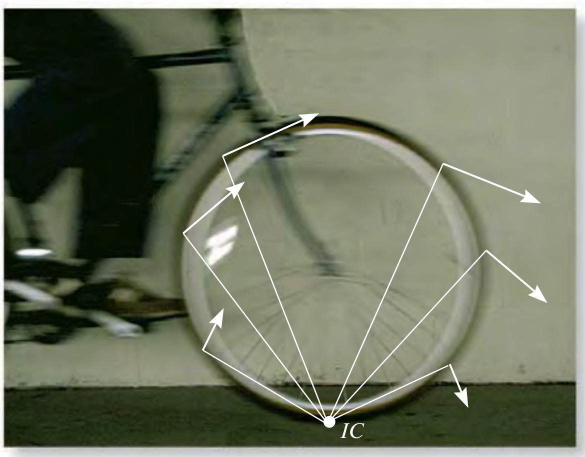


Prob. 16–80

16.6 Instantaneous Center of Zero Velocity

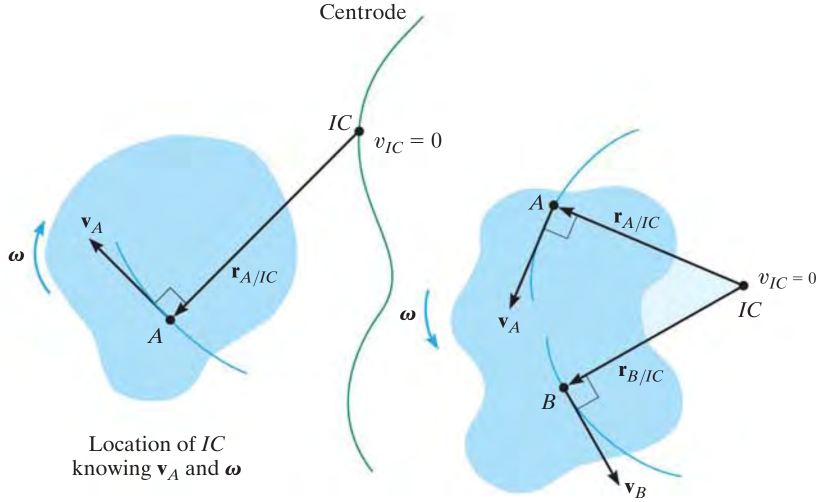
The velocity of any point B located on a rigid body can be obtained in a very direct way by choosing the base point A to be a point that has *zero velocity* at the instant considered. In this case, $\mathbf{v}_A = \mathbf{0}$, and therefore the velocity equation, $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, becomes $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. For a body having general plane motion, point A so chosen is called the *instantaneous center of zero velocity (IC)*, and it lies on the *instantaneous axis of zero velocity*. This axis is always perpendicular to the plane of motion, and the intersection of the axis with this plane defines the location of the IC. Since point A coincides with the IC, then $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/IC}$ and so point B moves momentarily about the IC in a *circular path;* in other words, the body appears to rotate about the instantaneous axis. The *magnitude* of $\mathbf{v}_B$ is simply $v_B = \omega r_{B/IC}$, where ω is the angular velocity of the body. Due to the circular motion, the *direction* of $\mathbf{v}_B$ must always be *perpendicular* to $\mathbf{r}_{B/IC}$.

For example, the IC for the bicycle wheel in Fig. 16–17 is at the contact point with the ground. There the spokes are somewhat visible, whereas at the top of the wheel they become blurred. If one imagines that the wheel is momentarily pinned at this point, the velocities of various points can be found using $v = \omega r$. Here the radial distances shown in the photo, Fig. 16–17, must be determined from the geometry of the wheel.



(© R.C. Hibbeler)

Fig. 16–17

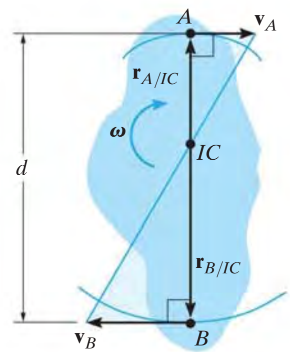


Location of IC knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

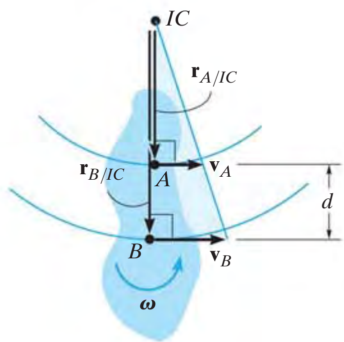
(a)

Location of IC knowing the directions of $\mathbf{v}_A$ and $\mathbf{v}_B$

(b)



(c)



Location of IC knowing $\mathbf{v}_A$ and $\mathbf{v}_B$

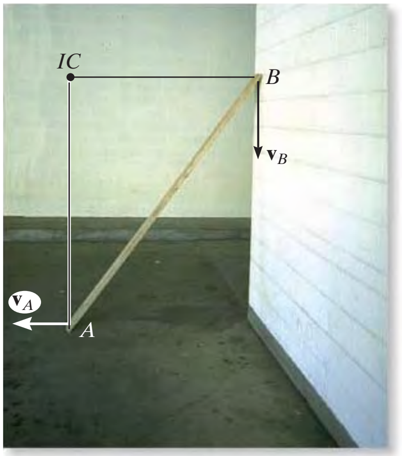
(d)

Fig. 16–18

Location of the *IC*. To locate the *IC* we can use the fact that the *velocity* of a point on the body is *always perpendicular* to the *relative-position vector* directed from the *IC* to the point. Several possibilities exist:

- *The velocity* $\mathbf{v}_A$ *of a point A on the body and the angular velocity* $\boldsymbol{\omega}$ *of the body are known*, Fig. 16–18*a*. In this case, the *IC* is located along the line drawn perpendicular to $\mathbf{v}_A$ at A, such that the distance from A to the *IC* is $r_{A/IC} = v_A/\omega$. Note that the *IC* lies up and to the right of A since $\mathbf{v}_A$ must cause a clockwise angular velocity $\boldsymbol{\omega}$ about the *IC*.
- *The lines of action of two nonparallel velocities* $\mathbf{v}_A$ and $\mathbf{v}_B$ *are known*, Fig. 16–18*b*. Construct at points A and B line segments that are perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$. Extending these perpendiculars to their *point of intersection* as shown locates the *IC* at the instant considered.
- *The magnitude and direction of two parallel velocities* $\mathbf{v}_A$ *and* $\mathbf{v}_B$ *are known*. Here the location of the *IC* is determined by proportional triangles. Examples are shown in Fig. 16–18*c* and *d*. In both cases $r_{A/IC} = v_A/\omega$ and $r_{B/IC} = v_B/\omega$. If d is a known distance between points A and B, then in Fig. 16–18*c*, $r_{A/IC} + r_{B/IC} = d$ and in Fig. 16–18*d*, $r_{B/IC} - r_{A/IC} = d$.

As the board slides downward to the left it is subjected to general plane motion. Since the directions of the velocities of its ends A and B are known, the IC is located as shown. At this instant the board will momentarily rotate about this point. Draw the board in several other positions and establish the IC for each case. (© R.C. Hibbeler)



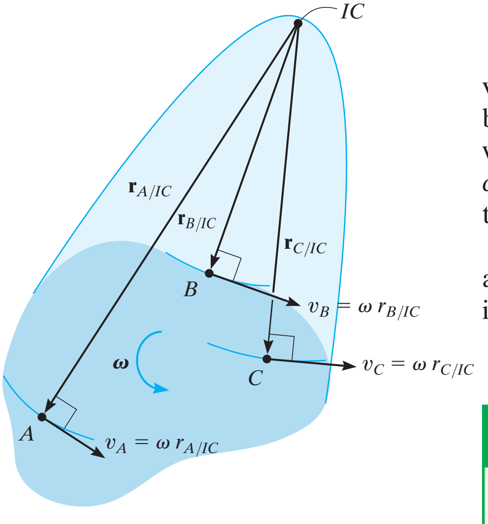


Fig. 16–19

Realize that the point chosen as the instantaneous center of zero velocity for the body *can only be used at the instant considered* since the body changes its position from one instant to the next. The locus of points which define the location of the IC during the body's motion is called a *centrode*, Fig. 16–18*a*, and so each point on the centrode acts as the IC for the body only for an instant.

Although the IC may be conveniently used to determine the velocity of any point in a body, it generally *does not have zero acceleration* and therefore it *should not* be used for finding the accelerations of points in a body.

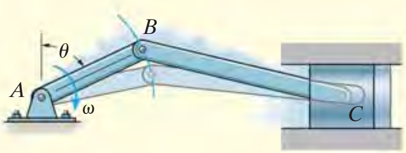
Procedure for Analysis

The velocity of a point on a body which is subjected to general plane motion can be determined with reference to its instantaneous center of zero velocity provided the location of the IC is first established using one of the three methods described above.

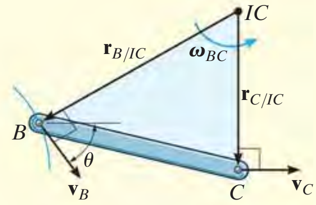
- As shown on the kinematic diagram in Fig. 16–19, the body is imagined as "extended and pinned" at the IC so that, at the instant considered, it rotates about this pin with its angular velocity $\boldsymbol{\omega}$.
- The *magnitude* of velocity for each of the arbitrary points A, B, and C on the body can be determined by using the equation $v = \omega r$, where r is the radial distance from the IC to each point.
- The line of action of each velocity vector $\mathbf{v}$ is *perpendicular* to its associated radial line $\mathbf{r}$, and the velocity has a *sense of direction* which tends to move the point in a manner consistent with the angular rotation $\boldsymbol{\omega}$ of the radial line, Fig. 16–19.

EXAMPLE 16.9

Show how to determine the location of the instantaneous center of zero velocity for (a) member *BC* shown in Fig. 16–20*a*; and (b) the link *CB* shown in Fig. 16–20*c*.



(a)

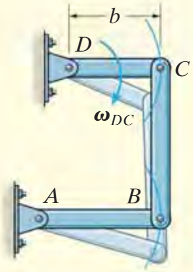


(b)

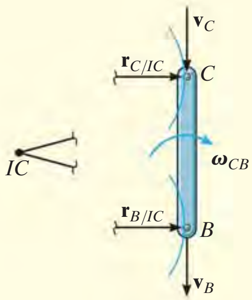
SOLUTION

Part (a). As shown in Fig. 16–20*a*, point *B* moves in a circular path such that $\mathbf{v}_B$ is perpendicular to *AB*. Therefore, it acts at an angle θ from the horizontal as shown in Fig. 16–20*b*. The motion of point *B* causes the piston to move forward *horizontally* with a velocity $\mathbf{v}_C$. When lines are drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$, Fig. 16–20*b*, they intersect at the *IC*.

Part (b). Points *B* and *C* follow circular paths of motion since links *AB* and *DC* are each subjected to rotation about a fixed axis, Fig. 16–20*c*. Since the velocity is always tangent to the path, at the instant considered, $\mathbf{v}_C$ on rod *DC* and $\mathbf{v}_B$ on rod *AB* are both directed vertically downward, along the axis of link *CB*, Fig. 16–20*d*. Radial lines drawn perpendicular to these two velocities form parallel lines which intersect at "infinity;" i.e., $r_{C/IC} \rightarrow \infty$ and $r_{B/IC} \rightarrow \infty$. Thus, $\omega_{CB} = (v_C/r_{C/IC}) \rightarrow 0$. As a result, link *CB* momentarily *translates*. An instant later, however, *CB* will move to a tilted position, causing the *IC* to move to some finite location.



(c)

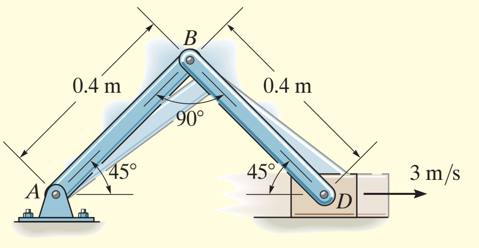


(d)

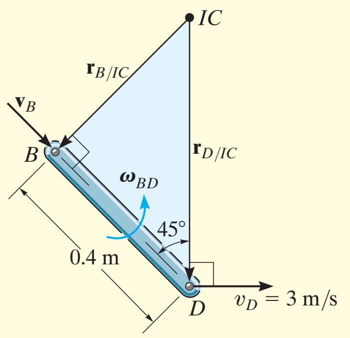
Fig. 16–20

EXAMPLE 16.10

Block D shown in Fig. 16–21a moves with a speed of 3 m/s. Determine the angular velocities of links BD and AB, at the instant shown.



(a)



(b)

SOLUTION

As D moves to the right, it causes AB to rotate clockwise about point A. Hence, $\mathbf{v}_B$ is directed perpendicular to AB. The instantaneous center of zero velocity for BD is located at the intersection of the line segments drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_D$, Fig. 16–21b. From the geometry,

$$r_{B/IC} = 0.4 \tan 45^\circ \text{ m} = 0.4 \text{ m}$$

$$r_{D/IC} = \frac{0.4 \text{ m}}{\cos 45^\circ} = 0.5657 \text{ m}$$

Since the magnitude of $\mathbf{v}_D$ is known, the angular velocity of link BD is

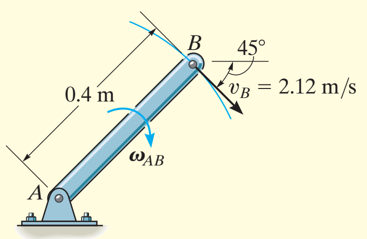
$$\omega_{BD} = \frac{v_D}{r_{D/IC}} = \frac{3 \text{ m/s}}{0.5657 \text{ m}} = 5.30 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

The velocity of B is therefore

$$v_B = \omega_{BD}(r_{B/IC}) = 5.30 \text{ rad/s } (0.4 \text{ m}) = 2.12 \text{ m/s} \quad \measuredangle 45^\circ$$

From Fig. 16–21c, the angular velocity of AB is

$$\omega_{AB} = \frac{v_B}{r_{B/A}} = \frac{2.12 \text{ m/s}}{0.4 \text{ m}} = 5.30 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$



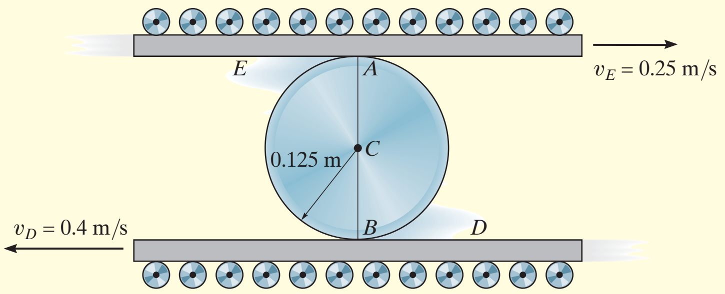
(c)

Fig. 16–21

NOTE: Try to solve this problem by applying $\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$ to member BD.

EXAMPLE 16.11

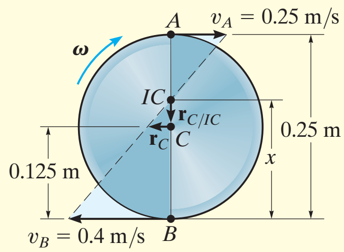
The cylinder shown in Fig. 16–22*a* rolls without slipping between the two moving plates E and D. Determine the angular velocity of the cylinder and the velocity of its center C.



(a)

SOLUTION

Since no slipping occurs, the contact points A and B on the cylinder have the same velocities as the plates E and D, respectively. Furthermore, the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are *parallel*, so that by the proportionality of right triangles the IC is located at a point on line AB, Fig. 16–22*b*. Assuming this point to be a distance x from B, we have



(b)

Fig. 16–22

$v_B = \omega x;$ $\qquad 0.4 \text{ m/s} = \omega x$

$v_A = \omega(0.25 \text{ m} - x);$ $\qquad 0.25 \text{ m/s} = \omega(0.25 \text{ m} - x)$

Dividing one equation into the other eliminates ω and yields

$$0.4(0.25 - x) = 0.25x$$

$$x = \frac{0.1}{0.65} = 0.1538 \text{ m}$$

Hence, the angular velocity of the cylinder is

$$\omega = \frac{v_B}{x} = \frac{0.4 \text{ m/s}}{0.1538 \text{ m}} = 2.60 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

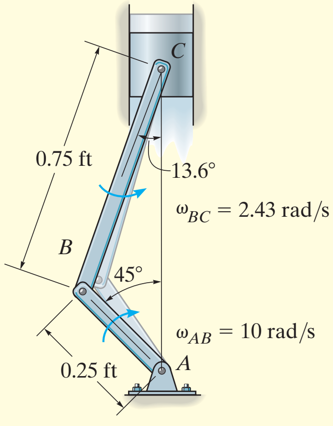
The velocity of point C is therefore

$$v_C = \omega r_{C/IC} = 2.60 \text{ rad/s}\,(0.1538 \text{ m} - 0.125 \text{ m})$$

$$= 0.0750 \text{ m/s} \leftarrow \qquad \textit{Ans.}$$

EXAMPLE 16.12

The crankshaft AB turns with a clockwise angular velocity of 10 rad/s, Fig. 16–23a. Determine the velocity of the piston at the instant shown.



(a)

SOLUTION

The crankshaft rotates about a fixed axis, and so the velocity of point B is

$$v_B = 10 \text{ rad/s } (0.25 \text{ ft}) = 2.50 \text{ ft/s} \measuredangle 45^\circ$$

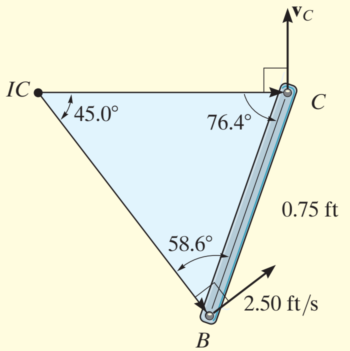
Since the directions of the velocities of B and C are known, then the location of the IC for the connecting rod BC is at the intersection of the lines extended from these points, perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$, Fig. 16–23b. The magnitudes of $\mathbf{r}_{B/IC}$ and $\mathbf{r}_{C/IC}$ can be obtained from the geometry of the triangle and the law of sines, i.e.,

$$\frac{0.75 \text{ ft}}{\sin 45^\circ} = \frac{r_{B/IC}}{\sin 76.4^\circ}$$

$$r_{B/IC} = 1.031 \text{ ft}$$

$$\frac{0.75 \text{ ft}}{\sin 45^\circ} = \frac{r_{C/IC}}{\sin 58.6^\circ}$$

$$r_{C/IC} = 0.9056 \text{ ft}$$



(b)

Fig. 16–23

The rotational sense of $\boldsymbol{\omega}_{BC}$ must be the same as the rotation caused by $\mathbf{v}_B$ about the IC, which is counterclockwise. Therefore,

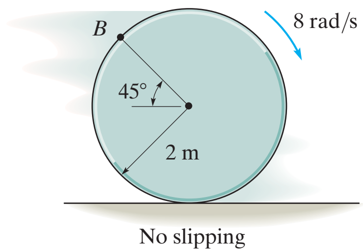
$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{2.5 \text{ ft/s}}{1.031 \text{ ft}} = 2.425 \text{ rad/s}$$

Using this result, the velocity of the piston is

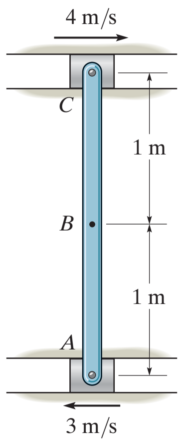
$$v_C = \omega_{BC} r_{C/IC} = (2.425 \text{ rad/s})(0.9056 \text{ ft}) = 2.20 \text{ ft/s} \qquad \textit{Ans.}$$

PRELIMINARY PROBLEM

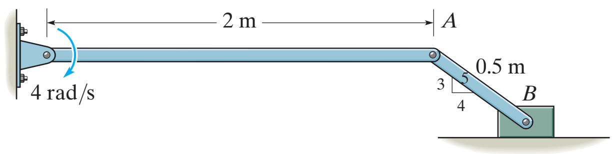
P16–2. Establish the location of the instantaneous center of zero velocity for finding the velocity of point *B*.



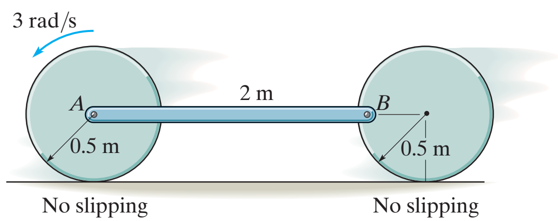
(a)



(d)



(b)



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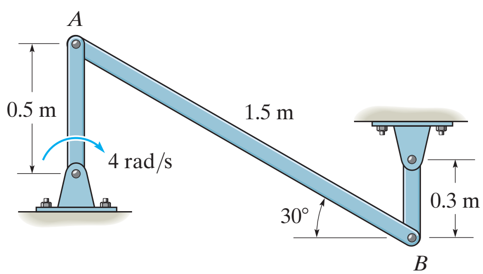

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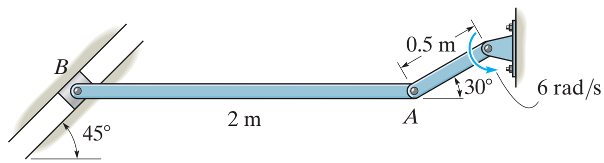
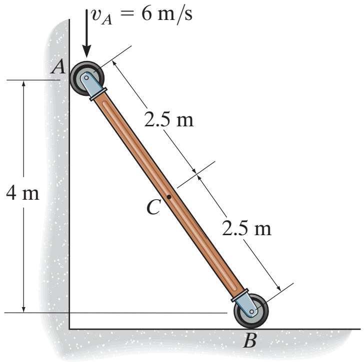


(f)

Prob. P16–2

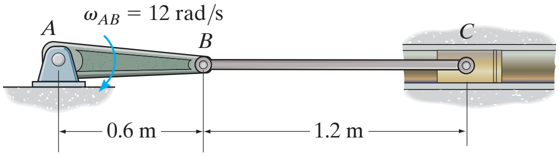
FUNDAMENTAL PROBLEMS

F16–13. Determine the angular velocity of the rod and the velocity of point C at the instant shown.



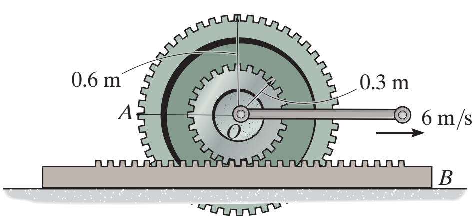
Prob. F16–13

F16–14. Determine the angular velocity of link BC and velocity of the piston C at the instant shown.



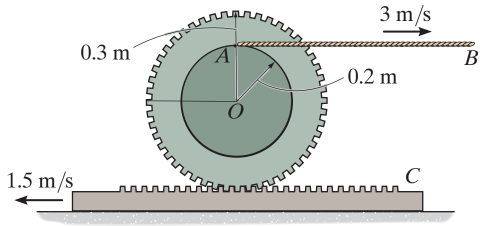
Prob. F16–14

F16–15. If the center O of the wheel is moving with a speed of $v_O = 6$ m/s, determine the velocity of point A on the wheel. The gear rack B is fixed.



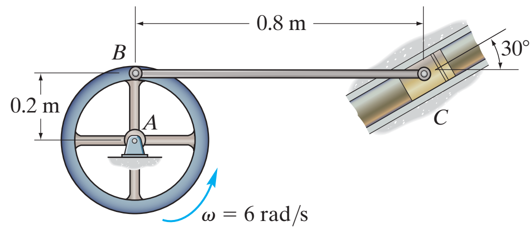
Prob. F16–15

F16–16. If cable AB is unwound with a speed of 3 m/s, and the gear rack C has a speed of 1.5 m/s, determine the angular velocity of the gear and the velocity of its center O.



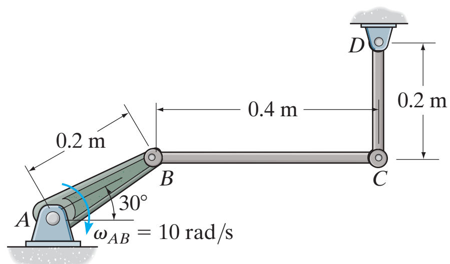
Prob. F16–16

F16–17. Determine the angular velocity of link BC and the velocity of the piston C at the instant shown.



Prob. F16–17

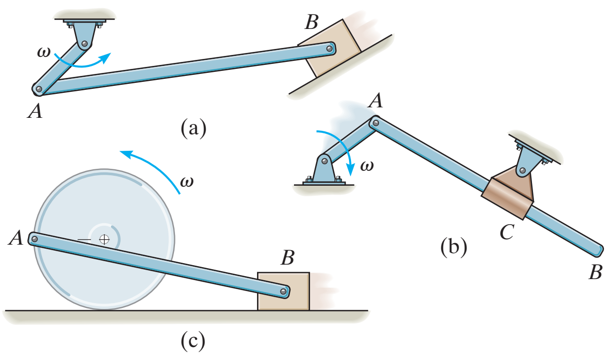
F16–18. Determine the angular velocity of links BC and CD at the instant shown.



Prob. F16–18

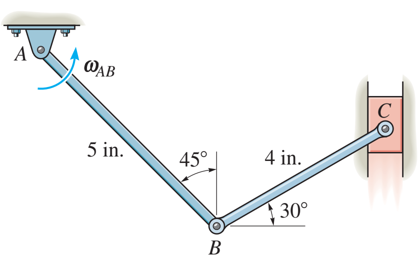
PROBLEMS

16–81. In each case show graphically how to locate the instantaneous center of zero velocity of link AB. Assume the geometry is known.



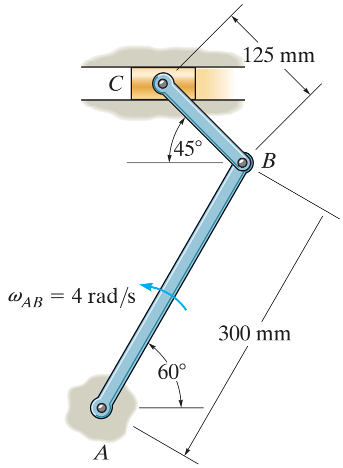
Prob. 16–81

16–82. Determine the angular velocity of link AB at the instant shown if block C is moving upward at 12 in/s.



Prob. 16–82

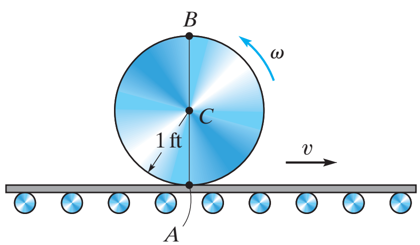
16–83. The shaper mechanism is designed to give a slow cutting stroke and a quick return to a blade attached to the slider at C. Determine the angular velocity of the link CB at the instant shown, if the link AB is rotating at 4 rad/s.



Prob. 16–83

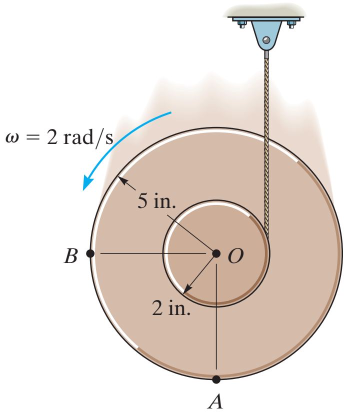
***16–84.** The conveyor belt is moving to the right at $v = 8$ ft/s, and at the same instant the cylinder is rolling counterclockwise at $\omega = 2$ rad/s without slipping. Determine the velocities of the cylinder's center C and point B at this instant.

16–85. The conveyor belt is moving to the right at $v = 12$ ft/s, and at the same instant the cylinder is rolling counterclockwise at $\omega = 6$ rad/s while its center has a velocity of 4 ft/s to the left. Determine the velocities of points A and B on the disk at this instant. Does the cylinder slip on the conveyor?



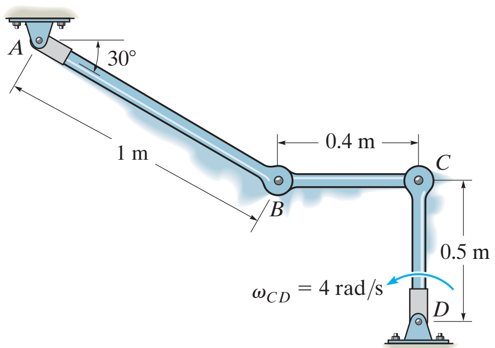
Probs. 16–84/85

16–86. As the cord unravels from the wheel's inner hub, the wheel is rotating at $\omega = 2$ rad/s at the instant shown. Determine the velocities of points A and B.



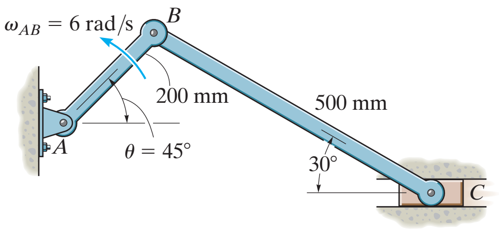
Prob. 16–86

16–87. If rod CD is rotating with an angular velocity $\omega_{CD} = 4$ rad/s, determine the angular velocities of rods AB and CB at the instant shown.



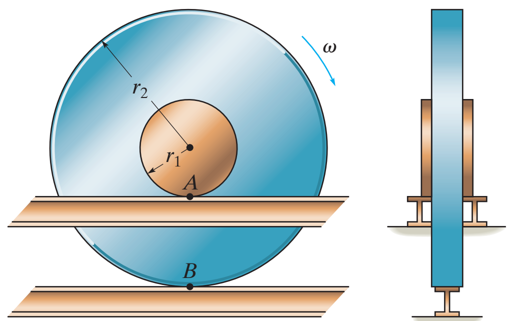
Prob. 16–87

***16–88.** If bar AB has an angular velocity $\omega_{AB} = 6$ rad/s, determine the velocity of the slider block C at the instant shown.



Prob. 16–88

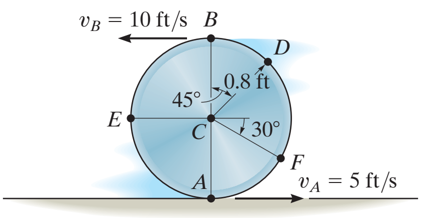
16–89. Show that if the rim of the wheel and its hub maintain contact with the three tracks as the wheel rolls, it is necessary that slipping occurs at the hub A if no slipping occurs at B. Under these conditions, what is the speed at A if the wheel has angular velocity $\boldsymbol{\omega}$?



Prob. 16–89

16–90. Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the center point C and point D at this instant.

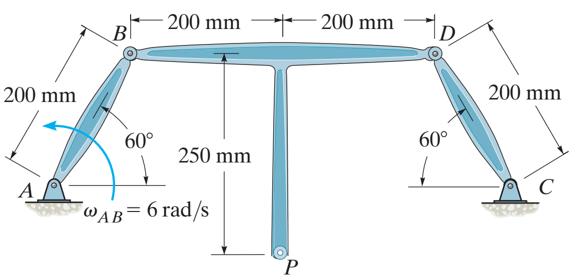
16–91. Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the center point C and point E at this instant.



Probs. 16–90/91

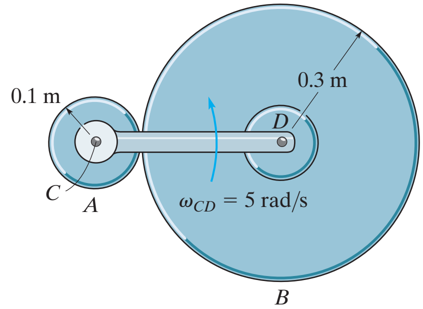
***16–92.** Member AB is rotating at $\omega_{AB} = 6$ rad/s. Determine the velocity of point D and the angular velocity of members BPD and CD.

16–93. Member AB is rotating at $\omega_{AB} = 6$ rad/s. Determine the velocity of point P, and the angular velocity of member BPD.



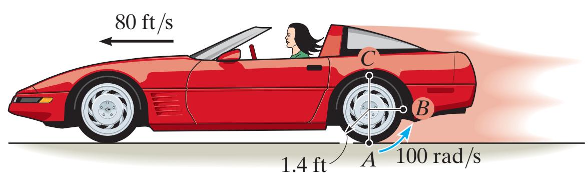
Probs. 16–92/93

16–94. The cylinder B rolls on the fixed cylinder A without slipping. If connected bar CD is rotating with an angular velocity $\omega_{CD} = 5$ rad/s, determine the angular velocity of cylinder B. Point C is a fixed point.



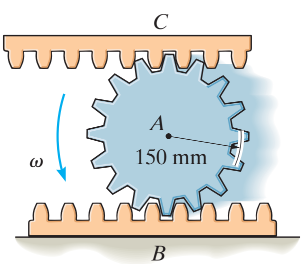
Prob. 16–94

16–95. As the car travels forward at 80 ft/s on a wet road, due to slipping, the rear wheels have an angular velocity $\omega = 100$ rad/s. Determine the speeds of points A, B, and C caused by the motion.



Prob. 16–95

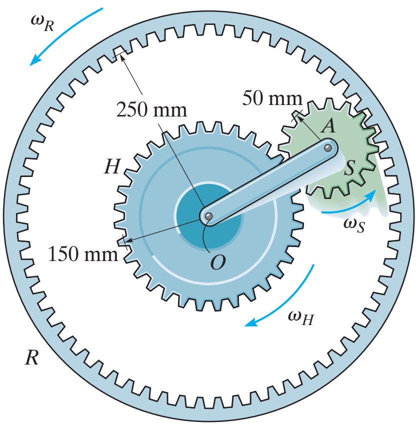
***16–96.** The pinion gear A rolls on the fixed gear rack B with an angular velocity $\omega = 8$ rad/s. Determine the velocity of the gear rack C.



Prob. 16–96

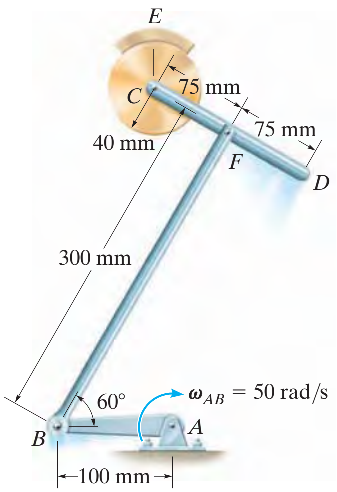
16–97. If the hub gear H and ring gear R have angular velocities $\omega_H = 5$ rad/s and $\omega_R = 20$ rad/s, respectively, determine the angular velocity ω_S of the spur gear S and the angular velocity of its attached arm OA.

16–98. If the hub gear H has an angular velocity $\omega_H = 5$ rad/s, determine the angular velocity of the ring gear R so that the arm OA attached to the spur gear S remains stationary ($\omega_{OA} = 0$). What is the angular velocity of the spur gear?



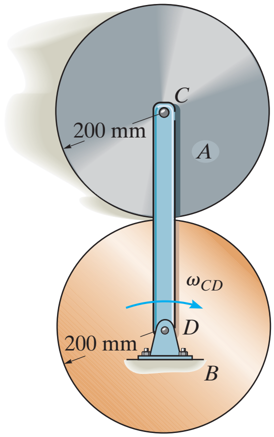
Probs. 16–97/98

16–99. The crankshaft AB rotates at $\omega_{AB} = 50$ rad/s about the fixed axis through point A, and the disk at C is held fixed in its support at E. Determine the angular velocity of rod CD at the instant shown.



Prob. 16–99

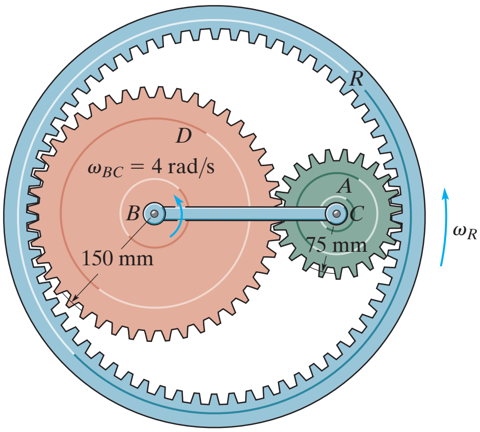
***16–100.** Cylinder A rolls on the *fixed cylinder* B without slipping. If bar CD is rotating with an angular velocity of $\omega_{CD} = 3$ rad/s, determine the angular velocity of A.



Prob. 16–100

16–101. The planet gear A is pin connected to the end of the link BC. If the link rotates about the fixed point B at 4 rad/s, determine the angular velocity of the ring gear R. The sun gear D is fixed from rotating.

16–102. Solve Prob. 16–101 if the sun gear D is rotating clockwise at $\omega_D = 5$ rad/s while link BC rotates counterclockwise at $\omega_{BC} = 4$ rad/s.



Probs. 16–101/102

16.7 Relative-Motion Analysis: Acceleration

An equation that relates the accelerations of two points on a bar (rigid body) subjected to general plane motion may be determined by differentiating $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ with respect to time. This yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms $d\mathbf{v}_B/dt = \mathbf{a}_B$ and $d\mathbf{v}_A/dt = \mathbf{a}_A$ are measured with respect to a set of *fixed x, y axes* and represent the *absolute accelerations* of points B and A. The last term represents the acceleration of B with respect to A as measured by an observer fixed to translating x', y' axes which have their origin at the base point A. In Sec. 16.5 it was shown that to this observer point B appears to move along a *circular arc* that has a radius of curvature $r_{B/A}$. Consequently, $\mathbf{a}_{B/A}$ can be expressed in terms of its tangential and normal components; i.e., $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$, where $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Hence, the relative-acceleration equation can be written in the form

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n \qquad (16\text{–}17)$$

where

$\mathbf{a}_B$ = acceleration of point B

$\mathbf{a}_A$ = acceleration of point A

$(\mathbf{a}_{B/A})_t$ = tangential acceleration component of B with respect to A. The *magnitude* is $(a_{B/A})_t = \alpha r_{B/A}$, and the *direction* is perpendicular to $\mathbf{r}_{B/A}$.

$(\mathbf{a}_{B/A})_n$ = normal acceleration component of B with respect to A. The *magnitude* is $(a_{B/A})_n = \omega^2 r_{B/A}$, and the *direction* is always from B toward A.

The terms in Eq. 16–17 are represented graphically in Fig. 16–24. Here it is seen that at a given instant the acceleration of B, Fig. 16–24*a*, is determined by considering the bar to translate with an acceleration $\mathbf{a}_A$, Fig. 16–24*b*, and simultaneously rotate about the base point A with an instantaneous angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$, Fig. 16–24*c*. Vector addition of these two effects, applied to B, yields $\mathbf{a}_B$, as shown in Fig. 16–24*d*. It should be noted from Fig. 16–24*a* that since points A and B move along *curved paths*, the accelerations of these points will have *both tangential and normal components*. (Recall that the acceleration of a point is *tangent to the path only* when the path is *rectilinear* or when it is an inflection point on a curve.)

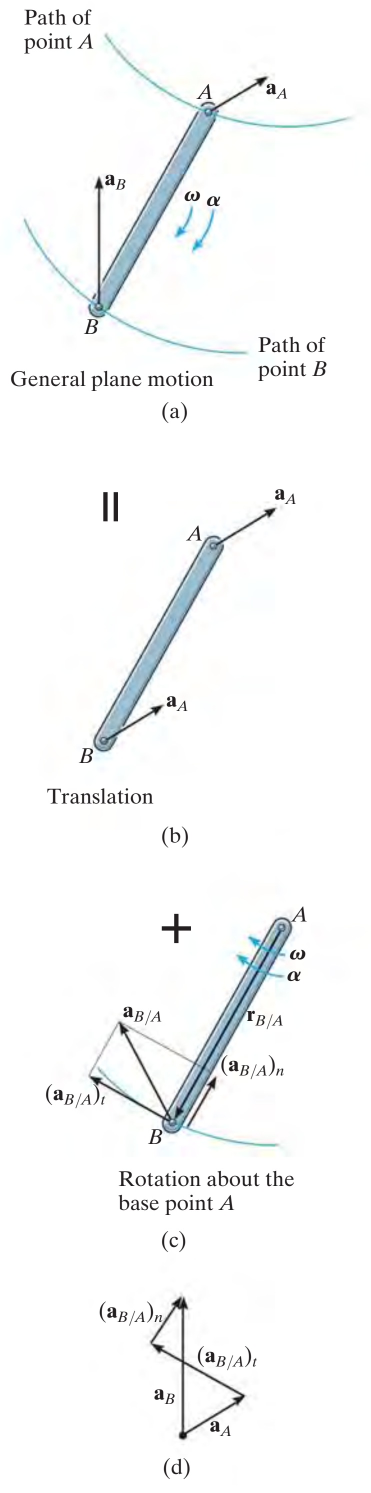
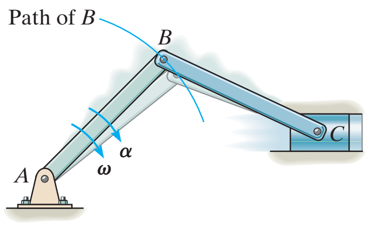
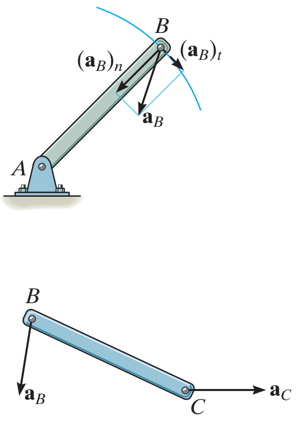


Fig. 16–24



(a)



(b)

Fig. 16–25

Since the relative-acceleration components represent the effect of *circular motion* observed from translating axes having their origin at the base point A, these terms can be expressed as $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$ and $(\mathbf{a}_{B/A})_n = -\omega^2\mathbf{r}_{B/A}$, Eq. 16–14. Hence, Eq. 16–17 becomes

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A} \tag{16–18}$$

where

$\mathbf{a}_B$ = acceleration of point B
$\mathbf{a}_A$ = acceleration of the base point A
$\boldsymbol{\alpha}$ = angular acceleration of the body
$\boldsymbol{\omega}$ = angular velocity of the body
$\mathbf{r}_{B/A}$ = position vector directed from A to B

If Eq. 16–17 or 16–18 is applied in a practical manner to study the accelerated motion of a rigid body which is *pin connected* to two other bodies, it should be realized that points which are *coincident at the pin* move with the *same acceleration*, since the path of motion over which they travel is the *same*. For example, point B lying on either rod BA or BC of the crank mechanism shown in Fig. 16–25*a* has the same acceleration, since the rods are pin connected at B. Here the motion of B is along a *circular path*, so that $\mathbf{a}_B$ can be expressed in terms of its tangential and normal components. At the other end of rod BC point C moves along a *straight-lined path*, which is defined by the piston. Hence, $\mathbf{a}_C$ is horizontal, Fig. 16–25*b*.

Finally, consider a disk that rolls without slipping as shown in Fig. 16–26*a*. As a result, $v_A = 0$ and so from the kinematic diagram in Fig. 16–26*b*, the velocity of the mass center G is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\omega\mathbf{k}) \times (r\mathbf{j})$$

So that

$$v_G = \omega r \tag{16–19}$$

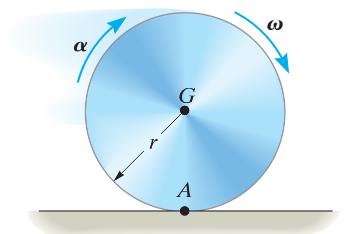
This same result can also be determined using the IC method where point A is the *IC*.

Since G moves along a *straight line*, its acceleration in this case can be determined from the time derivative of its velocity.

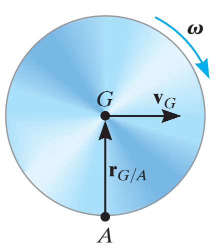
$$\frac{dv_G}{dt} = \frac{d\omega}{dt}r$$

$$a_G = \alpha r \tag{16–20}$$

These two important results were also obtained in Example 16–4. They apply as well to any circular object, such as a ball, gear, wheel, etc., that *rolls without slipping*.



(a)



(b)

Fig. 16–26

Procedure for Analysis

The relative acceleration equation can be applied between any two points A and B on a body either by using a Cartesian vector analysis, or by writing the x and y scalar component equations directly.

Velocity Analysis.

- Determine the angular velocity $\boldsymbol{\omega}$ of the body by using a velocity analysis as discussed in Sec. 16.5 or 16.6. Also, determine the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points A and B *if these points move along curved paths*.

Vector Analysis

Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw the kinematic diagram of the body. Indicate on it $\mathbf{a}_A$, $\mathbf{a}_B$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{r}_{B/A}$.
- If points A and B move along *curved paths*, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\mathbf{a}_A = (\mathbf{a}_A)_t + (\mathbf{a}_A)_n$ and $\mathbf{a}_B = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n$.

Acceleration Equation.

- To apply $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective **i** and **j** components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates that the sense of direction of the vector is opposite to that shown on the kinematic diagram.

Scalar Analysis

Kinematic Diagram.

- If the acceleration equation is applied in scalar form, then the magnitudes and directions of the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$ must be established. To do this draw a kinematic diagram such as shown in Fig. 16–24*c*. Since the body is considered to be momentarily "pinned" at the base point A, the *magnitudes* of these components are $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Their *sense of direction* is established from the diagram such that $(\mathbf{a}_{B/A})_t$ acts perpendicular to $\mathbf{r}_{B/A}$, in accordance with the rotational motion $\boldsymbol{\alpha}$ of the body, and $(\mathbf{a}_{B/A})_n$ is directed from B toward A.*

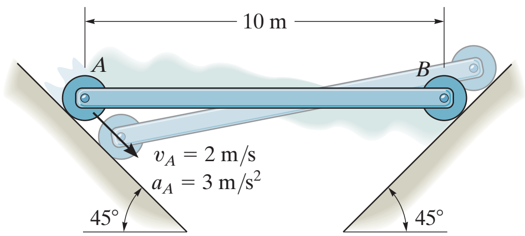
Acceleration Equation.

- Represent the vectors in $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$ graphically by showing their magnitudes and directions underneath each term. The scalar equations are determined from the x and y components of these vectors.

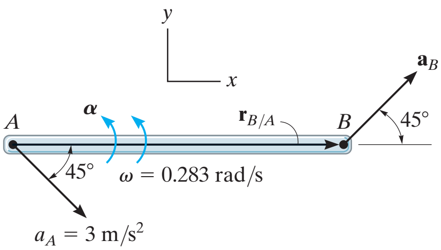
*The notation $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A(\text{pin})})_t + (\mathbf{a}_{B/A(\text{pin})})_n$ may be helpful in recalling that A is assumed to be pinned.

The mechanism for a window is shown. Here CA rotates about a fixed axis through C, and AB undergoes general plane motion. Since point A moves along a curved path it has two components of acceleration, whereas point B moves along a straight track and the direction of its acceleration is specified. (© R.C. Hibbeler)

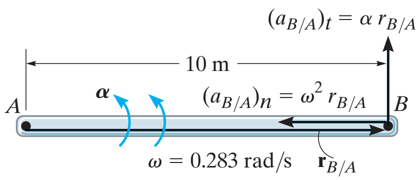
EXAMPLE 16.13



(a)



(b)



(c)

Fig. 16–27

The rod AB shown in Fig. 16–27a is confined to move along the inclined planes at A and B. If point A has an acceleration of $3\ \text{m/s}^2$ and a velocity of $2\ \text{m/s}$, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points A and B on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283\ \text{rad/s}$ ↺ using either the velocity equation or the method of instantaneous centers.

Kinematic Diagram. Since points A and B both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns in Fig. 16–27b, namely, a_B and α.

Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

$$a_B \cos 45^\circ \mathbf{i} + a_B \sin 45^\circ \mathbf{j} = 3 \cos 45^\circ \mathbf{i} - 3 \sin 45^\circ \mathbf{j} + (\alpha \mathbf{k}) \times (10\mathbf{i}) - (0.283)^2(10\mathbf{i})$$

Carrying out the cross product and equating the **i** and **j** components yields

$$a_B \cos 45^\circ = 3 \cos 45^\circ - (0.283)^2(10) \qquad (1)$$

$$a_B \sin 45^\circ = -3 \sin 45^\circ + \alpha(10) \qquad (2)$$

Solving, we have

$$a_B = 1.87\ \text{m/s}^2 \measuredangle 45^\circ$$

$$\alpha = 0.344\ \text{rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$, Fig. 16–27c, we have

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$\begin{bmatrix} a_B \\ \measuredangle 45^\circ \end{bmatrix} = \begin{bmatrix} 3\ \text{m/s}^2 \\ \searrow 45^\circ \end{bmatrix} + \begin{bmatrix} \alpha(10\ \text{m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283\ \text{rad/s})^2(10\ \text{m}) \\ \leftarrow \end{bmatrix}$$

Equating the x and y components yields Eqs. 1 and 2, and the solution proceeds as before.

EXAMPLE 16.14

The disk rolls without slipping and has the angular motion shown in Fig. 16–28*a*. Determine the acceleration of point *A* at this instant.

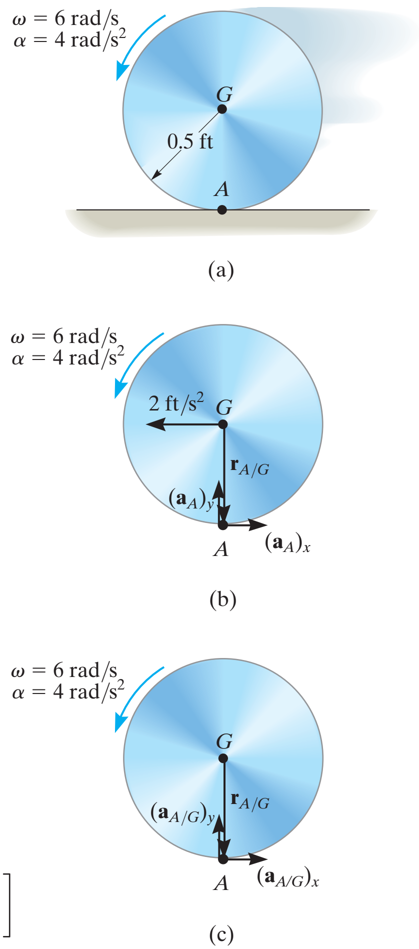


Fig. 16–28

SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs, applying Eq. 16–20,

$$a_G = \alpha r = (4\ \text{rad/s}^2)(0.5\ \text{ft}) = 2\ \text{ft/s}^2$$

Acceleration Equation.

We will apply the acceleration equation to points *G* and *A*, Fig. 16–28*b*,

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} - \omega^2 \mathbf{r}_{A/G}$$
$$\mathbf{a}_A = -2\mathbf{i} + (4\mathbf{k}) \times (-0.5\mathbf{j}) - (6)^2(-0.5\mathbf{j})$$
$$= \{18\mathbf{j}\}\ \text{ft/s}^2$$

SOLUTION II (SCALAR ANALYSIS)

Using the result for $a_G = 2\ \text{ft/s}^2$ determined above, and from the kinematic diagram, showing the relative motion $\mathbf{a}_{A/G}$, Fig. 16–28*c*, we have

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_x + (\mathbf{a}_{A/G})_y$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 2\ \text{ft/s}^2 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (4\ \text{rad/s}^2)(0.5\ \text{ft}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6\ \text{rad/s})^2(0.5\ \text{ft}) \\ \uparrow \end{bmatrix}$$

$$\overset{+}{\rightarrow} \qquad (a_A)_x = -2 + 2 = 0$$

$$+\uparrow \qquad (a_A)_y = 18\ \text{ft/s}^2$$

Therefore,

$$a_A = \sqrt{(0)^2 + (18\ \text{ft/s}^2)^2} = 18\ \text{ft/s}^2$$ *Ans.*

NOTE: The fact that $a_A = 18\ \text{ft/s}^2$ indicates that the instantaneous center of zero velocity, point *A*, is *not* a point of zero acceleration.

EXAMPLE 16.15

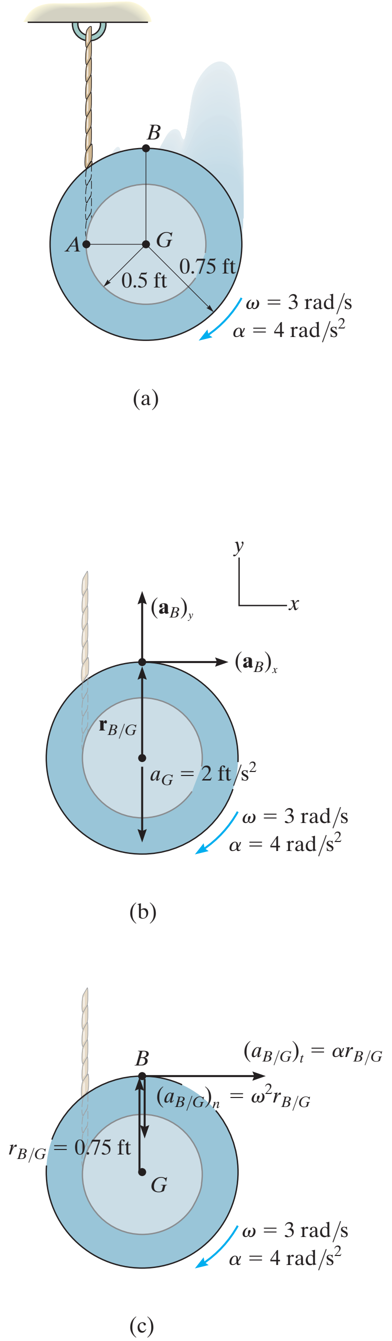


Fig. 16–29

The spool shown in Fig. 16–29*a* unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s^2. Determine the acceleration of point *B*.

SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point *A*. Therefore, we can use the results of Eq. 16–20 to determine the acceleration of point *G*, i.e.,

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

We will apply the acceleration equation to points *G* and *B*.

Kinematic Diagram. Point *B* moves along a *curved path* having an *unknown* radius of curvature.* Its acceleration will be represented by its unknown *x* and *y* components as shown in Fig. 16–29*b*.

Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \omega^2 \mathbf{r}_{B/G}$$

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -2\mathbf{j} + (-4\mathbf{k}) \times (0.75\mathbf{j}) - (3)^2(0.75\mathbf{j})$$

Equating the **i** and **j** terms, the component equations are

$$(a_B)_x = 4(0.75) = 3 \text{ ft/s}^2 \rightarrow \qquad (1)$$

$$(a_B)_y = -2 - 6.75 = -8.75 \text{ ft/s}^2 = 8.75 \text{ ft/s}^2 \downarrow \qquad (2)$$

The magnitude and direction of $\mathbf{a}_B$ are therefore

$$a_B = \sqrt{(3)^2 + (8.75)^2} = 9.25 \text{ ft/s}^2 \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{8.75}{3} = 71.1° \qquad \textit{Ans.}$$

SOLUTION II (SCALAR ANALYSIS)

This problem may be solved by writing the scalar component equations directly. The kinematic diagram in Fig. 16–29*c* shows the relative-acceleration components $(\mathbf{a}_{B/G})_t$ and $(\mathbf{a}_{B/G})_n$. Thus,

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$

$$\begin{bmatrix}(a_B)_x \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_B)_y \\ \uparrow\end{bmatrix}$$

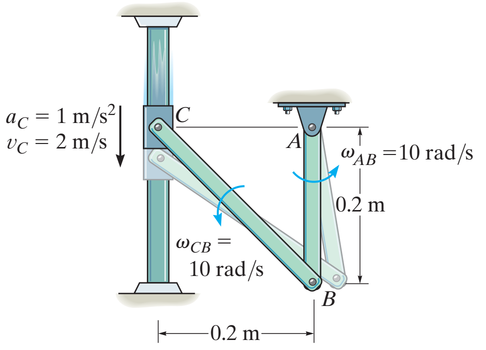
$$= \begin{bmatrix}2 \text{ ft/s}^2 \\ \downarrow\end{bmatrix} + \begin{bmatrix}4 \text{ rad/s}^2 \text{ } (0.75 \text{ ft}) \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(3 \text{ rad/s})^2(0.75 \text{ ft}) \\ \downarrow\end{bmatrix}$$

The *x* and *y* components yield Eqs. 1 and 2 above.

*Realize that the path's radius of curvature ρ is not equal to the radius of the spool since the spool is not rotating about point *G*. Furthermore, ρ is not defined as the distance from *A* (*IC*) to *B*, since the location of the *IC* depends only on the velocity of a point and not the geometry of its path.

EXAMPLE 16.16

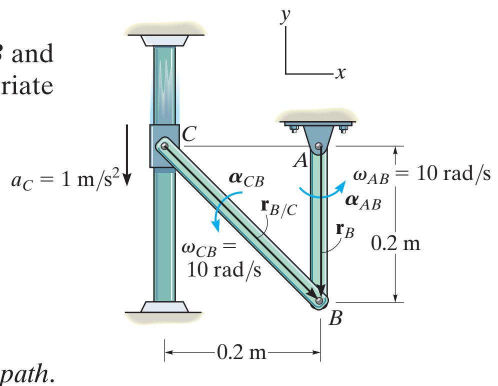
The collar C in Fig. 16–30*a* moves downward with an acceleration of 1 m/s^2. At the instant shown, it has a speed of 2 m/s which gives links CB and AB an angular velocity $\omega_{AB} = \omega_{CB} = 10\text{ rad/s}$. (See Example 16.8.) Determine the angular accelerations of CB and AB at this instant.



(a)

SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams of *both* links AB and CB are shown in Fig. 16–30*b*. To solve, we will apply the appropriate kinematic equation to each link.



(b)

Fig. 16–30

Acceleration Equation.

Link AB (rotation about a fixed axis):

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$

$$\mathbf{a}_B = (\alpha_{AB}\mathbf{k}) \times (-0.2\mathbf{j}) - (10)^2(-0.2\mathbf{j})$$

$$\mathbf{a}_B = 0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j}$$

Note that $\mathbf{a}_B$ has n and t components since it moves along a *circular path*.

Link BC (general plane motion): Using the result for $\mathbf{a}_B$ and applying Eq. 16–18, we have

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \omega_{CB}^2 \mathbf{r}_{B/C}$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^2(0.2\mathbf{i} - 0.2\mathbf{j})$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\alpha_{CB}\mathbf{j} + 0.2\alpha_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$

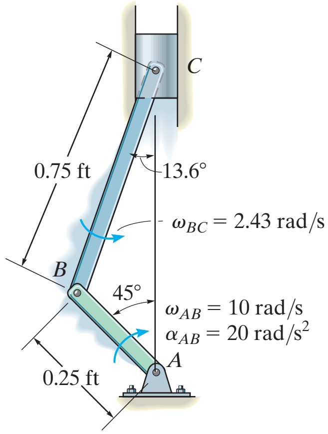
$$20 = -1 + 0.2\alpha_{CB} + 20$$

Solving,

$$\alpha_{CB} = 5\text{ rad/s}^2 ↺ \qquad \textit{Ans.}$$

$$\alpha_{AB} = -95\text{ rad/s}^2 = 95\text{ rad/s}^2 ↻ \qquad \textit{Ans.}$$

EXAMPLE 16.17



(a)

The crankshaft AB turns with a clockwise angular acceleration of 20 rad/s^2, Fig. 16–31a. Determine the acceleration of the piston at the instant AB is in the position shown. At this instant $\omega_{AB} = 10$ rad/s and $\omega_{BC} = 2.43$ rad/s. (See Example 16.12.)

SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams for both AB and BC are shown in Fig. 16–31b. Here $\mathbf{a}_C$ is vertical since C moves along a straight-line path.

Acceleration Equation. Expressing each of the position vectors in Cartesian vector form

$$\mathbf{r}_B = \{-0.25 \sin 45°\mathbf{i} + 0.25 \cos 45°\mathbf{j}\} \text{ ft} = \{-0.177\mathbf{i} + 0.177\mathbf{j}\} \text{ ft}$$

$$\mathbf{r}_{C/B} = \{0.75 \sin 13.6°\mathbf{i} + 0.75 \cos 13.6°\mathbf{j}\} \text{ ft} = \{0.177\mathbf{i} + 0.729\mathbf{j}\} \text{ ft}$$

Crankshaft AB (rotation about a fixed axis):

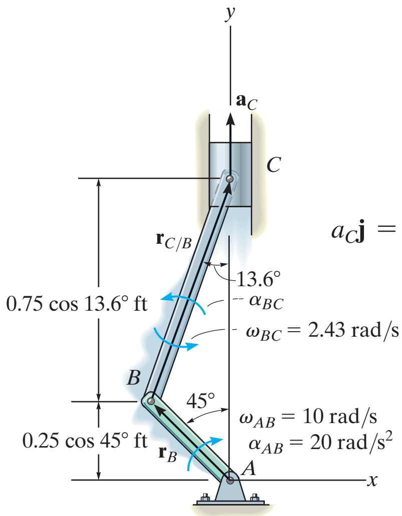
$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B$$
$$= (-20\mathbf{k}) \times (-0.177\mathbf{i} + 0.177\mathbf{j}) - (10)^2(-0.177\mathbf{i} + 0.177\mathbf{j})$$
$$= \{21.21\mathbf{i} - 14.14\mathbf{j}\} \text{ ft/s}^2$$

Connecting Rod BC (general plane motion): Using the result for $\mathbf{a}_B$ and noting that $\mathbf{a}_C$ is in the vertical direction, we have

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$
$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.177\mathbf{i} + 0.729\mathbf{j}) - (2.43)^2(0.177\mathbf{i} + 0.729\mathbf{j})$$
$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + 0.177\alpha_{BC}\mathbf{j} - 0.729\alpha_{BC}\mathbf{i} - 1.04\mathbf{i} - 4.30\mathbf{j}$$
$$0 = 20.17 - 0.729\alpha_{BC}$$
$$a_C = 0.177\alpha_{BC} - 18.45$$

Solving yields

$$\alpha_{BC} = 27.7 \text{ rad/s}^2 \circlearrowleft$$
$$a_C = -13.5 \text{ ft/s}^2$$ *Ans.*



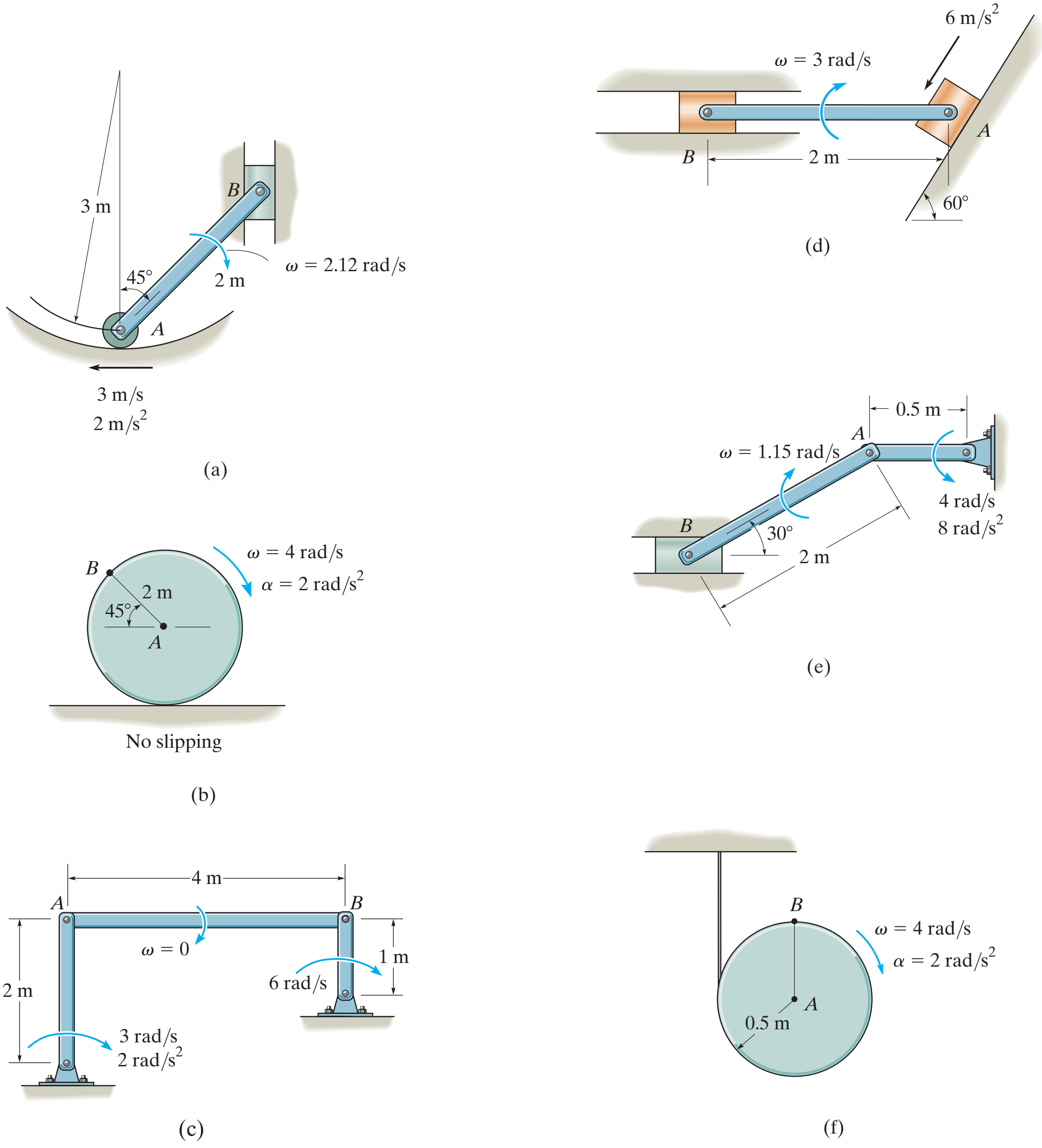
(b)

Fig. 16–31

NOTE: Since the piston is moving upward, the negative sign for a_C indicates that the piston is decelerating, i.e., $\mathbf{a}_C = \{-13.5\mathbf{j}\}$ ft/s^2. This causes the speed of the piston to decrease until AB becomes vertical, at which time the piston is momentarily at rest.

PRELIMINARY PROBLEM

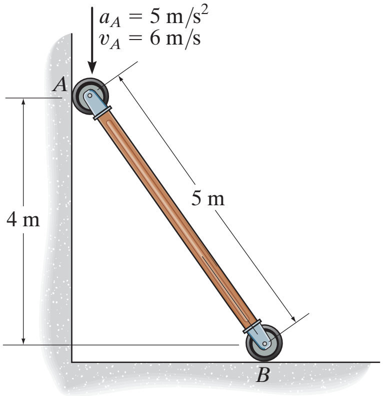
P16–3. Set up the relative acceleration equation between points A and B. The angular velocity is given.



Prob. P16–3

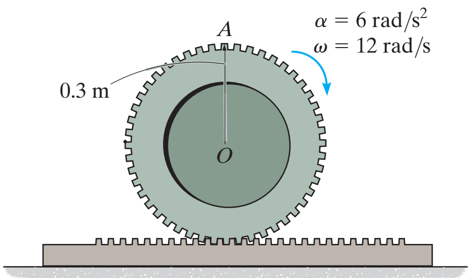
FUNDAMENTAL PROBLEMS

F16–19. At the instant shown, end A of the rod has the velocity and acceleration shown. Determine the angular acceleration of the rod and acceleration of end B of the rod.



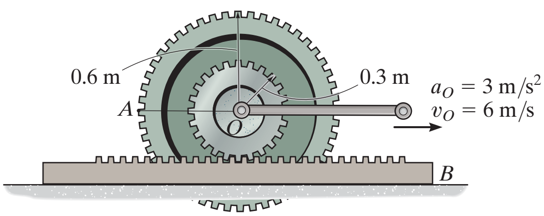
Prob. F16–19

F16–20. The gear rolls on the fixed rack with an angular velocity of $\omega = 12\ \text{rad/s}$ and angular acceleration of $\alpha = 6\ \text{rad/s}^2$. Determine the acceleration of point A.



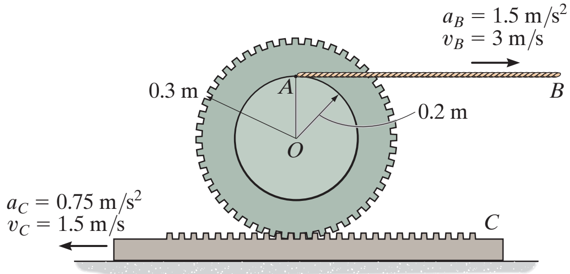
Prob. F16–20

F16–21. The gear rolls on the fixed rack B. At the instant shown, the center O of the gear moves with a velocity of $v_O = 6\ \text{m/s}$ and acceleration of $a_O = 3\ \text{m/s}^2$. Determine the angular acceleration of the gear and acceleration of point A at this instant.



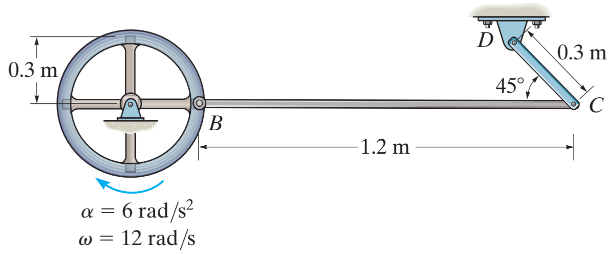
Prob. F16–21

F16–22. At the instant shown, cable AB has a velocity of $3\ \text{m/s}$ and acceleration of $1.5\ \text{m/s}^2$, while the gear rack has a velocity of $1.5\ \text{m/s}$ and acceleration of $0.75\ \text{m/s}^2$. Determine the angular acceleration of the gear at this instant.



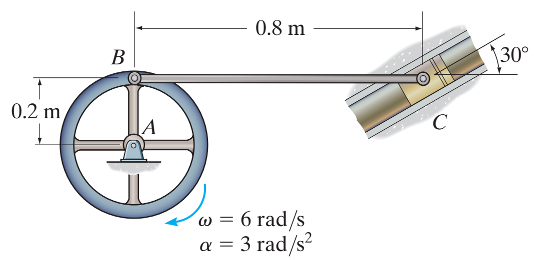
Prob. F16–22

F16–23. At the instant shown, the wheel rotates with an angular velocity of $\omega = 12\ \text{rad/s}$ and an angular acceleration of $\alpha = 6\ \text{rad/s}^2$. Determine the angular acceleration of link BC at the instant shown.



Prob. F16–23

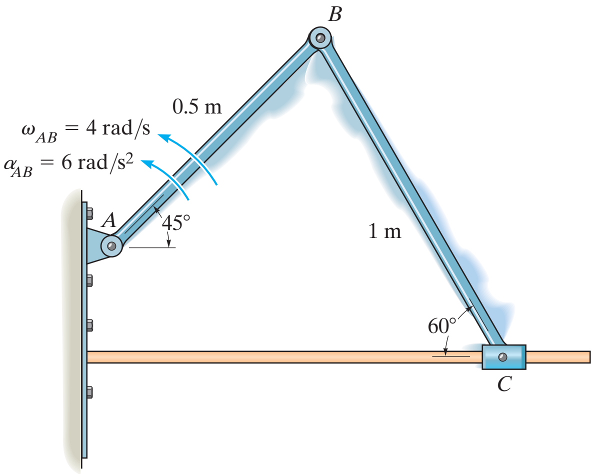
F16–24. At the instant shown, wheel A rotates with an angular velocity of $\omega = 6\ \text{rad/s}$ and an angular acceleration of $\alpha = 3\ \text{rad/s}^2$. Determine the angular acceleration of link BC and the acceleration of piston C.



Prob. F16–24

PROBLEMS

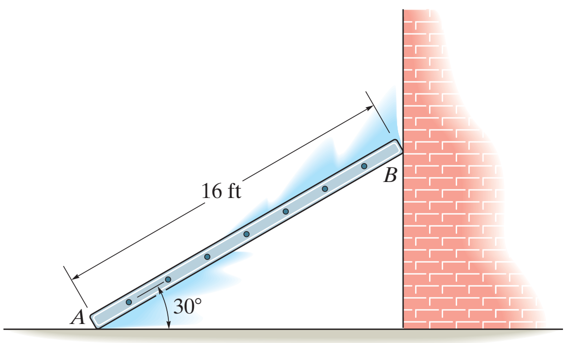
16–103. Bar AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.



Prob. 16–103

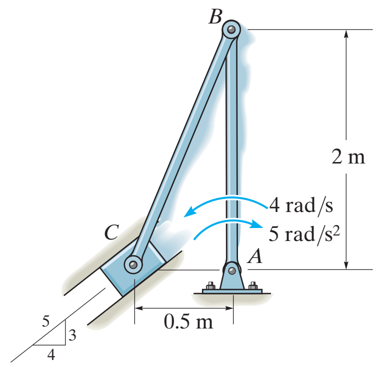
***16–104.** At a given instant the bottom A of the ladder has an acceleration $a_A = 4\ \text{ft/s}^2$ and velocity $v_A = 6$ ft/s, both acting to the left. Determine the acceleration of the top of the ladder, B, and the ladder's angular acceleration at this same instant.

16–105. At a given instant the top B of the ladder has an acceleration $a_B = 2\ \text{ft/s}^2$ and a velocity of $v_B = 4$ ft/s, both acting downward. Determine the acceleration of the bottom A of the ladder, and the ladder's angular acceleration at this instant.



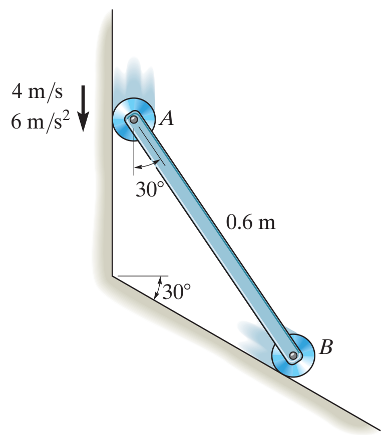
Probs. 16–104/105

16–106. Member AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.



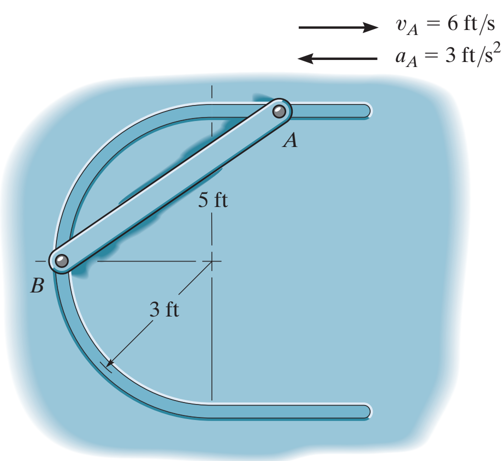
Prob. 16–106

16–107. At a given instant the roller A on the bar has the velocity and acceleration shown. Determine the velocity and acceleration of the roller B, and the bar's angular velocity and angular acceleration at this instant.



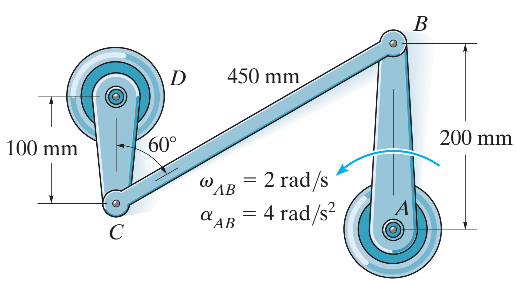
Prob. 16–107

***16–108.** The rod is confined to move along the path due to the pins at its ends. At the instant shown, point A has the motion shown. Determine the velocity and acceleration of point B at this instant.



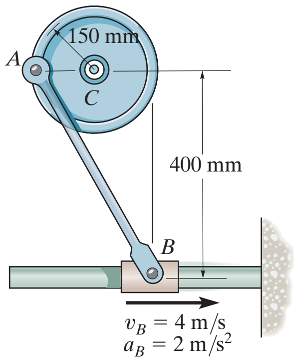
Prob. 16–108

16–109. Member AB has the angular motions shown. Determine the angular velocity and angular acceleration of members CB and DC.



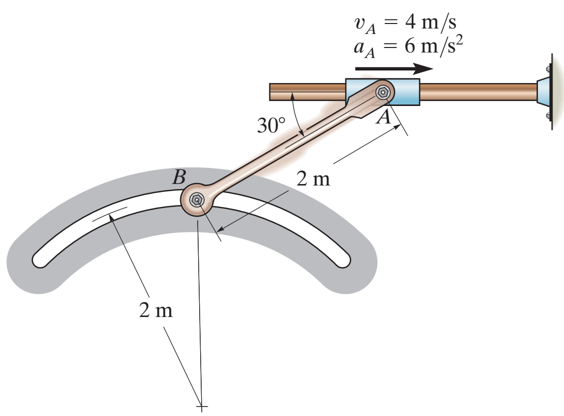
Prob. 16–109

16–110. The slider block has the motion shown. Determine the angular velocity and angular acceleration of the wheel at this instant.



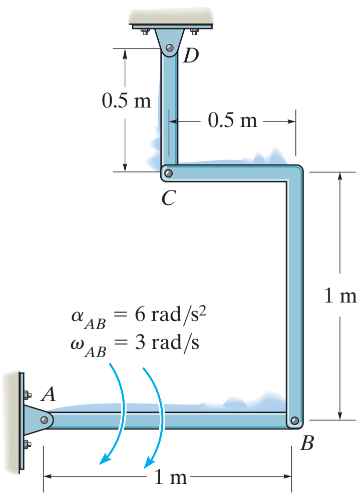
Prob. 16–110

16–111. At a given instant the slider block A is moving to the right with the motion shown. Determine the angular acceleration of link AB and the acceleration of point B at this instant.



Prob. 16–111

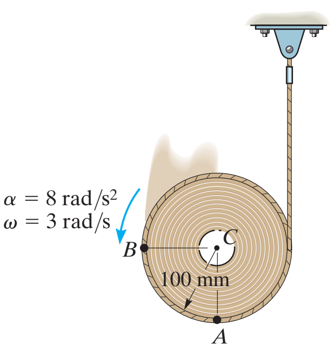
*16–112. Determine the angular acceleration of link CD if link AB has the angular velocity and angular acceleration shown.



Prob. 16–112

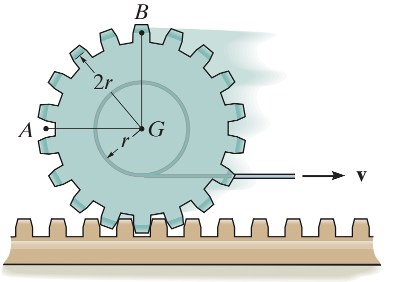
16–113. The reel of rope has the angular motion shown. Determine the velocity and acceleration of point A at the instant shown.

16–114. The reel of rope has the angular motion shown. Determine the velocity and acceleration of point B at the instant shown.



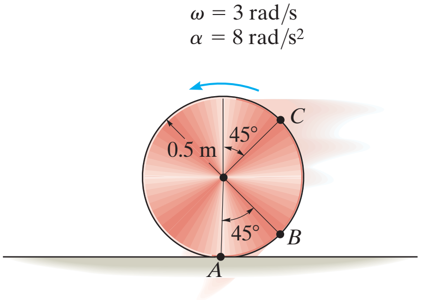
Probs. 16–113/114

16–115. A cord is wrapped around the inner spool of the gear. If it is pulled with a constant velocity **v**, determine the velocities and accelerations of points A and B. The gear rolls on the fixed gear rack.



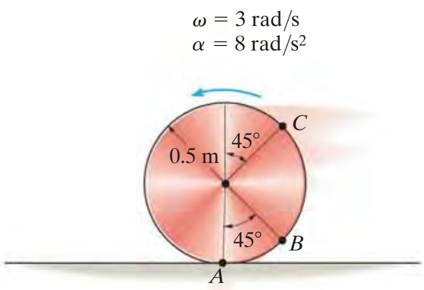
Prob. 16–115

*16–116. The disk has an angular acceleration $\alpha = 8\ \text{rad/s}^2$ and angular velocity $\omega = 3\ \text{rad/s}$ at the instant shown. If it does not slip at A, determine the acceleration of point B.



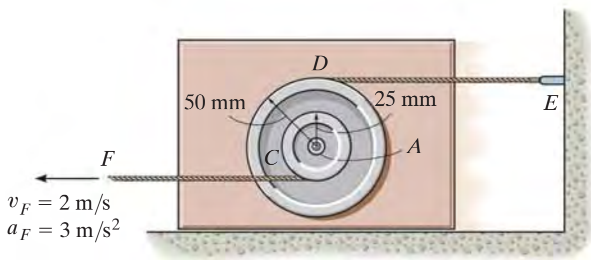
Prob. 16–116

16–117. The disk has an angular acceleration $\alpha = 8 \text{ rad/s}^2$ and angular velocity $\omega = 3 \text{ rad/s}$ at the instant shown. If it does not slip at A, determine the acceleration of point C.



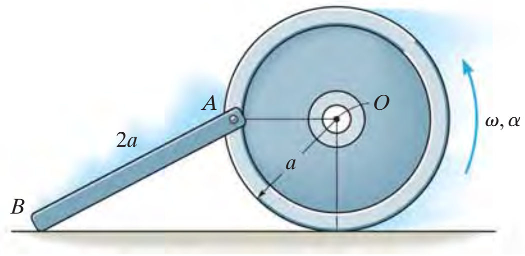
Prob. 16–117

16–118. A single pulley having both an inner and outer rim is pin connected to the block at A. As cord CF unwinds from the inner rim of the pulley with the motion shown, cord DE unwinds from the outer rim. Determine the angular acceleration of the pulley and the acceleration of the block at the instant shown.



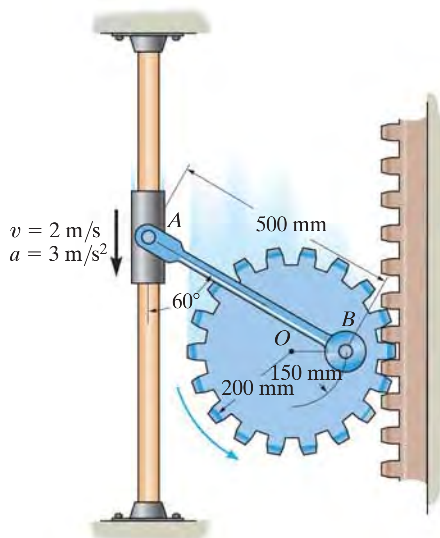
Prob. 16–118

16–119. The wheel rolls without slipping such that at the instant shown it has an angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Determine the velocity and acceleration of point B on the rod at this instant.



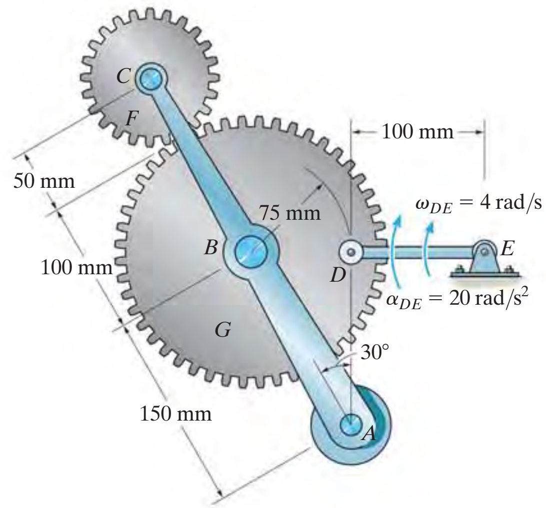
Prob. 16–119

***16–120.** The collar is moving downward with the motion shown. Determine the angular velocity and angular acceleration of the gear at the instant shown as it rolls along the fixed gear rack.



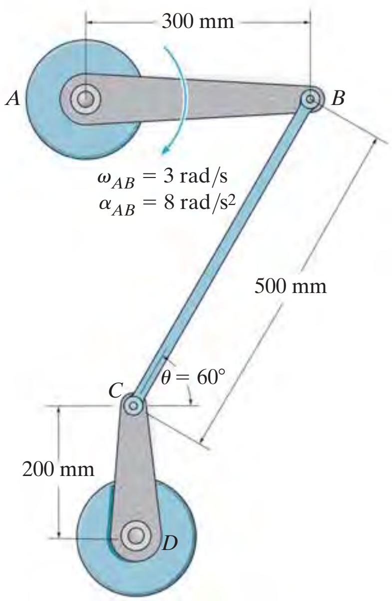
Prob. 16–120

16–121. The tied crank and gear mechanism gives rocking motion to crank AC, necessary for the operation of a printing press. If link DE has the angular motion shown, determine the respective angular velocities of gear F and crank AC at this instant, and the angular acceleration of crank AC.



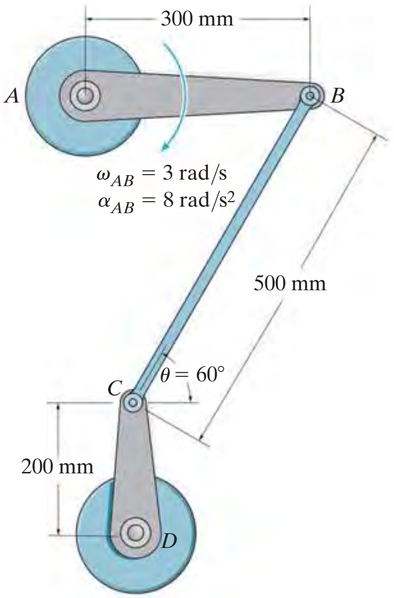
Prob. 16–121

16–122. If member AB has the angular motion shown, determine the angular velocity and angular acceleration of member CD at the instant shown.



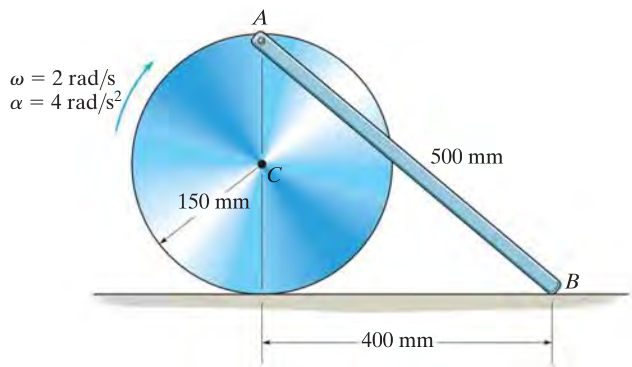
Prob. 16–122

16–123. If member AB has the angular motion shown, determine the velocity and acceleration of point C at the instant shown.



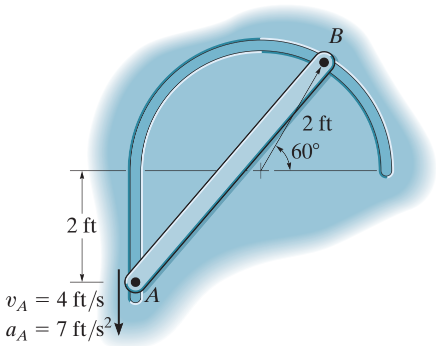
Prob. 16–123

***16–124.** The disk rolls without slipping such that it has an angular acceleration of $\alpha = 4$ rad/s^2 and angular velocity of $\omega = 2$ rad/s at the instant shown. Determine the acceleration of points A and B on the link and the link's angular acceleration at this instant. Assume point A lies on the periphery of the disk, 150 mm from C.



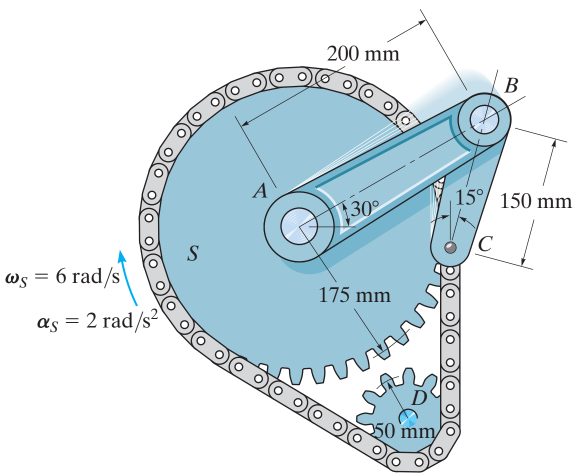
Prob. 16–124

16–125. The ends of the bar AB are confined to move along the paths shown. At a given instant, A has a velocity of $v_A = 4\text{ ft/s}$ and an acceleration of $a_A = 7\text{ ft/s}^2$. Determine the angular velocity and angular acceleration of AB at this instant.



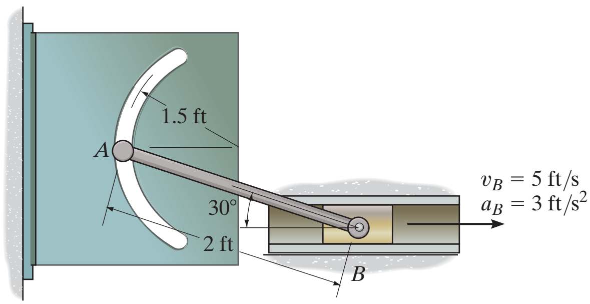
Prob. 16–125

16–126. The mechanism produces intermittent motion of link AB. If the sprocket S is turning with an angular acceleration $\alpha_S = 2\text{ rad/s}^2$ and has an angular velocity $\omega_S = 6\text{ rad/s}$ at the instant shown, determine the angular velocity and angular acceleration of link AB at this instant. The sprocket S is mounted on a shaft which is *separate* from a collinear shaft attached to AB at A. The pin at C is attached to one of the chain links such that it moves vertically downward.



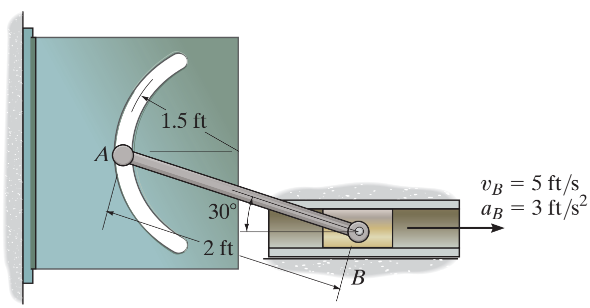
Prob. 16–126

16–127. The slider block moves with a velocity of $v_B = 5\text{ ft/s}$ and an acceleration of $a_B = 3\text{ ft/s}^2$. Determine the angular acceleration of rod AB at the instant shown.



Prob. 16–127

***16–128.** The slider block moves with a velocity of $v_B = 5\text{ ft/s}$ and an acceleration of $a_B = 3\text{ ft/s}^2$. Determine the acceleration of A at the instant shown.



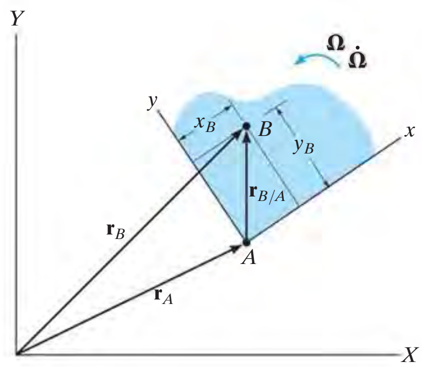
Prob. 16–128

16.8 Relative-Motion Analysis using Rotating Axes

In the previous sections the relative-motion analysis for velocity and acceleration was described using a translating coordinate system. This type of analysis is useful for determining the motion of points on the *same* rigid body, or the motion of points located on several pin-connected bodies. In some problems, however, rigid bodies (mechanisms) are constructed such that *sliding* will occur at their connections. The kinematic analysis for such cases is best performed if the motion is analyzed using a coordinate system which both *translates* and *rotates*. Furthermore, this frame of reference is useful for analyzing the motions of two points on a mechanism which are *not* located in the *same* body and for specifying the kinematics of particle motion when the particle moves along a rotating path.

In the following analysis two equations will be developed which relate the velocity and acceleration of two points, one of which is the origin of a moving frame of reference subjected to both a translation and a rotation in the plane.*

Position. Consider the two points A and B shown in Fig. 16–32*a*. Their location is specified by the position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, which are measured with respect to the fixed X, Y, Z coordinate system. As shown in the figure, the "base point" A represents the origin of the x, y, z coordinate system, which is assumed to be both translating and rotating with respect to the X, Y, Z system. The position of B with respect to A is specified by the relative-position vector $\mathbf{r}_{B/A}$. The components of this vector may be expressed either in terms of unit vectors along the X, Y axes, i.e., $\mathbf{I}$ and $\mathbf{J}$, or by unit vectors along the x, y axes, i.e., $\mathbf{i}$ and $\mathbf{j}$. For the development which follows, $\mathbf{r}_{B/A}$ will be measured with respect to the moving x, y frame of reference. Thus, if B has coordinates (x_B, y_B), Fig. 16–32*a*, then



(a)

Fig. 16–32

$$\mathbf{r}_{B/A} = x_B\mathbf{i} + y_B\mathbf{j}$$

Using vector addition, the three position vectors in Fig. 16–32*a* are related by the equation

$$\boxed{\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}} \qquad (16\text{–}21)$$

At the instant considered, point A has a velocity $\mathbf{v}_A$ and an acceleration $\mathbf{a}_A$, while the angular velocity and angular acceleration of the x, y axes are $\mathbf{\Omega}$ (omega) and $\dot{\mathbf{\Omega}} = d\mathbf{\Omega}/dt$, respectively.

*The more general, three-dimensional motion of the points is developed in Sec. 20.4.

Velocity. The velocity of point B is determined by taking the time derivative of Eq. 16–21, which yields

$$\mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt} \tag{16–22}$$

The last term in this equation is evaluated as follows:

$$\begin{aligned}
\frac{d\mathbf{r}_{B/A}}{dt} &= \frac{d}{dt}(x_B\mathbf{i} + y_B\mathbf{j}) \\
&= \frac{dx_B}{dt}\mathbf{i} + x_B\frac{d\mathbf{i}}{dt} + \frac{dy_B}{dt}\mathbf{j} + y_B\frac{d\mathbf{j}}{dt} \\
&= \left(\frac{dx_B}{dt}\mathbf{i} + \frac{dy_B}{dt}\mathbf{j}\right) + \left(x_B\frac{d\mathbf{i}}{dt} + y_B\frac{d\mathbf{j}}{dt}\right)
\end{aligned} \tag{16–23}$$

The two terms in the first set of parentheses represent the components of velocity of point B as measured by an observer attached to the moving x, y, z coordinate system. These terms will be denoted by vector $(\mathbf{v}_{B/A})_{xyz}$. In the second set of parentheses the instantaneous time rate of change of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ is measured by an observer located in the fixed X, Y, Z coordinate system. These changes, $d\mathbf{i}$ and $d\mathbf{j}$, are due *only* to the *rotation* $d\theta$ of the x, y, z axes, causing $\mathbf{i}$ to become $\mathbf{i}' = \mathbf{i} + d\mathbf{i}$ and $\mathbf{j}$ to become $\mathbf{j}' = \mathbf{j} + d\mathbf{j}$, Fig. 16–32*b*. As shown, the *magnitudes* of both $d\mathbf{i}$ and $d\mathbf{j}$ equal 1 $d\theta$, since $i = i' = j = j' = 1$. The *direction* of $d\mathbf{i}$ is defined by $+\mathbf{j}$, since $d\mathbf{i}$ is tangent to the path described by the arrowhead of $\mathbf{i}$ in the limit as $\Delta t \rightarrow dt$. Likewise, $d\mathbf{j}$ acts in the $-\mathbf{i}$ direction, Fig. 16–32*b*. Hence,

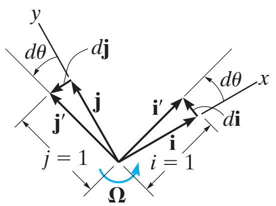
$$\frac{d\mathbf{i}}{dt} = \frac{d\theta}{dt}(\mathbf{j}) = \Omega\mathbf{j} \qquad \frac{d\mathbf{j}}{dt} = \frac{d\theta}{dt}(-\mathbf{i}) = -\Omega\mathbf{i}$$

Viewing the axes in three dimensions, Fig. 16–32*c*, and noting that $\boldsymbol{\Omega} = \Omega\mathbf{k}$, we can express the above derivatives in terms of the cross product as

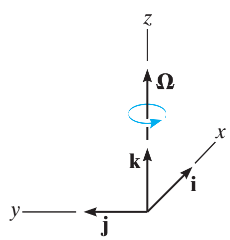
$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\Omega} \times \mathbf{i} \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\Omega} \times \mathbf{j} \tag{16–24}$$

Substituting these results into Eq. 16–23 and using the distributive property of the vector cross product, we obtain

$$\frac{d\mathbf{r}_{B/A}}{dt} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (x_B\mathbf{i} + y_B\mathbf{j}) = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} \tag{16–25}$$



(b)



(c)

Fig. 16–32 (cont.)

Hence, Eq. 16–22 becomes

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz} \tag{16–26}$$

where

$\mathbf{v}_B$ = velocity of B, measured from the X, Y, Z reference

$\mathbf{v}_A$ = velocity of the origin A of the x, y, z reference, measured from the X, Y, Z reference

$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A," as measured by an observer attached to the rotating x, y, z reference

$\mathbf{\Omega}$ = angular velocity of the x, y, z reference, measured from the X, Y, Z reference

$\mathbf{r}_{B/A}$ = position of B with respect to A

Comparing Eq. 16–26 with Eq. 16–16 ($\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{B/A}$), which is valid for a translating frame of reference, it can be seen that the only difference between these two equations is represented by the term $(\mathbf{v}_{B/A})_{xyz}$.

When applying Eq. 16–26 it is often useful to understand what each of the terms represents. In order of appearance, they are as follows:

$\mathbf{v}_B$	absolute velocity of B	motion of B observed from the X, Y, Z frame
	(equals)	
$\mathbf{v}_A$	absolute velocity of the origin of x, y, z frame	motion of x, y, z frame observed from the X, Y, Z frame
	(plus)	
$\mathbf{\Omega} \times \mathbf{r}_{B/A}$	angular velocity effect caused by rotation of x, y, z frame	
	(plus)	
$(\mathbf{v}_{B/A})_{xyz}$	velocity of B with respect to A	motion of B observed from the x, y, z frame

Acceleration. The acceleration of B, observed from the X, Y, Z coordinate system, may be expressed in terms of its motion measured with respect to the rotating system of coordinates by taking the time derivative of Eq. 16–26.

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} \qquad (16\text{–}27)$$

Here $\dot{\mathbf{\Omega}} = d\mathbf{\Omega}/dt$ is the angular acceleration of the x, y, z coordinate system. Since $\mathbf{\Omega}$ is always perpendicular to the plane of motion, then $\dot{\mathbf{\Omega}}$ measures *only the change in magnitude* of $\mathbf{\Omega}$. The derivative $d\mathbf{r}_{B/A}/dt$ is defined by Eq. 16–25, so that

$$\mathbf{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} = \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) \qquad (16\text{–}28)$$

Finding the time derivative of $(\mathbf{v}_{B/A})_{xyz} = (v_{B/A})_x\mathbf{i} + (v_{B/A})_y\mathbf{j}$,

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = \left[\frac{d(v_{B/A})_x}{dt}\mathbf{i} + \frac{d(v_{B/A})_y}{dt}\mathbf{j}\right] + \left[(v_{B/A})_x\frac{d\mathbf{i}}{dt} + (v_{B/A})_y\frac{d\mathbf{j}}{dt}\right]$$

The two terms in the first set of brackets represent the components of acceleration of point B as measured by an observer attached to the rotating coordinate system. These terms will be denoted by $(\mathbf{a}_{B/A})_{xyz}$. The terms in the second set of brackets can be simplified using Eqs. 16–24.

$$\frac{d(\mathbf{v}_{B/A})_{xyz}}{dt} = (\mathbf{a}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

Substituting this and Eq. 16–28 into Eq. 16–27 and rearranging terms,

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \qquad (16\text{–}29)$$

where

$\mathbf{a}_B$ = acceleration of B, measured from the X, Y, Z reference

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z reference, measured from the X, Y, Z reference

$(\mathbf{a}_{B/A})_{xyz}, (\mathbf{v}_{B/A})_{xyz}$ = acceleration and velocity of B with respect to A, as measured by an observer attached to the rotating x, y, z reference

$\dot{\mathbf{\Omega}}, \mathbf{\Omega}$ = angular acceleration and angular velocity of the x, y, z reference, measured from the X, Y, Z reference

$\mathbf{r}_{B/A}$ = position of B with respect to A

If Eq. 16–29 is compared with Eq. 16–18, written in the form $\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A})$, which is valid for a translating frame of reference, it can be seen that the difference between these two equations is represented by the terms $2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ and $(\mathbf{a}_{B/A})_{xyz}$. In particular, $2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ is called the *Coriolis acceleration*, named after the French engineer G. C. Coriolis, who was the first to determine it. This term represents the difference in the acceleration of B as measured from nonrotating and rotating x, y, z axes. As indicated by the vector cross product, the Coriolis acceleration will *always* be perpendicular to both $\mathbf{\Omega}$ and $(\mathbf{v}_{B/A})_{xyz}$. It is an important component of the acceleration which must be considered whenever rotating reference frames are used. This often occurs, for example, when studying the accelerations and forces which act on rockets, long-range projectiles, or other bodies having motions whose measurements are significantly affected by the rotation of the earth.

The following interpretation of the terms in Eq. 16–29 may be useful when applying this equation to the solution of problems.

Term	Interpretation	
$\mathbf{a}_B$	absolute acceleration of B	motion of B observed from the X, Y, Z frame
	(equals)	
$\mathbf{a}_A$	absolute acceleration of the origin of x, y, z frame	motion of x, y, z frame observed from the X, Y, Z frame
	(plus)	
$\dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A}$	angular acceleration effect caused by rotation of x, y, z frame	
	(plus)	
$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A})$	angular velocity effect caused by rotation of x, y, z frame	
	(plus)	
$2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$	combined effect of B moving relative to x, y, z coordinates and rotation of x, y, z frame	interacting motion
	(plus)	
$(\mathbf{a}_{B/A})_{xyz}$	acceleration of B with respect to A	motion of B observed from the x, y, z frame

Procedure for Analysis

Equations 16–26 and 16–29 can be applied to the solution of problems involving the planar motion of particles or rigid bodies using the following procedure.

Coordinate Axes.

- Choose an appropriate location for the origin and proper orientation of the axes for both fixed X, Y, Z and moving x, y, z reference frames.

- Most often solutions are easily obtained if at the instant considered:
  1. the origins are coincident
  2. the corresponding axes are collinear
  3. the corresponding axes are parallel

- The moving frame should be selected fixed to the body or device along which the relative motion occurs.

Kinematic Equations.

- After defining the origin A of the moving reference and specifying the moving point B, Eqs. 16–26 and 16–29 should be written in symbolic form

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$
$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- The Cartesian components of all these vectors may be expressed along either the X, Y, Z axes or the x, y, z axes. The choice is arbitrary provided a consistent set of unit vectors is used.

- Motion of the moving reference is expressed by $\mathbf{v}_A$, $\mathbf{a}_A$, $\boldsymbol{\Omega}$, and $\dot{\boldsymbol{\Omega}}$; and motion of B with respect to the moving reference is expressed by $\mathbf{r}_{B/A}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$.

The rotation of the dumping bin of the truck about point C is operated by the extension of the hydraulic cylinder AB. To determine the rotation of the bin due to this extension, we can use the equations of relative motion and fix the x, y axes to the cylinder so that the relative motion of the cylinder's extension occurs along the y axis. (© R.C. Hibbeler)

EXAMPLE 16.18

At the instant $\theta = 60°$, the rod in Fig. 16–33 has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2. At this same instant, collar C travels outward along the rod such that when $x = 0.2$ m the velocity is 2 m/s and the acceleration is 3 m/s^2, both measured relative to the rod. Determine the Coriolis acceleration and the velocity and acceleration of the collar at this instant.

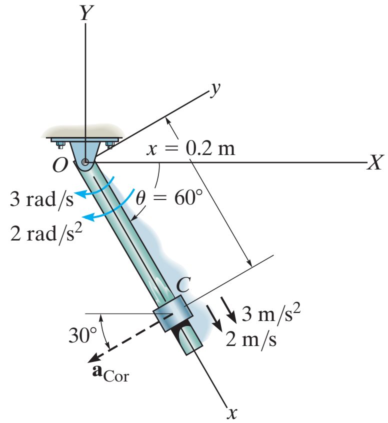


Fig. 16–33

SOLUTION

Coordinate Axes. The origin of both coordinate systems is located at point O, Fig. 16–33. Since motion of the collar is reported relative to the rod, the moving x, y, z frame of reference is *attached* to the rod.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_O + \mathbf{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz} \tag{1}$$

$$\mathbf{a}_C = \mathbf{a}_O + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/O} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/O}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz} \tag{2}$$

It will be simpler to express the data in terms of **i**, **j**, **k** component vectors rather than **I**, **J**, **K** components. Hence,

Motion of moving reference	Motion of C with respect to moving reference
$\mathbf{v}_O = \mathbf{0}$	$\mathbf{r}_{C/O} = \{0.2\mathbf{i}\}$ m
$\mathbf{a}_O = \mathbf{0}$	$(\mathbf{v}_{C/O})_{xyz} = \{2\mathbf{i}\}$ m/s
$\mathbf{\Omega} = \{-3\mathbf{k}\}$ rad/s	$(\mathbf{a}_{C/O})_{xyz} = \{3\mathbf{i}\}$ m/s^2
$\dot{\mathbf{\Omega}} = \{-2\mathbf{k}\}$ rad/s^2	

The Coriolis acceleration is defined as

$$\mathbf{a}_{Cor} = 2\mathbf{\Omega} \times (\mathbf{v}_{C/O})_{xyz} = 2(-3\mathbf{k}) \times (2\mathbf{i}) = \{-12\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

This vector is shown dashed in Fig. 16–33. If desired, it may be resolved into **I**, **J** components acting along the X and Y axes, respectively.

The velocity and acceleration of the collar are determined by substituting the data into Eqs. 1 and 2 and evaluating the cross products, which yields

$$\mathbf{v}_C = \mathbf{v}_O + \mathbf{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz}$$
$$= \mathbf{0} + (-3\mathbf{k}) \times (0.2\mathbf{i}) + 2\mathbf{i}$$
$$= \{2\mathbf{i} - 0.6\mathbf{j}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_O + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/O} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/O}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz}$$
$$= \mathbf{0} + (-2\mathbf{k}) \times (0.2\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.2\mathbf{i})] + 2(-3\mathbf{k}) \times (2\mathbf{i}) + 3\mathbf{i}$$
$$= \mathbf{0} - 0.4\mathbf{j} - 1.80\mathbf{i} - 12\mathbf{j} + 3\mathbf{i}$$
$$= \{1.20\mathbf{i} - 12.4\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

EXAMPLE 16.19

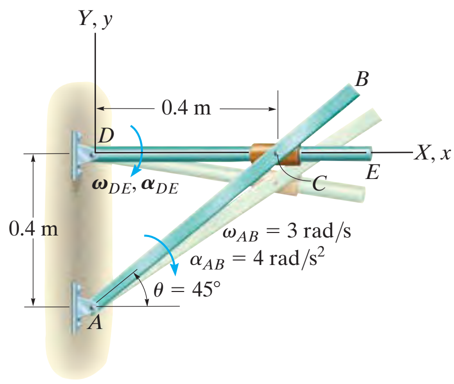


Fig. 16–34

Rod AB, shown in Fig. 16–34, rotates clockwise such that it has an angular velocity $\omega_{AB} = 3$ rad/s and angular acceleration $\alpha_{AB} = 4$ rad/s^2 when $\theta = 45°$. Determine the angular motion of rod DE at this instant. The collar at C is pin connected to AB and slides over rod DE.

SOLUTION

Coordinate Axes. The origin of both the fixed and moving frames of reference is located at D, Fig. 16–34. Furthermore, the x, y, z reference is attached to and rotates with rod DE so that the relative motion of the collar is easy to follow.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz} \tag{1}$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz} \tag{2}$$

All vectors will be expressed in terms of **i**, **j**, **k** components.

Motion of moving reference	Motion of C with respect to moving reference
$\mathbf{v}_D = \mathbf{0}$	$\mathbf{r}_{C/D} = \{0.4\mathbf{i}\}$ m
$\mathbf{a}_D = \mathbf{0}$	$(\mathbf{v}_{C/D})_{xyz} = (v_{C/D})_{xyz}\mathbf{i}$
$\boldsymbol{\Omega} = -\omega_{DE}\mathbf{k}$	$(\mathbf{a}_{C/D})_{xyz} = (a_{C/D})_{xyz}\mathbf{i}$
$\dot{\boldsymbol{\Omega}} = -\alpha_{DE}\mathbf{k}$	

Motion of C: Since the collar moves along a *circular path* of radius AC, its velocity and acceleration can be determined using Eqs. 16–9 and 16–14.

$$\mathbf{v}_C = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{C/A} = (-3\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) = \{1.2\mathbf{i} - 1.2\mathbf{j}\} \text{ m/s}$$

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{C/A} - \omega_{AB}^2 \mathbf{r}_{C/A}$$
$$= (-4\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) - (3)^2(0.4\mathbf{i} + 0.4\mathbf{j}) = \{-2\mathbf{i} - 5.2\mathbf{j}\} \text{ m/s}^2$$

Substituting the data into Eqs. 1 and 2, we have

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz}$$
$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} + (-\omega_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (v_{C/D})_{xyz}\mathbf{i}$$
$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} - 0.4\omega_{DE}\mathbf{j} + (v_{C/D})_{xyz}\mathbf{i}$$
$$(v_{C/D})_{xyz} = 1.2 \text{ m/s}$$
$$\omega_{DE} = 3 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz}$$
$$-2\mathbf{i} - 5.2\mathbf{j} = \mathbf{0} + (-\alpha_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.4\mathbf{i})]$$
$$+ 2(-3\mathbf{k}) \times (1.2\mathbf{i}) + (a_{C/D})_{xyz}\mathbf{i}$$
$$-2\mathbf{i} - 5.2\mathbf{j} = -0.4\alpha_{DE}\mathbf{j} - 3.6\mathbf{i} - 7.2\mathbf{j} + (a_{C/D})_{xyz}\mathbf{i}$$
$$(a_{C/D})_{xyz} = 1.6 \text{ m/s}^2$$
$$\alpha_{DE} = -5 \text{ rad/s}^2 = 5 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

EXAMPLE 16.20

Planes A and B fly at the same elevation and have the motions shown in Fig. 16–35. Determine the velocity and acceleration of A as measured by the pilot of B.

SOLUTION

Coordinate Axes. Since the relative motion of A with respect to the pilot in B is being sought, the x, y, z axes are attached to plane B, Fig. 16–35. At the *instant* considered, the origin B coincides with the origin of the fixed X, Y, Z frame.

Kinematic Equations.

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + (\mathbf{v}_{A/B})_{xyz} \qquad (1)$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{xyz} + (\mathbf{a}_{A/B})_{xyz} \qquad (2)$$

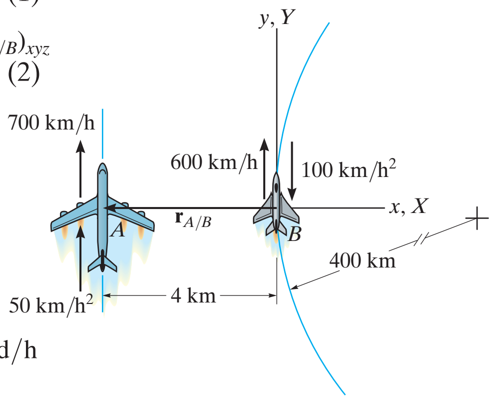


Fig. 16–35

Motion of Moving Reference:

$$\mathbf{v}_B = \{600\mathbf{j}\} \text{ km/h}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(600)^2}{400} = 900 \text{ km/h}^2$$

$$\mathbf{a}_B = (\mathbf{a}_B)_n + (\mathbf{a}_B)_t = \{900\mathbf{i} - 100\mathbf{j}\} \text{ km/h}^2$$

$$\boldsymbol{\Omega} = \frac{v_B}{\rho} = \frac{600 \text{ km/h}}{400 \text{ km}} = 1.5 \text{ rad/h} \circlearrowright \qquad \boldsymbol{\Omega} = \{-1.5\mathbf{k}\} \text{ rad/h}$$

$$\dot{\boldsymbol{\Omega}} = \frac{(a_B)_t}{\rho} = \frac{100 \text{ km/h}^2}{400 \text{ km}} = 0.25 \text{ rad/h}^2 \circlearrowleft \qquad \dot{\boldsymbol{\Omega}} = \{0.25\mathbf{k}\} \text{ rad/h}^2$$

Motion of A with Respect to Moving Reference:

$$\mathbf{r}_{A/B} = \{-4\mathbf{i}\} \text{ km} \quad (\mathbf{v}_{A/B})_{xyz} = ? \quad (\mathbf{a}_{A/B})_{xyz} = ?$$

Substituting the data into Eqs. 1 and 2, realizing that $\mathbf{v}_A = \{700\mathbf{j}\}$ km/h and $\mathbf{a}_A = \{50\mathbf{j}\}$ km/h^2, we have

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + (\mathbf{v}_{A/B})_{xyz}$$

$$700\mathbf{j} = 600\mathbf{j} + (-1.5\mathbf{k}) \times (-4\mathbf{i}) + (\mathbf{v}_{A/B})_{xyz}$$

$$(\mathbf{v}_{A/B})_{xyz} = \{94\mathbf{j}\} \text{ km/h} \qquad \textit{Ans.}$$

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{xyz} + (\mathbf{a}_{A/B})_{xyz}$$

$$50\mathbf{j} = (900\mathbf{i} - 100\mathbf{j}) + (0.25\mathbf{k}) \times (-4\mathbf{i})$$

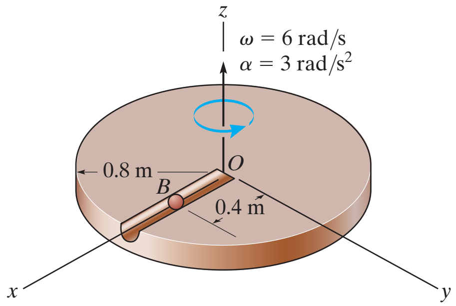
$$+ (-1.5\mathbf{k}) \times [(-1.5\mathbf{k}) \times (-4\mathbf{i})] + 2(-1.5\mathbf{k}) \times (94\mathbf{j}) + (\mathbf{a}_{A/B})_{xyz}$$

$$(\mathbf{a}_{A/B})_{xyz} = \{-1191\mathbf{i} + 151\mathbf{j}\} \text{ km/h}^2 \qquad \textit{Ans.}$$

NOTE: The solution of this problem should be compared with that of Example 12.26, where it is seen that $(v_{B/A})_{xyz} \neq (v_{A/B})_{xyz}$ and $(a_{B/A})_{xyz} \neq (a_{A/B})_{xyz}$.

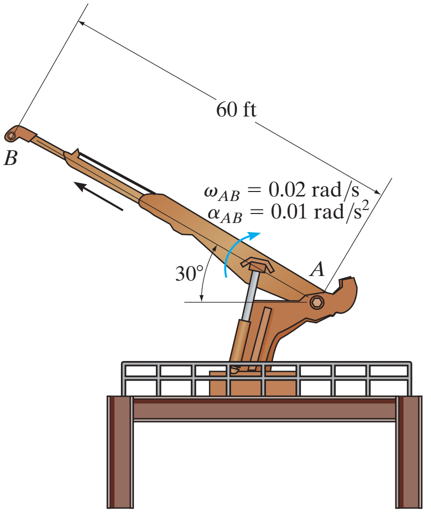
PROBLEMS

16–129. At the instant shown, ball B is rolling along the slot in the disk with a velocity of 600 mm/s and an acceleration of 150 mm/s^2, both measured relative to the disk and directed away from O. If at the same instant the disk has the angular velocity and angular acceleration shown, determine the velocity and acceleration of the ball at this instant.



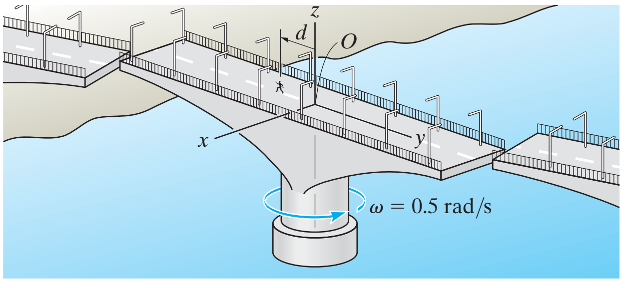
Prob. 16–129

16–130. The crane's telescopic boom rotates with the angular velocity and angular acceleration shown. At the same instant, the boom is extending with a constant speed of 0.5 ft/s, measured relative to the boom. Determine the magnitudes of the velocity and acceleration of point B at this instant.



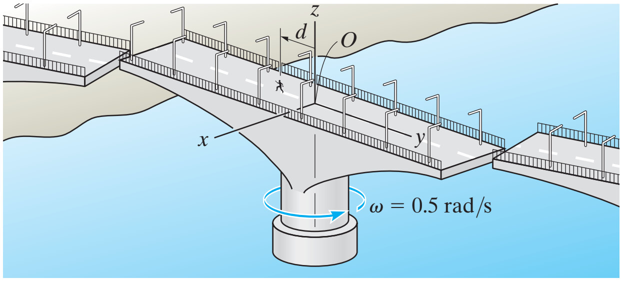
Prob. 16–130

16–131. While the swing bridge is closing with a constant rotation of 0.5 rad/s, a man runs along the roadway at a constant speed of 5 ft/s relative to the roadway. Determine his velocity and acceleration at the instant $d = 15$ ft.



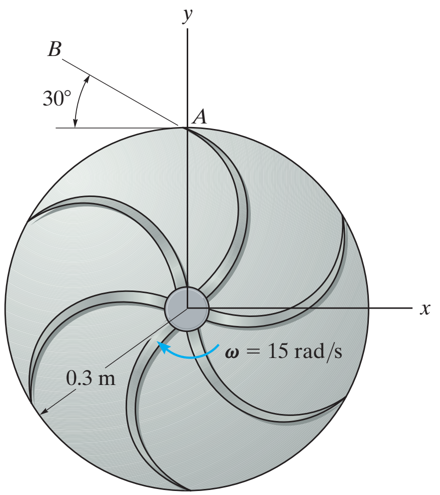
Prob. 16–131

***16–132.** While the swing bridge is closing with a constant rotation of 0.5 rad/s, a man runs along the roadway such that when $d = 10$ ft he is running outward from the center at 5 ft/s with an acceleration of 2 ft/s^2, both measured relative to the roadway. Determine his velocity and acceleration at this instant.



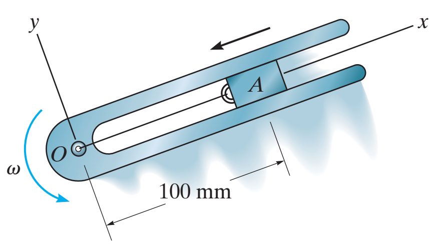
Prob. 16–132

16–133. Water leaves the impeller of the centrifugal pump with a velocity of 25 m/s and acceleration of 30 m/s^2, both measured relative to the impeller along the blade line AB. Determine the velocity and acceleration of a water particle at A as it leaves the impeller at the instant shown. The impeller rotates with a constant angular velocity of $\omega = 15$ rad/s.



Prob. 16–133

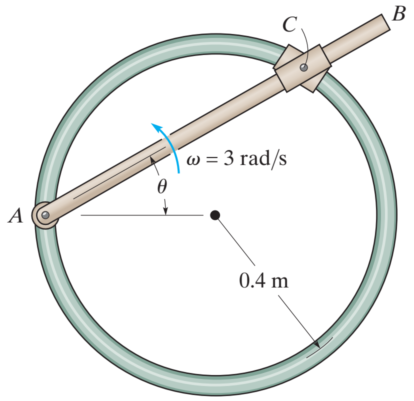
16–134. Block A, which is attached to a cord, moves along the slot of a horizontal forked rod. At the instant shown, the cord is pulled down through the hole at O with an acceleration of 4 m/s^2 and its velocity is 2 m/s. Determine the acceleration of the block at this instant. The rod rotates about O with a constant angular velocity $\omega = 4$ rad/s.



Prob. 16–134

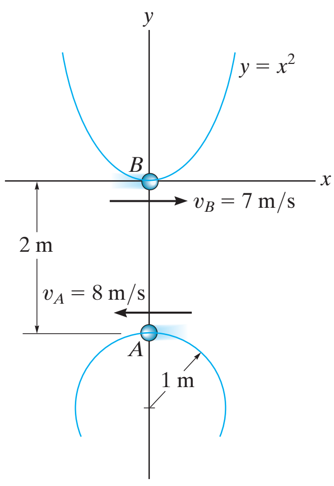
16–135. Rod AB rotates counterclockwise with a constant angular velocity $\omega = 3$ rad/s. Determine the velocity of point C located on the double collar when $\theta = 30°$. The collar consists of two pin-connected slider blocks which are constrained to move along the circular path and the rod AB.

***16–136.** Rod AB rotates counterclockwise with a constant angular velocity $\omega = 3$ rad/s. Determine the velocity and acceleration of point C located on the double collar when $\theta = 45°$. The collar consists of two pin-connected slider blocks which are constrained to move along the circular path and the rod AB.



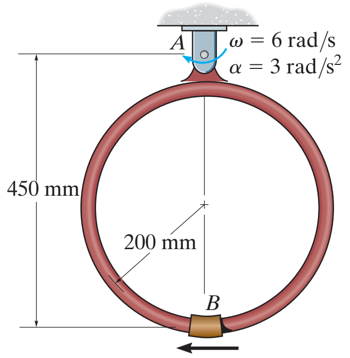
Probs. 16–135/136

16–137. Particles B and A move along the parabolic and circular paths, respectively. If B has a velocity of 7 m/s in the direction shown and its speed is increasing at 4 m/s^2, while A has a velocity of 8 m/s in the direction shown and its speed is decreasing at 6 m/s^2, determine the relative velocity and relative acceleration of B with respect to A.



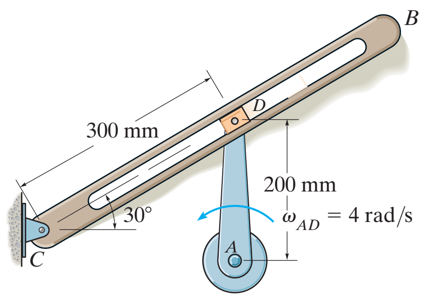
Prob. 16–137

16–138. Collar B moves to the left with a speed of 5 m/s, which is increasing at a constant rate of 1.5 m/s^2, relative to the hoop, while the hoop rotates with the angular velocity and angular acceleration shown. Determine the magnitudes of the velocity and acceleration of the collar at this instant.



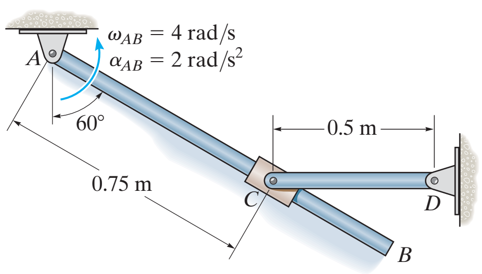
Prob. 16–138

16–139. Block D of the mechanism is confined to move within the slot of member CB. If link AD is rotating at a constant rate of $\omega_{AD} = 4$ rad/s, determine the angular velocity and angular acceleration of member CB at the instant shown.



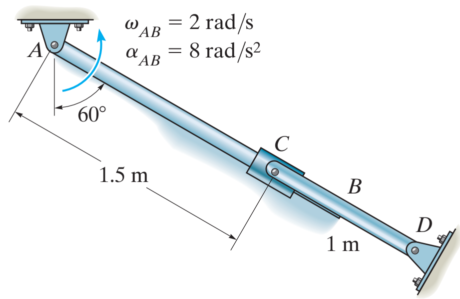
Prob. 16–139

***16–140.** At the instant shown rod AB has an angular velocity $\omega_{AB} = 4$ rad/s and an angular acceleration $\alpha_{AB} = 2$ rad/s^2. Determine the angular velocity and angular acceleration of rod CD at this instant. The collar at C is pin connected to CD and slides freely along AB.



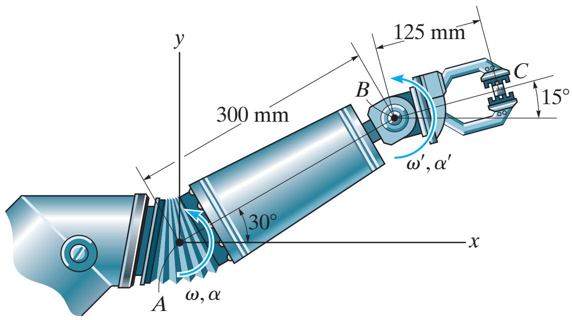
Prob. 16–140

16–141. The collar C is pinned to rod CD while it slides on rod AB. If rod AB has an angular velocity of 2 rad/s and an angular acceleration of 8 rad/s^2, both acting counterclockwise, determine the angular velocity and the angular acceleration of rod CD at the instant shown.



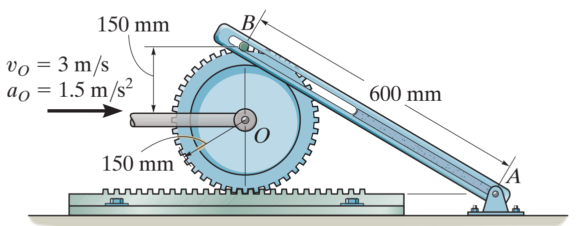
Prob. 16–141

16–142. At the instant shown, the robotic arm AB is rotating counterclockwise at $\omega = 5$ rad/s and has an angular acceleration $\alpha = 2$ rad/s^2. Simultaneously, the grip BC is rotating counterclockwise at $\omega' = 6$ rad/s and $\alpha' = 2$ rad/s^2, both measured relative to a *fixed* reference. Determine the velocity and acceleration of the object held at the grip C.



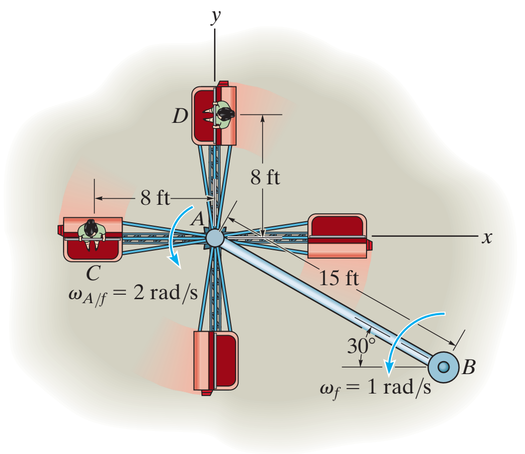
Prob. 16–142

16–143. Peg B on the gear slides freely along the slot in link AB. If the gear's center O moves with the velocity and acceleration shown, determine the angular velocity and angular acceleration of the link at this instant.



Prob. 16–143

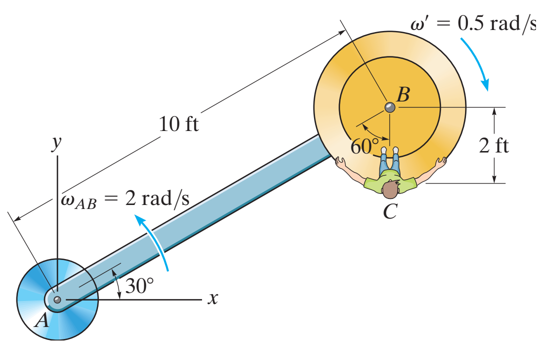
***16–144.** The cars on the amusement-park ride rotate around the axle at A with a constant angular velocity $\omega_{A/f} = 2$ rad/s, measured relative to the frame AB. At the same time the frame rotates around the main axle support at B with a constant angular velocity $\omega_f = 1$ rad/s. Determine the velocity and acceleration of the passenger at C at the instant shown.



Prob. 16–144

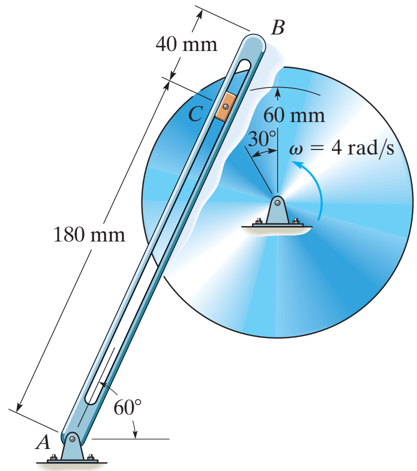
16–145. A ride in an amusement park consists of a rotating arm AB having a constant angular velocity $\omega_{AB} = 2$ rad/s point A and a car mounted at the end of the arm which has a constant angular velocity $\boldsymbol{\omega}' = \{-0.5\mathbf{k}\}$ rad/s, measured relative to the arm. At the instant shown, determine the velocity and acceleration of the passenger at C.

16–146. A ride in an amusement park consists of a rotating arm AB that has an angular acceleration of $\alpha_{AB} = 1$ rad/s^2 when $\omega_{AB} = 2$ rad/s at the instant shown. Also at this instant the car mounted at the end of the arm has an angular acceleration of $\boldsymbol{\alpha} = \{-0.6\mathbf{k}\}$ rad/s^2 and angular velocity of $\boldsymbol{\omega}' = \{-0.5\mathbf{k}\}$ rad/s, measured relative to the arm. Determine the velocity and acceleration of the passenger C at this instant.



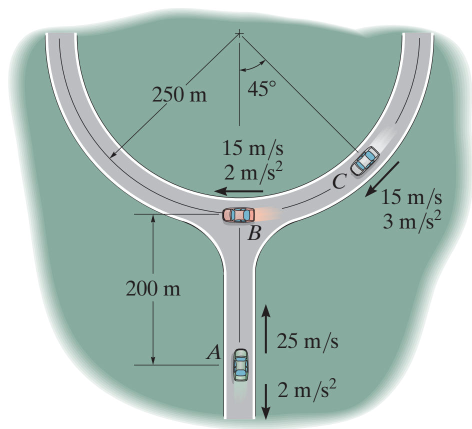
Probs. 16–145/146

16–147. If the slider block C is fixed to the disk that has a constant counterclockwise angular velocity of 4 rad/s, determine the angular velocity and angular acceleration of the slotted arm AB at the instant shown.



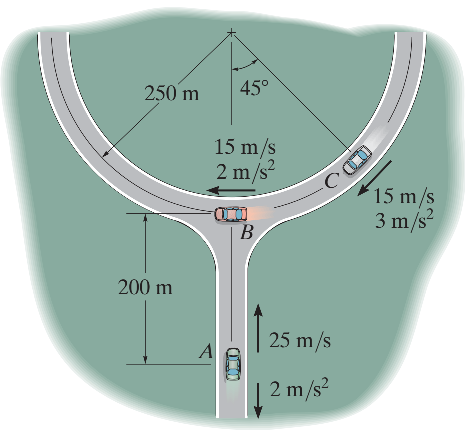
Prob. 16–147

***16–148.** At the instant shown, car A travels with a speed of 25 m/s, which is decreasing at a constant rate of 2 m/s^2, while car C travels with a speed of 15 m/s, which is increasing at a constant rate of 3 m/s. Determine the velocity and acceleration of car A with respect to car C.



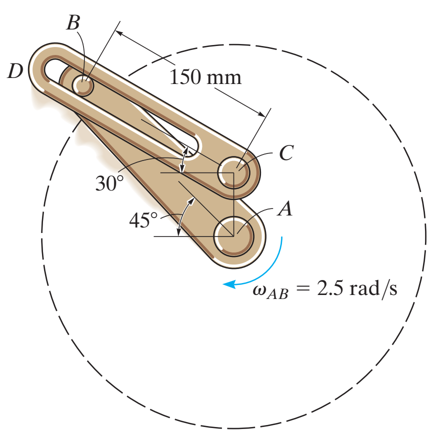
Prob. 16–148

16–149. At the instant shown, car B travels with a speed of 15 m/s, which is increasing at a constant rate of 2 m/s^2, while car C travels with a speed of 15 m/s, which is increasing at a constant rate of 3 m/s^2. Determine the velocity and acceleration of car B with respect to car C.



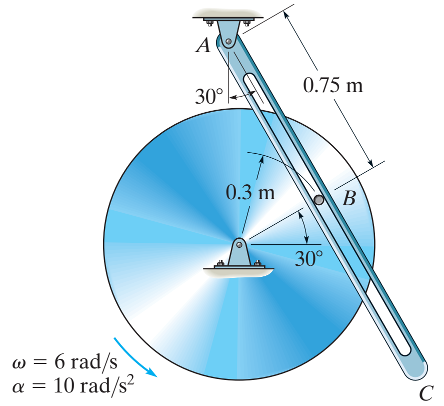
Prob. 16–149

16–150. The two-link mechanism serves to amplify angular motion. Link AB has a pin at B which is confined to move within the slot of link CD. If at the instant shown, AB (input) has an angular velocity of $\omega_{AB} = 2.5$ rad/s, determine the angular velocity of CD (output) at this instant.



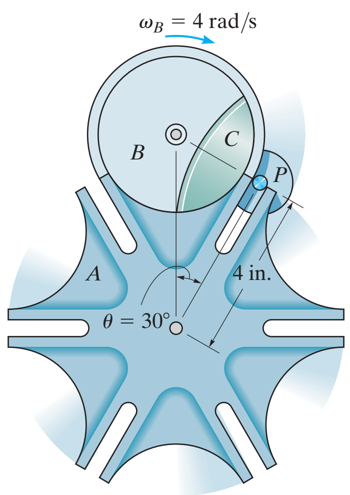
Prob. 16–150

16–151. The disk rotates with the angular motion shown. Determine the angular velocity and angular acceleration of the slotted link AC at this instant. The peg at B is fixed to the disk.



Prob. 16–151

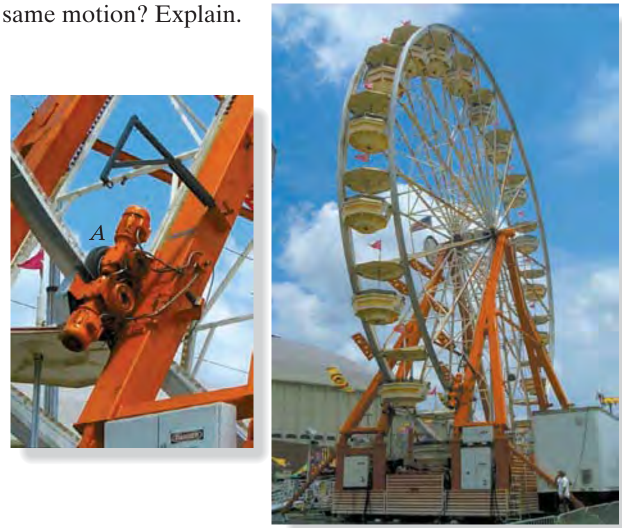
***16–152.** The Geneva mechanism is used in a packaging system to convert constant angular motion into intermittent angular motion. The star wheel A makes one sixth of a revolution for each full revolution of the driving wheel B and the attached guide C. To do this, pin P, which is attached to B, slides into one of the radial slots of A, thereby turning wheel A, and then exits the slot. If B has a constant angular velocity of $\omega_B = 4$ rad/s, determine $\boldsymbol{\omega}_A$ and $\boldsymbol{\alpha}_A$ of wheel A at the instant shown.



Prob. 16–152

CONCEPTUAL PROBLEMS

C16–1. An electric motor turns the tire at A at a constant angular velocity, and friction then causes the tire to roll without slipping on the inside rim of the Ferris wheel. Using appropriate numerical values, determine the magnitude of the velocity and acceleration of passengers in one of the baskets. Do passengers in the other baskets experience this same motion? Explain.



Prob. C16–1 (© R.C. Hibbeler)

C16–2. The crank AB turns counterclockwise at a constant rate $\boldsymbol{\omega}$ causing the connecting arm CD and rocking beam DE to move. Draw a sketch showing the location of the IC for the connecting arm when $\theta = 0°, 90°, 180°$, and $270°$. Also, how was the curvature of the head at E determined, and why is it curved in this way?

Prob. C16–2 (© R.C. Hibbeler)

C16–3. The bi-fold hangar door is opened by cables that move upward at a constant speed of 0.5 m/s. Determine the angular velocity of BC and the angular velocity of AB when $\theta = 45°$. Panel BC is pinned at C and has a height which is the same as the height of BA. Use appropriate numerical values to explain your result.

Prob. C16–3 (© R.C. Hibbeler)

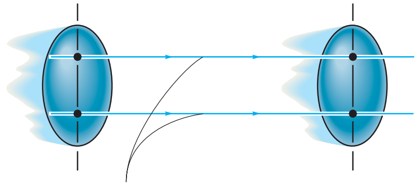
C16–4. If the tires do not slip on the pavement, determine the points on the tire that have a maximum and minimum speed and the points that have a maximum and minimum acceleration. Use appropriate numerical values for the car's speed and tire size to explain your result.

Prob. C16–4 (© R.C. Hibbeler)

CHAPTER REVIEW

Rigid-Body Planar Motion

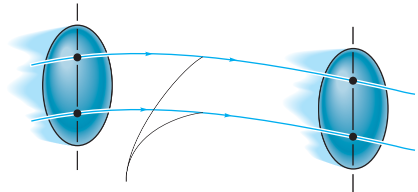
A rigid body undergoes three types of planar motion: translation, rotation about a fixed axis, and general plane motion.



Path of rectilinear translation

Translation

When a body has rectilinear translation, all the particles of the body travel along parallel straight-line paths. If the paths have the same radius of curvature, then curvilinear translation occurs. Provided we know the motion of one of the particles, then the motion of all of the others is also known.



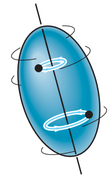
Path of curvilinear translation

Rotation about a Fixed Axis

For this type of motion, all of the particles move along circular paths. Here, all line segments in the body undergo the same angular displacement, angular velocity, and angular acceleration.

Once the angular motion of the body is known, then the velocity of any particle a distance r from the axis can be obtained.

The acceleration of any particle has two components. The tangential component accounts for the change in the magnitude of the velocity, and the normal component accounts for the change in the velocity's direction.



Rotation about a fixed axis

$$\omega = d\theta/dt \qquad \qquad \omega = \omega_0 + \alpha_c t$$

$$\alpha = d\omega/dt \qquad \text{or} \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$

$$\alpha\, d\theta = \omega\, d\omega \qquad \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

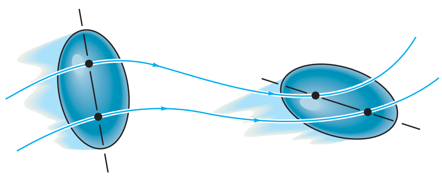
Constant α_c

$$v = \omega r \qquad \qquad a_t = \alpha r, \quad a_n = \omega^2 r$$

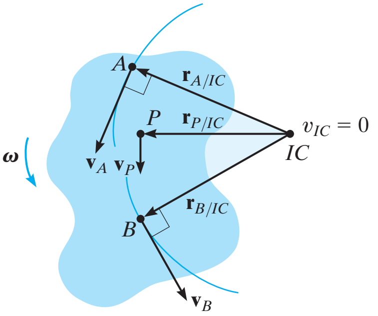
General Plane Motion

When a body undergoes general plane motion, it simultaneously translates and rotates. There are several methods for analyzing this motion.

Absolute Motion Analysis
If the motion of a point on a body or the angular motion of a line is known, then it may be possible to relate this motion to that of another point or line using an absolute motion analysis. To do so, linear position coordinates s or angular position coordinates θ are established (measured from a fixed point or line). These position coordinates are then related using the geometry of the body. The time derivative of this equation gives the relationship between the velocities and/or the angular velocities. A second time derivative relates the accelerations and/or the angular accelerations.

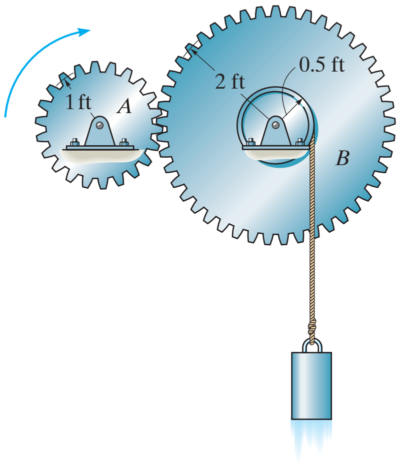


General plane motion

Relative-Motion using Translating Axes General plane motion can also be analyzed using a relative-motion analysis between two points A and B located on the body. This method considers the motion in parts: first a translation of the selected base point A, then a relative "rotation" of the body about point A, which is measured from a translating axis. Since the relative motion is viewed as circular motion about the base point, point B will have a velocity $\mathbf{v}_{B/A}$ that is tangent to the circle. It also has two components of acceleration, $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$. It is also important to realize that $\mathbf{a}_A$ and $\mathbf{a}_B$ will have tangential and normal components if these points move along curved paths.	$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$ $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$
Instantaneous Center of Zero Velocity If the base point A is selected as having zero velocity, then the relative velocity equation becomes $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. In this case, motion appears as if the body rotates about an instantaneous axis passing through A. The instantaneous center of rotation (IC) can be established provided the directions of the velocities of any two points on the body are known, or the velocity of a point and the angular velocity are known. Since a radial line r will always be perpendicular to each velocity, then the IC is at the point of intersection of these two radial lines. Its measured location is determined from the geometry of the body. Once it is established, then the velocity of any point P on the body can be determined from $v = \omega r$, where r extends from the IC to point P.	
Relative Motion using Rotating Axes Problems that involve connected members that slide relative to one another or points not located on the same body can be analyzed using a relative-motion analysis referenced from a rotating frame. This gives rise to the term $2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ that is called the Coriolis acceleration.	$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$ $\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$

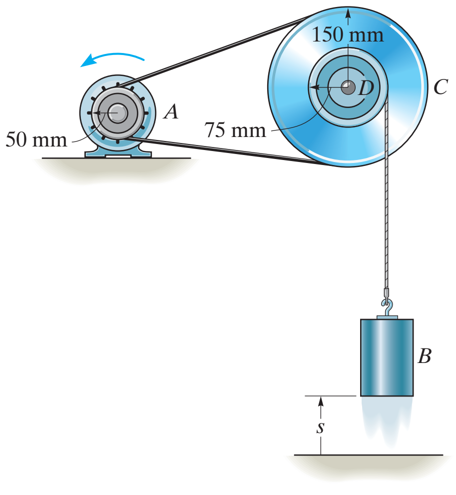
REVIEW PROBLEMS

R16–1. The hoisting gear A has an initial angular velocity of 60 rad/s and a constant deceleration of 1 rad/s^2. Determine the velocity and deceleration of the block which is being hoisted by the hub on gear B when $t = 3$ s.



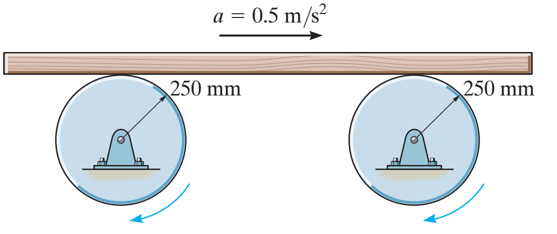
Prob. R16–1

R16–2. Starting at $(\omega_A)_0 = 3$ rad/s, when $\theta = 0$, $s = 0$, pulley A is given an angular acceleration $\alpha = (0.6\theta)$ rad/s^2, where θ is in radians. Determine the speed of block B when it has risen $s = 0.5$ m. The pulley has an inner hub D which is fixed to C and turns with it.



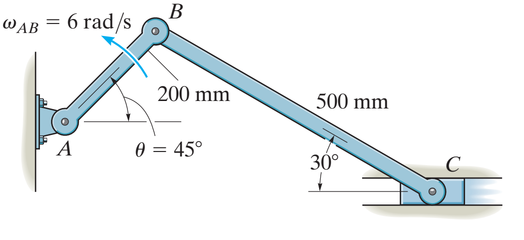
Prob. R16–2

R16–3. The board rests on the surface of two drums. At the instant shown, it has an acceleration of 0.5 m/s^2 to the right, while at the same instant points on the outer rim of each drum have an acceleration with a magnitude of 3 m/s^2. If the board does not slip on the drums, determine its speed due to the motion.



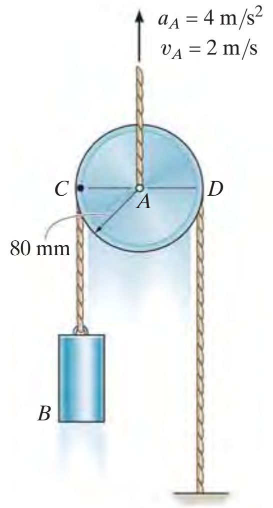
Prob. R16–3

R16–4. If bar AB has an angular velocity $\omega_{AB} = 6$ rad/s, determine the velocity of the slider block C at the instant shown.



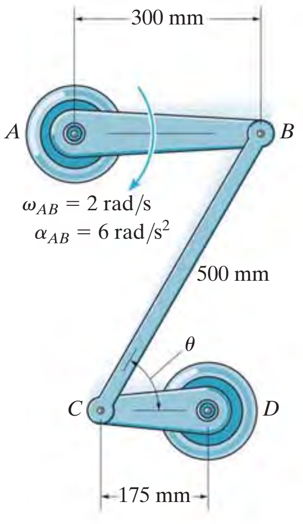
Prob. R16–4

R16–5. The center of the pulley is being lifted vertically with an acceleration of 4 m/s^2 at the instant it has a velocity of 2 m/s. If the cable does not slip on the pulley's surface, determine the accelerations of the cylinder B and point C on the pulley.



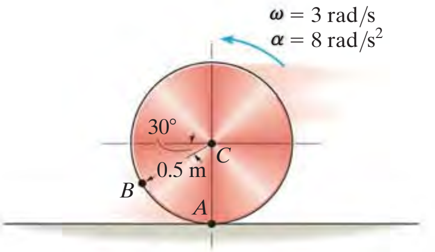
Prob. R16–5

R16–6. At the instant shown, link AB has an angular velocity $\omega_{AB} = 2$ rad/s and an angular acceleration $\alpha_{AB} = 6\ rad/s^2$. Determine the acceleration of the pin at C and the angular acceleration of link CB at this instant, when $\theta = 60°$.



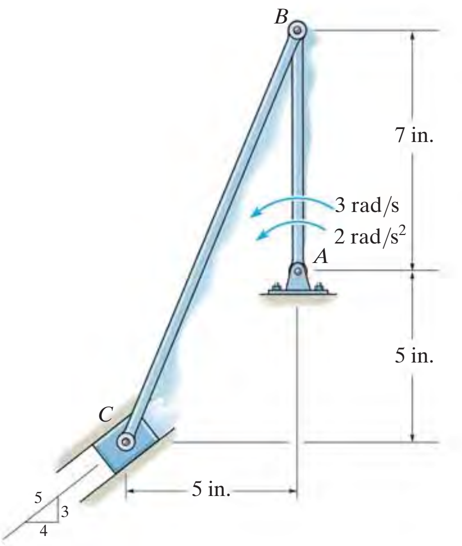
Prob. R16–6

R16–7. The disk is moving to the left such that it has an angular acceleration $\alpha = 8\ rad/s^2$ and angular velocity $\omega = 3$ rad/s at the instant shown. If it does not slip at A, determine the acceleration of point B.



Prob. R16–7

R16–8. At the given instant member AB has the angular motions shown. Determine the velocity and acceleration of the slider block C at this instant.



Prob. R16–8

Chapter 17

(© Surasaki/Fotolia)

Tractors and other heavy equipment can be subjected to severe loadings due to dynamic loadings as they accelerate. In this chapter we will show how to determine these loadings for planar motion.

Planar Kinetics of a Rigid Body: Force and Acceleration

CHAPTER OBJECTIVES

- To introduce the methods used to determine the mass moment of inertia of a body.
- To develop the planar kinetic equations of motion for a symmetric rigid body.
- To discuss applications of these equations to bodies undergoing translation, rotation about a fixed axis, and general plane motion.

17.1 Mass Moment of Inertia

Since a body has a definite size and shape, an applied nonconcurrent force system can cause the body to both translate and rotate. The translational aspects of the motion were studied in Chapter 13 and are governed by the equation $\mathbf{F} = m\mathbf{a}$. It will be shown in the next section that the rotational aspects, caused by a moment $\mathbf{M}$, are governed by an equation of the form $\mathbf{M} = I\boldsymbol{\alpha}$. The symbol I in this equation is termed the mass moment of inertia. By comparison, the *moment of inertia* is a measure of the resistance of a body to *angular acceleration* ($\mathbf{M} = I\boldsymbol{\alpha}$) in the same way that *mass* is a measure of the body's resistance to *acceleration* ($\mathbf{F} = m\mathbf{a}$).

The flywheel on the engine of this tractor has a large moment of inertia about its axis of rotation. Once it is set into motion, it will be difficult to stop, and this in turn will prevent the engine from stalling and instead will allow it to maintain a constant power.

(© R.C. Hibbeler)

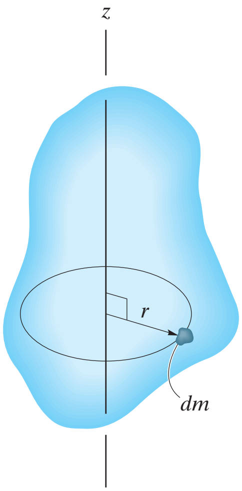


Fig. 17–1

We define the *moment of inertia* as the integral of the "second moment" about an axis of all the elements of mass dm which compose the body.* For example, the body's moment of inertia about the z axis in Fig. 17–1 is

$$I = \int_m r^2 \, dm \tag{17–1}$$

Here the "moment arm" r is the perpendicular distance from the z axis to the arbitrary element dm. Since the formulation involves r, the value of I is different for each axis about which it is computed. In the study of planar kinetics, the axis chosen for analysis generally passes through the body's mass center G and is always perpendicular to the plane of motion. The moment of inertia about this axis will be denoted as I_G. Since r is squared in Eq. 17–1, the mass moment of inertia is always a *positive* quantity. Common units used for its measurement are $\text{kg} \cdot \text{m}^2$ or $\text{slug} \cdot \text{ft}^2$.

If the body consists of material having a variable density, $\rho = \rho(x, y, z)$, the elemental mass dm of the body can be expressed in terms of its density and volume as $dm = \rho \, dV$. Substituting dm into Eq. 17–1, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho \, dV \tag{17–2}$$

*Another property of the body, which measures the symmetry of the body's mass with respect to a coordinate system, is the product of inertia. This property applies to the three-dimensional motion of a body and will be discussed in Chapter 21.

In the special case of ρ being a *constant*, this term may be factored out of the integral, and the integration is then purely a function of geometry,

$$I = \rho \int_V r^2 \, dV \tag{17–3}$$

When the volume element chosen for integration has infinitesimal dimensions in all three directions, Fig. 17–2*a*, the moment of inertia of the body must be determined using "triple integration." The integration process can, however, be simplified to a *single integration* provided the chosen volume element has a differential size or thickness in only *one direction*. Shell or disk elements are often used for this purpose.

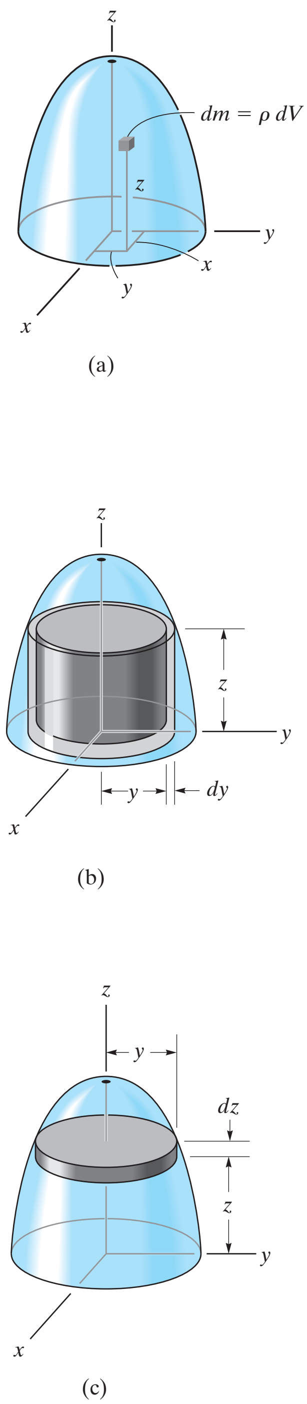


Fig. 17–2

Procedure for Analysis

To obtain the moment of inertia by integration, we will consider only symmetric bodies having volumes which are generated by revolving a curve about an axis. An example of such a body is shown in Fig. 17–2*a*. Two types of differential elements can be chosen.

Shell Element.

- If a *shell element* having a height z, radius $r = y$, and thickness dy is chosen for integration, Fig. 17–2*b*, then the volume is $dV = (2\pi y)(z)dy$.

- This element may be used in Eq. 17–2 or 17–3 for determining the moment of inertia I_z of the body about the z axis, since the *entire element*, due to its "thinness," lies at the *same* perpendicular distance $r = y$ from the z axis (see Example 17.1).

Disk Element.

- If a disk element having a radius y and a thickness dz is chosen for integration, Fig. 17–2*c*, then the volume is $dV = (\pi y^2)dz$.

- This element is *finite* in the radial direction, and consequently its parts *do not* all lie at the *same radial distance* r from the z axis. As a result, Eq. 17–2 or 17–3 *cannot* be used to determine I_z directly. Instead, to perform the integration it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result (see Example 17.2).

Determine the moment of inertia of the cylinder shown in Fig. 17–3*a* about the z axis. The density of the material, ρ, is constant.

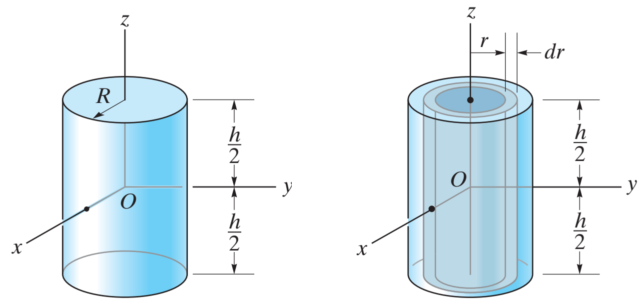


Fig. 17–3

SOLUTION

Shell Element. This problem can be solved using the *shell element* in Fig. 17–3*b* and a single integration. The volume of the element is $dV = (2\pi r)(h)\,dr$, so that its mass is $dm = \rho dV = \rho(2\pi hr\,dr)$. Since the *entire element* lies at the same distance r from the z axis, the moment of inertia *of the element* is

$$dI_z = r^2 dm = \rho 2\pi h r^3\,dr$$

Integrating over the entire region of the cylinder yields

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2}R^4 h$$

The mass of the cylinder is

$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

so that

$$I_z = \frac{1}{2}mR^2 \qquad \textit{Ans.}$$

If the density of the material is 5 slug/ft^3, determine the moment of inertia of the solid in Fig. 17–4*a* about the *y* axis.

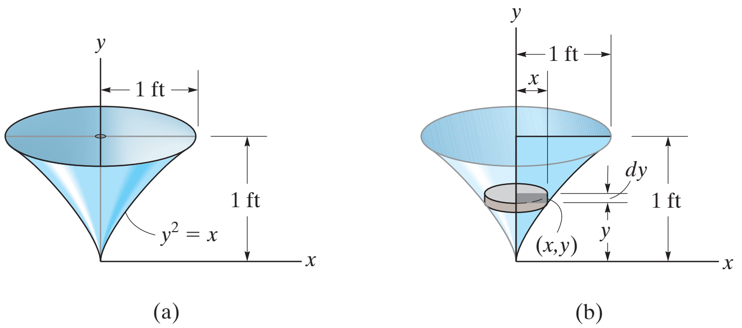


Fig. 17–4

SOLUTION

Disk Element. The moment of inertia will be found using a *disk element*, as shown in Fig. 17–4*b*. Here the element intersects the curve at the arbitrary point (x,y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all portions of the element are *not* located at the same distance from the *y* axis, it is still possible to determine the moment of inertia dI_y *of the element* about the *y* axis. In the preceding example it was shown that the moment of inertia of a cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height is not involved in this formula, the disk itself can be thought of as a cylinder. Thus, for the disk element in Fig. 17–4*b*, we have

$$dI_y = \tfrac{1}{2}(dm)x^2 = \tfrac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

Substituting $x = y^2$, $\rho = 5$ slug/ft^3, and integrating with respect to y, from $y = 0$ to $y = 1$ ft, yields the moment of inertia for the entire solid.

$$I_y = \frac{\pi(5\text{ slug/ft}^3)}{2}\int_0^{1\text{ ft}} x^4\, dy = \frac{\pi(5)}{2}\int_0^{1\text{ ft}} y^8\, dy = 0.873\text{ slug}\cdot\text{ft}^2 \quad \textit{Ans.}$$

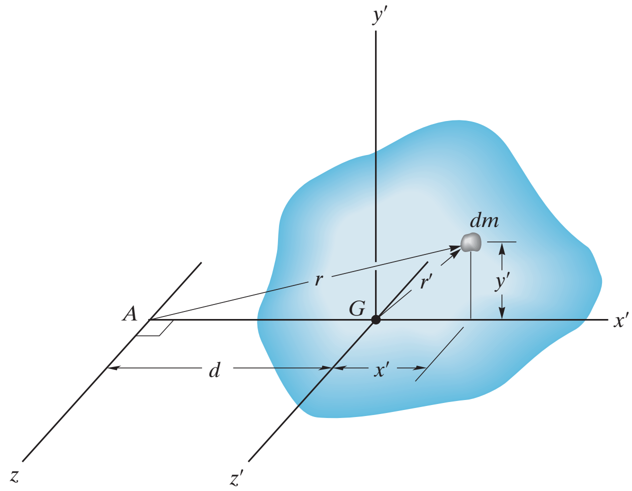


Fig. 17–5

17

Parallel-Axis Theorem. If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the *parallel-axis theorem*. This theorem can be derived by considering the body shown in Fig. 17–5. Here the z' axis passes through the mass center G, whereas the corresponding *parallel* z *axis* lies at a constant distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, we can express the moment of inertia of the body about the z axis as

$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral equals *zero*, since the z' axis passes through the body's mass center, i.e., $\int x' dm = \bar{x}' m = 0$ since $\bar{x}' = 0$. Finally, the third integral represents the total mass m of the body. Hence, the moment of inertia about the z axis can be written as

$$I = I_G + md^2 \tag{17–4}$$

where

I_G = moment of inertia about the z' axis passing through the mass center G
m = mass of the body
d = perpendicular distance between the parallel z and z' axes

Radius of Gyration. Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the *radius of gyration, k*. This is a geometrical property which has units of length. When it and the body's mass m are known, the body's moment of inertia is determined from the equation

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}} \tag{17–5}$$

Note the *similarity* between the definition of k in this formula and r in the equation $dI = r^2\,dm$, which defines the moment of inertia of an elemental mass dm of the body about an axis.

Composite Bodies. If a body consists of a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis can be determined by adding algebraically the moments of inertia of all the composite shapes computed about the axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been counted as a piece of another part—for example, a "hole" subtracted from a solid plate. The parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the axis. For the calculation, then, $I = \Sigma(I_G + md^2)$. Here I_G for each of the composite parts is determined by integration, or for simple shapes, such as rods and disks, it can be found from a table, such as the one given on the inside back cover of this book.

If the plate shown in Fig. 17–6*a* has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its moment of inertia about an axis directed perpendicular to the page and passing through point O.

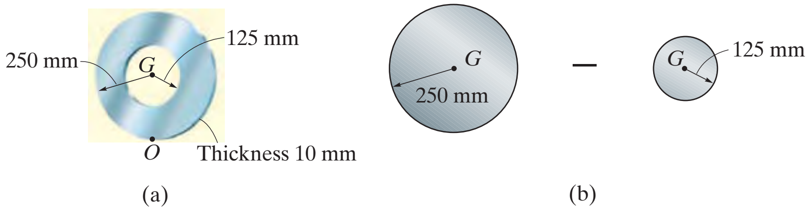


Fig. 17–6

SOLUTION

The plate consists of two composite parts, the 250-mm-radius disk *minus* a 125-mm-radius disk, Fig. 17–6*b*. The moment of inertia about O can be determined by computing the moment of inertia of each of these parts about O and then adding the results *algebraically*. The calculations are performed by using the parallel-axis theorem in conjunction with the data listed in the table on the inside back cover.

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Disk. The moment of inertia of a disk about the centroidal axis perpendicular to the plane of the disk is $I_G = \frac{1}{2}mr^2$. The mass center of the disk is located at a distance of 0.25 m from point O. Thus,

$$m_d = \rho_d V_d = 8000 \text{ kg/m}^3 \, [\pi(0.25 \text{ m})^2(0.01 \text{ m})] = 15.71 \text{ kg}$$

$$(I_d)_O = \tfrac{1}{2}m_d r_d^2 + m_d d^2$$

$$= \frac{1}{2}(15.71 \text{ kg})(0.25 \text{ m})^2 + (15.71 \text{ kg})(0.25 \text{ m})^2$$

$$= 1.473 \text{ kg} \cdot \text{m}^2$$

Hole. For the 125-mm-radius disk (hole), we have

$$m_h = \rho_h V_h = 8000 \text{ kg/m}^3 \, [\pi(0.125 \text{ m})^2(0.01 \text{ m})] = 3.927 \text{ kg}$$

$$(I_h)_O = \tfrac{1}{2}m_h r_h^2 + m_h d^2$$

$$= \frac{1}{2}(3.927 \text{ kg})(0.125 \text{ m})^2 + (3.927 \text{ kg})(0.25 \text{ m})^2$$

$$= 0.276 \text{ kg} \cdot \text{m}^2$$

The moment of inertia of the plate about point O is therefore

$$I_O = (I_d)_O - (I_h)_O$$

$$= 1.473 \text{ kg} \cdot \text{m}^2 - 0.276 \text{ kg} \cdot \text{m}^2$$

$$= 1.20 \text{ kg} \cdot \text{m}^2 \qquad \textit{Ans.}$$

The pendulum in Fig. 17–7 is suspended from the pin at O and consists of two thin rods. Rod OA weighs 10 lb, and BC weighs 8 lb. Determine the moment of inertia of the pendulum about an axis passing through (a) point O, and (b) the mass center G of the pendulum.

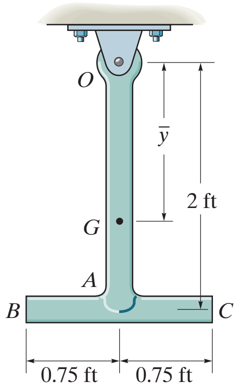


Fig. 17–7

SOLUTION

Part (a). Using the table on the inside back cover, the moment of inertia of rod OA about an axis perpendicular to the page and passing through point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10\text{ lb}}{32.2\text{ ft/s}^2}\right)(2\text{ ft})^2 = 0.414\text{ slug}\cdot\text{ft}^2$$

This same value can be obtained using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem.

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10\text{ lb}}{32.2\text{ ft/s}^2}\right)(2\text{ ft})^2 + \left(\frac{10\text{ lb}}{32.2\text{ ft/s}^2}\right)(1\text{ ft})^2$$
$$= 0.414\text{ slug}\cdot\text{ft}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{8\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.5\text{ ft})^2 + \left(\frac{8\text{ lb}}{32.2\text{ ft/s}^2}\right)(2\text{ ft})^2$$
$$= 1.040\text{ slug}\cdot\text{ft}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.414 + 1.040 = 1.454 = 1.45\text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to point O. Assuming this distance to be $\bar{y}$, Fig. 17–7, and using the formula for determining the mass center, we have

$$\bar{y} = \frac{\Sigma\tilde{y}m}{\Sigma m} = \frac{1(10/32.2) + 2(8/32.2)}{(10/32.2) + (8/32.2)} = 1.444\text{ ft}$$

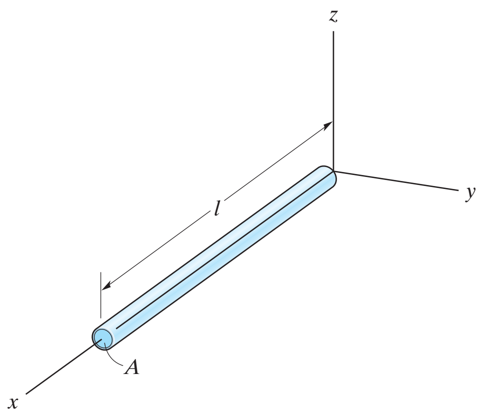
The moment of inertia I_G may be found in the same manner as I_O, which requires successive applications of the parallel-axis theorem to transfer the moments of inertia of rods OA and BC to G. A more direct solution, however, involves using the result for I_O, i.e.,

$$I_O = I_G + md^2; \quad 1.454\text{ slug}\cdot\text{ft}^2 = I_G + \left(\frac{18\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.444\text{ ft})^2$$

$$I_G = 0.288\text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

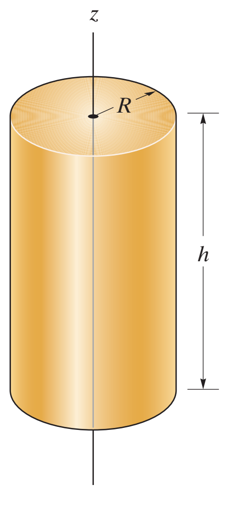
PROBLEMS

17–1. Determine the moment of inertia I_y for the slender rod. The rod's density ρ and cross-sectional area A are constant. Express the result in terms of the rod's total mass m.



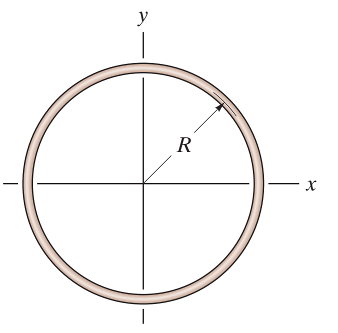
Prob. 17–1

17–2. The solid cylinder has an outer radius R, height h, and is made from a material having a density that varies from its center as $\rho = k + ar^2$, where k and a are constants. Determine the mass of the cylinder and its moment of inertia about the z axis.



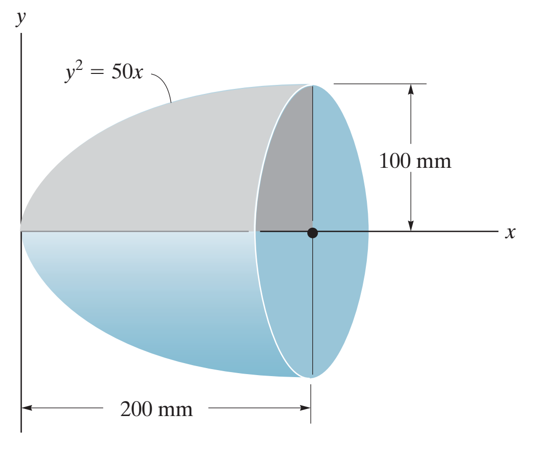
Prob. 17–2

17–3. Determine the moment of inertia of the thin ring about the z axis. The ring has a mass m.



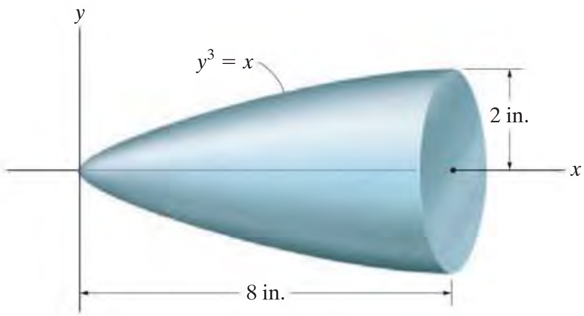
Prob. 17–3

***17–4.** The paraboloid is formed by revolving the shaded area around the x axis. Determine the radius of gyration k_x. The density of the material is $\rho = 5\ \text{Mg/m}^3$.



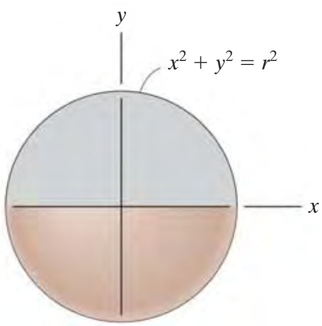
Prob. 17–4

17–5. Determine the radius of gyration k_x of the body. The specific weight of the material is $\gamma = 380 \text{ lb/ft}^3$.



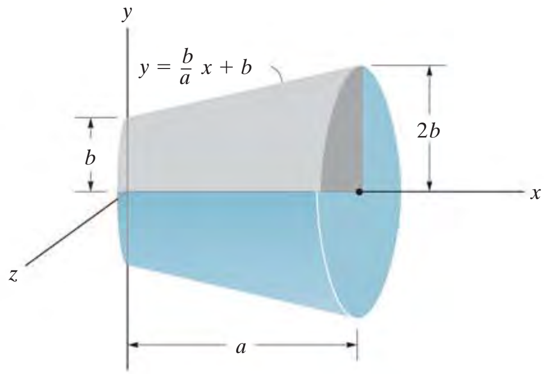
Prob. 17–5

17–6. The sphere is formed by revolving the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the sphere. The material has a constant density ρ.



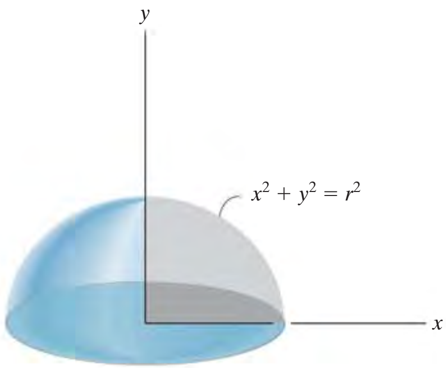
Prob. 17–6

17–7. The frustum is formed by rotating the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the frustum. The frustum has a constant density ρ.



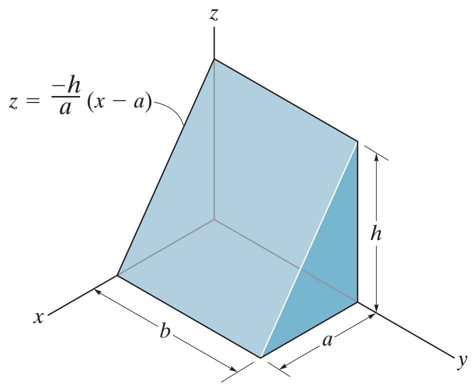
Prob. 17–7

***17–8.** The hemisphere is formed by rotating the shaded area around the y axis. Determine the moment of inertia I_y and express the result in terms of the total mass m of the hemisphere. The material has a constant density ρ.



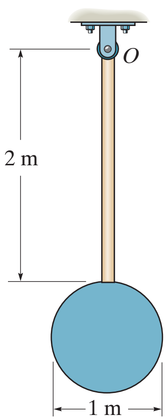
Prob. 17–8

17–9. Determine the moment of inertia of the homogeneous triangular prism with respect to the y axis. Express the result in terms of the mass m of the prism. *Hint*: For integration, use thin plate elements parallel to the x–y plane and having a thickness dz.



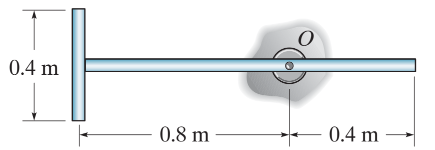
Prob. 17–9

17–10. The pendulum consists of a 4-kg circular plate and a 2-kg slender rod. Determine the radius of gyration of the pendulum about an axis perpendicular to the page and passing through point O.



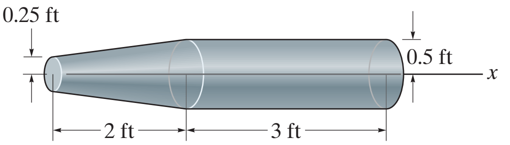
Prob. 17–10

17–11. The assembly is made of the slender rods that have a mass per unit length of 3 kg/m. Determine the mass moment of inertia of the assembly about an axis perpendicular to the page and passing through point O.



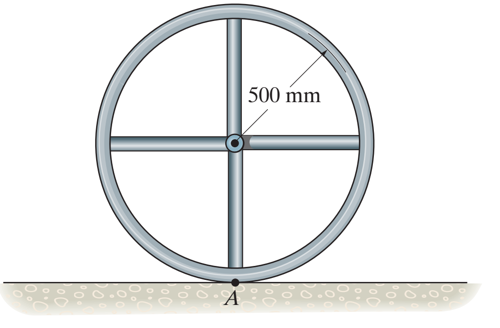
Prob. 17–11

***17–12.** Determine the moment of inertia of the solid steel assembly about the x axis. Steel has a specific weight of $\gamma_{st} = 490 \text{ lb/ft}^3$.



Prob. 17–12

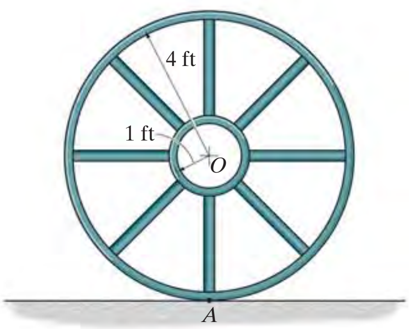
17–13. The wheel consists of a thin ring having a mass of 10 kg and four spokes made from slender rods and each having a mass of 2 kg. Determine the wheel's moment of inertia about an axis perpendicular to the page and passing through point A.



Prob. 17–13

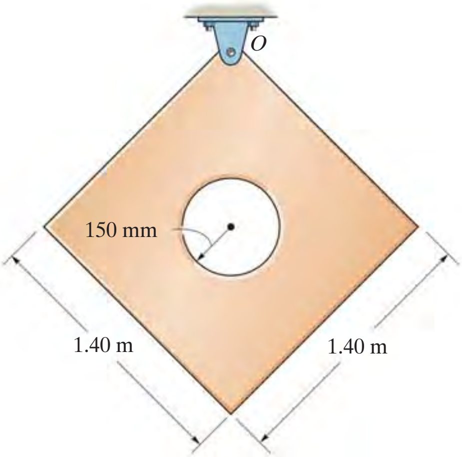
17

17–14. If the large ring, small ring and each of the spokes weigh 100 lb, 15 lb, and 20 lb, respectively, determine the mass moment of inertia of the wheel about an axis perpendicular to the page and passing through point A.



Prob. 17–14

17–15. Determine the moment of inertia about an axis perpendicular to the page and passing through the pin at O. The thin plate has a hole in its center. Its thickness is 50 mm, and the material has a density $\rho = 50\ \text{kg/m}^3$.



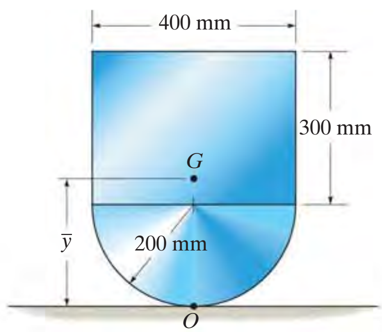
Prob. 17–15

***17–16.** Determine the mass moment of inertia of the thin plate about an axis perpendicular to the page and passing through point O. The material has a mass per unit area of $20\ \text{kg/m}^2$.

Prob. 17–16

17–17. Determine the location $\bar{y}$ of the center of mass G of the assembly and then calculate the moment of inertia about an axis perpendicular to the page and passing through G. The block has a mass of 3 kg and the semicylinder has a mass of 5 kg.

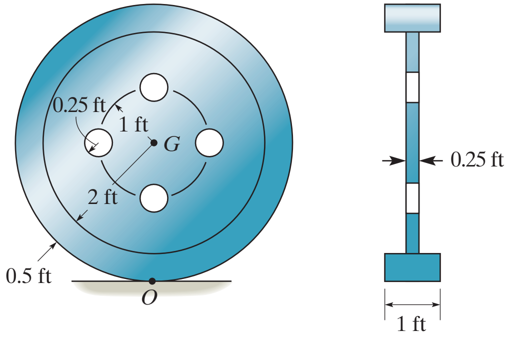
17–18. Determine the moment of inertia of the assembly about an axis perpendicular to the page and passing through point O. The block has a mass of 3 kg, and the semicylinder has a mass of 5 kg.



Probs. 17–17/18

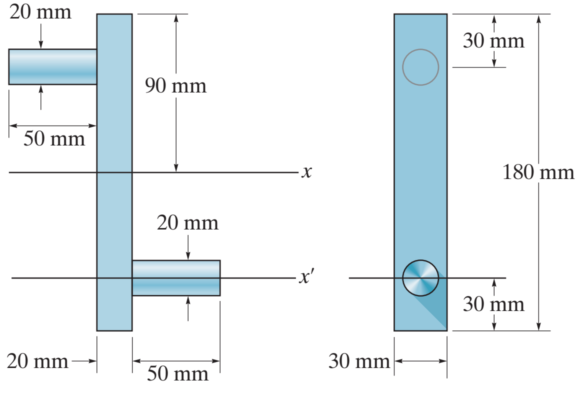
17–19. Determine the moment of inertia of the wheel about an axis which is perpendicular to the page and passes through the center of mass G. The material has a specific weight $\gamma = 90\ \text{lb/ft}^3$.

***17–20.** Determine the moment of inertia of the wheel about an axis which is perpendicular to the page and passes through point O. The material has a specific weight $\gamma = 90\ \text{lb/ft}^3$.



Probs. 17–19/20

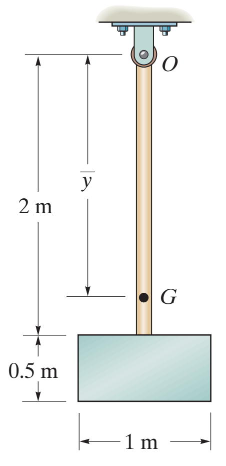
17–22. Determine the moment of inertia of the overhung crank about the x axis. The material is steel having a density of $\rho = 7.85\ \text{Mg/m}^3$.



Prob. 17–22

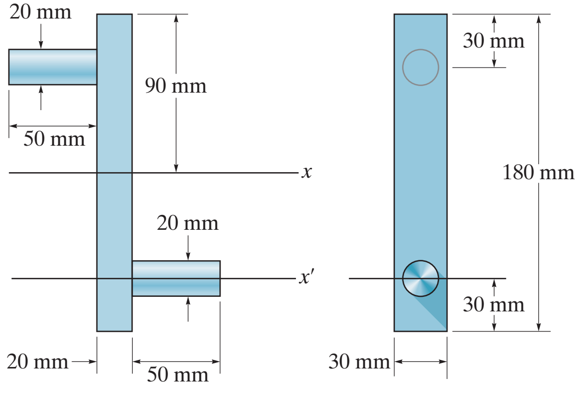
17

17–21. The pendulum consists of the 3-kg slender rod and the 5-kg thin plate. Determine the location $\bar{y}$ of the center of mass G of the pendulum; then calculate the moment of inertia of the pendulum about an axis perpendicular to the page and passing through G.



Prob. 17–21

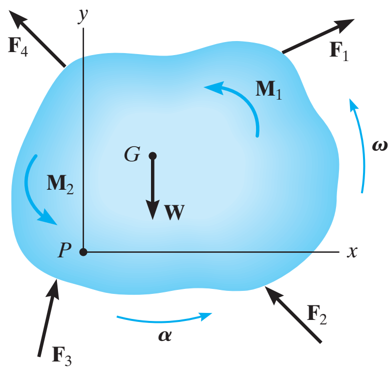
17–23. Determine the moment of inertia of the overhung crank about the x' axis. The material is steel having a density of $\rho = 7.85\ \text{Mg/m}^3$.



Prob. 17–23

17.2 Planar Kinetic Equations of Motion

In the following analysis we will limit our study of planar kinetics to rigid bodies which, along with their loadings, are considered to be *symmetrical* with respect to a fixed reference plane.* Since the motion of the body can be viewed within the reference plane, all the forces (and couple moments) acting on the body can then be projected onto the plane. An example of an arbitrary body of this type is shown in Fig. 17–8*a*. Here the *inertial frame of reference x, y, z* has its origin *coincident* with the arbitrary point *P* in the body. By definition, *these axes do not rotate and are either fixed or translate with constant velocity.*



(a)

Fig. 17–8

Equation of Translational Motion. The external forces acting on the body in Fig. 17–8*a* represent the effect of gravitational, electrical, magnetic, or contact forces between adjacent bodies. Since this force system has been considered previously in Sec. 13.3 for the analysis of a system of particles, the resulting Eq. 13–6 can be used here, in which case

$$\Sigma \mathbf{F} = m\mathbf{a}_G$$

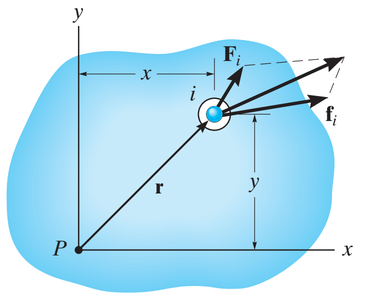
This equation is referred to as the *translational equation of motion* for the mass center of a rigid body. It states that *the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center G.*

For motion of the body in the x–y plane, the translational equation of motion may be written in the form of two independent scalar equations, namely,

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

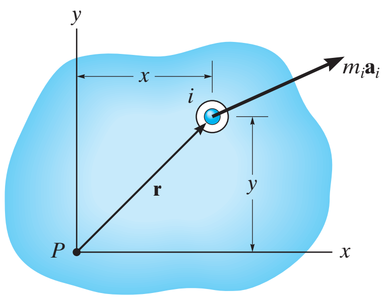
*By doing this, the rotational equation of motion reduces to a rather simplified form. The more general case of body shape and loading is considered in Chapter 21.



Particle free-body diagram

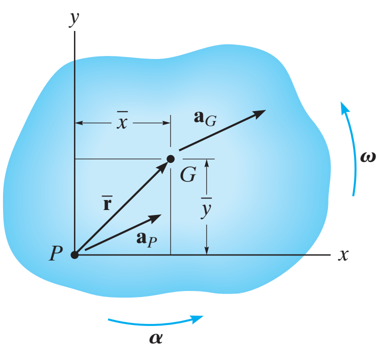
(b)

=



Particle kinetic diagram

(c)



(d)

Fig. 17–8 (cont.)

17

Equation of Rotational Motion. We will now determine the effects caused by the moments of the external force system computed about an axis perpendicular to the plane of motion (the z axis) and passing through point P. As shown on the free-body diagram of the ith particle, Fig. 17–8*b*, $\mathbf{F}_i$ represents the *resultant external force* acting on the particle, and $\mathbf{f}_i$ is the *resultant of the internal forces* caused by interactions with adjacent particles. If the particle has a mass m_i and its acceleration is $\mathbf{a}_i$, then its kinetic diagram is shown in Fig. 17–8*c*. Summing moments about point P, we require

$$\mathbf{r} \times \mathbf{F}_i + \mathbf{r} \times \mathbf{f}_i = \mathbf{r} \times m_i \mathbf{a}_i$$

or

$$(\mathbf{M}_P)_i = \mathbf{r} \times m_i \mathbf{a}_i$$

The moments about P can also be expressed in terms of the acceleration of point P, Fig. 17–8*d*. If the body has an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, then using Eq. 16–18 we have

$$\begin{aligned}(\mathbf{M}_P)_i &= m_i \mathbf{r} \times (\mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r}) \\ &= m_i[\mathbf{r} \times \mathbf{a}_P + \mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r}) - \omega^2(\mathbf{r} \times \mathbf{r})]\end{aligned}$$

The last term is zero, since $\mathbf{r} \times \mathbf{r} = \mathbf{0}$. Expressing the vectors with Cartesian components and carrying out the cross-product operations yields

$$\begin{aligned}(M_P)_i \mathbf{k} &= m_i\{(x\mathbf{i} + y\mathbf{j}) \times [(a_P)_x \mathbf{i} + (a_P)_y \mathbf{j}] \\ &\quad + (x\mathbf{i} + y\mathbf{j}) \times [\alpha\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]\} \\ (M_P)_i \mathbf{k} &= m_i[-y(a_P)_x + x(a_P)_y + \alpha x^2 + \alpha y^2]\mathbf{k} \\ \circlearrowleft (M_P)_i &= m_i[-y(a_P)_x + x(a_P)_y + \alpha r^2]\end{aligned}$$

Letting $m_i \rightarrow dm$ and integrating with respect to the entire mass m of the body, we obtain the resultant moment equation

$$\circlearrowleft \Sigma M_P = -\left(\int_m y\, dm\right)(a_P)_x + \left(\int_m x\, dm\right)(a_P)_y + \left(\int_m r^2 dm\right)\alpha$$

Here ΣM_P represents only the moment of the *external forces* acting on the body about point P. The resultant moment of the internal forces is zero, since for the entire body these forces occur in equal and opposite collinear pairs and thus the moment of each pair of forces about P cancels. The integrals in the first and second terms on the right are used to locate the body's center of mass G with respect to P, since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$, Fig. 17–8*d*. Also, the last integral represents the body's moment of inertia about the z axis, i.e., $I_P = \int r^2 dm$. Thus,

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_P)_x + \bar{x}m(a_P)_y + I_P\alpha \qquad (17\text{–}6)$$

It is possible to reduce this equation to a simpler form if point P coincides with the mass center G for the body. If this is the case, then $\bar{x} = \bar{y} = 0$, and therefore*

$$\Sigma M_G = I_G \alpha \qquad (17\text{–}7)$$

This rotational equation of motion states that the sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration.

Equation 17–6 can also be rewritten in terms of the x and y components of $\mathbf{a}_G$ and the body's moment of inertia I_G. If point G is located at $(\bar{x}, \bar{y})$, Fig. 17–8d, then by the parallel-axis theorem, $I_P = I_G + m(\bar{x}^2 + \bar{y}^2)$. Substituting into Eq. 17–6 and rearranging terms, we get

$$\circlearrowleft \Sigma M_P = \bar{y}m[-(a_P)_x + \bar{y}\alpha] + \bar{x}m[(a_P)_y + \bar{x}\alpha] + I_G\alpha \qquad (17\text{–}8)$$

From the kinematic diagram of Fig. 17–8d, $\mathbf{a}_P$ can be expressed in terms of $\mathbf{a}_G$ as

$$\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \bar{\mathbf{r}} - \omega^2 \bar{\mathbf{r}}$$

$$(a_G)_x\mathbf{i} + (a_G)_y\mathbf{j} = (a_P)_x\mathbf{i} + (a_P)_y\mathbf{j} + \alpha\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) - \omega^2(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

Carrying out the cross product and equating the respective $\mathbf{i}$ and $\mathbf{j}$ components yields the two scalar equations

$$(a_G)_x = (a_P)_x - \bar{y}\alpha - \bar{x}\omega^2$$
$$(a_G)_y = (a_P)_y + \bar{x}\alpha - \bar{y}\omega^2$$

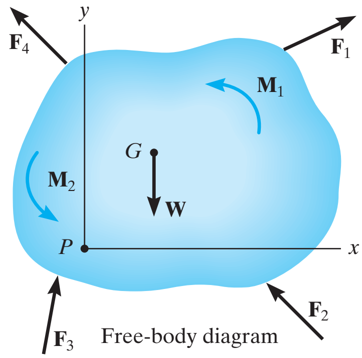
From these equations, $[-(a_P)_x + \bar{y}\alpha] = [-(a_G)_x - \bar{x}\omega^2]$ and $[(a_P)_y + \bar{x}\alpha] = [(a_G)_y + \bar{y}\omega^2]$. Substituting these results into Eq. 17–8 and simplifying gives

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha \qquad (17\text{–}9)$$

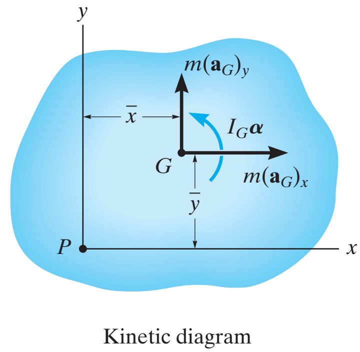
This important result indicates that when moments of the external forces shown on the free-body diagram are summed about point P, Fig. 17–8e, they are equivalent to the sum of the "kinetic moments" of the components of m$\mathbf{a}_G$ *about P plus the "kinetic moment" of* $I_G\boldsymbol{\alpha}$, *Fig. 17–8f*. In other words, when the "kinetic moments," $\Sigma(\mathcal{M}_k)_P$, are computed, Fig. 17–8f, the vectors $m(\mathbf{a}_G)_x$ and $m(\mathbf{a}_G)_y$ are treated as sliding vectors; that is, they can act at *any point along their line of action*. In a similar manner, $I_G\boldsymbol{\alpha}$ can be treated as a free vector and can therefore act at *any point*. It is important to keep in mind, however, that $m\mathbf{a}_G$ and $I_G\boldsymbol{\alpha}$ are not the same as a force or a couple moment. Instead, they are caused by the external effects of forces and couple moments acting on the body. With this in mind we can therefore write Eq. 17–9 in a more general form as

$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P \qquad (17\text{–}10)$$

*It also reduces to this same simple form $\Sigma M_P = I_P\alpha$ if point P is a *fixed point* (see Eq. 17–16) or the acceleration of point P is directed along the line PG.

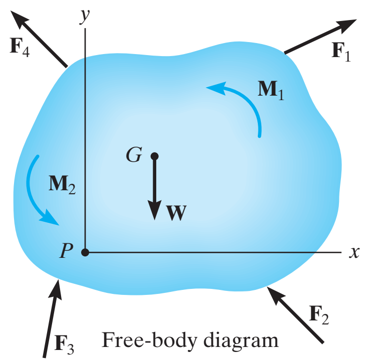


(e)

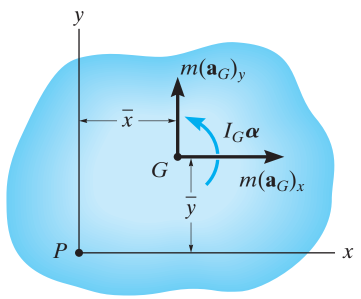


(f)

Fig. 17–8 (cont.)



(e)



Kinetic diagram

(f)

Fig. 17–8 (cont.)

General Application of the Equations of Motion. To summarize this analysis, *three* independent scalar equations can be written to describe the general plane motion of a symmetrical rigid body.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = I_G\alpha$$

or

$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P \qquad (17\text{–}11)$$

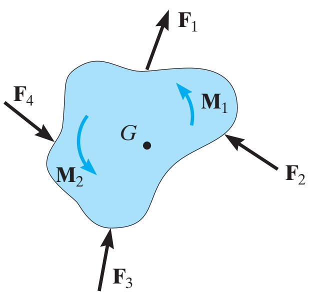
When applying these equations, one should *always* draw a free-body diagram, Fig. 17–8*e*, in order to account for the terms involved in ΣF_x, ΣF_y, ΣM_G, or ΣM_P. In some problems it may also be helpful to draw the *kinetic diagram* for the body, Fig. 17–8*f*. This diagram graphically accounts for the terms $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$. It is especially convenient when used to determine the components of $m\mathbf{a}_G$ and the moment of these components in $\Sigma(\mathcal{M}_k)_P$.*

17.3 Equations of Motion: Translation

When the rigid body in Fig. 17–9*a* undergoes a *translation*, all the particles of the body have the *same acceleration*. Furthermore, $\boldsymbol{\alpha} = \mathbf{0}$, in which case the rotational equation of motion applied at point G reduces to a simplified form, namely, $\Sigma M_G = 0$. Application of this and the force equations of motion will now be discussed for each of the two types of translation.

Rectilinear Translation. When a body is subjected to *rectilinear translation*, all the particles of the body (slab) travel along parallel straight-line paths. The free-body and kinetic diagrams are shown in Fig. 17–9*b*. Since $I_G\boldsymbol{\alpha} = \mathbf{0}$, only $m\mathbf{a}_G$ is shown on the kinetic diagram. Hence, the equations of motion which apply in this case become

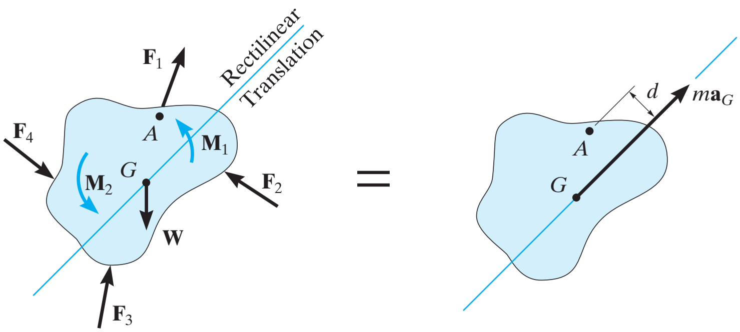
$$\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_G &= 0 \end{aligned} \qquad (17\text{–}12)$$



(a)

Fig. 17–9

*For this reason, the kinetic diagram will be used in the solution of an example problem whenever $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is applied.



(b)

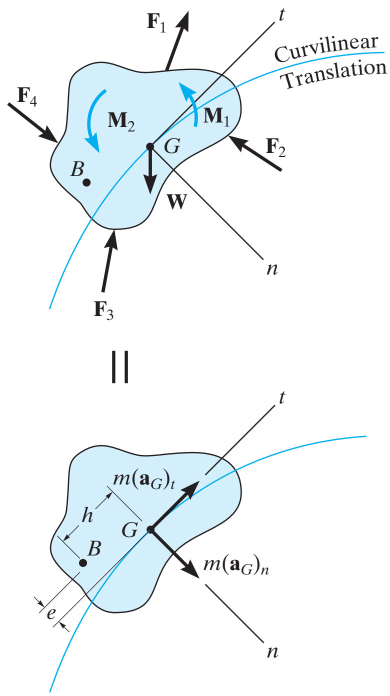
It is also possible to sum moments about other points on or off the body, in which case the moment of $m\mathbf{a}_G$ must be taken into account. For example, if point A is chosen, which lies at a perpendicular distance d from the line of action of $m\mathbf{a}_G$, the following moment equation applies:

$$\zeta + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad \Sigma M_A = (ma_G)d$$

Here the sum of moments of the external forces and couple moments about A (ΣM_A, free-body diagram) equals the moment of $m\mathbf{a}_G$ about A ($\Sigma(\mathcal{M}_k)_A$, kinetic diagram).

Curvilinear Translation. When a rigid body is subjected to *curvilinear translation*, all the particles of the body have the same accelerations as they travel along *curved paths* as noted in Sec. 16.1. For analysis, it is often convenient to use an inertial coordinate system having an origin which coincides with the body's mass center at the instant considered, and axes which are oriented in the normal and tangential directions to the path of motion, Fig. 17–9*c*. The three scalar equations of motion are then

$$\begin{aligned} \Sigma F_n &= m(a_G)_n \\ \Sigma F_t &= m(a_G)_t \\ \Sigma M_G &= 0 \end{aligned} \qquad (17\text{–}13)$$



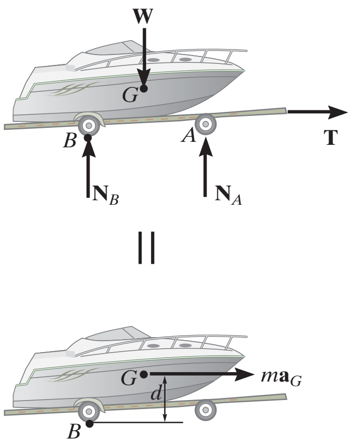
(c)

Fig. 17–9

If moments are summed about the arbitrary point B, Fig. 17–9*c*, then it is necessary to account for the moments, $\Sigma(\mathcal{M}_k)_B$, of the two components $m(\mathbf{a}_G)_n$ and $m(\mathbf{a}_G)_t$ about this point. From the kinetic diagram, h and e represent the perpendicular distances (or "moment arms") from B to the lines of action of the components. The required moment equation therefore becomes

$$\zeta + \Sigma M_B = \Sigma(\mathcal{M}_k)_B; \qquad \Sigma M_B = e[m(a_G)_t] - h[m(a_G)_n]$$

The free-body and kinetic diagrams for this boat and trailer are drawn first in order to apply the equations of motion. Here the forces on the free-body diagram cause the effect shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = 0$. However, if moments are summed about point B then $\circlearrowleft + \Sigma M_B = ma_G(d)$. (© R.C. Hibbeler)



Procedure for Analysis

Kinetic problems involving rigid-body *translation* can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y or n, t inertial coordinate system and draw the free-body diagram in order to account for all the external forces and couple moments that act on the body.
- The direction and sense of the acceleration of the body's mass center $\mathbf{a}_G$ should be established.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used in the solution, then consider drawing the kinetic diagram, since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$ or $m(\mathbf{a}_G)_t$, $m(\mathbf{a}_G)_n$ and is therefore convenient for "visualizing" the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- To simplify the analysis, the moment equation $\Sigma M_G = 0$ can be replaced by the more general equation $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$, where point P is usually located at the intersection of the lines of action of as many unknown forces as possible.
- If the body is in contact with a *rough surface* and slipping occurs, use the friction equation $F = \mu_k N$. Remember, $\mathbf{F}$ always acts on the body so as to oppose the motion of the body relative to the surface it contacts.

Kinematics.

- Use kinematics to determine the velocity and position of the body.
- For rectilinear translation with *variable acceleration*

  $$a_G = dv_G/dt \quad a_G ds_G = v_G dv_G$$

- For rectilinear translation with *constant acceleration*

  $$v_G = (v_G)_0 + a_G t \quad v_G^2 = (v_G)_0^2 + 2a_G[s_G - (s_G)_0]$$

  $$s_G = (s_G)_0 + (v_G)_0 t + \tfrac{1}{2}a_G t^2$$

- For curvilinear translation

  $$(a_G)_n = v_G^2/\rho$$

  $$(a_G)_t = dv_G/dt \quad (a_G)_t\, ds_G = v_G\, dv_G$$

The car shown in Fig. 17–10*a* has a mass of 2 Mg and a center of mass at G. Determine the acceleration if the rear "driving" wheels are always slipping, whereas the front wheels are free to rotate. Neglect the mass of the wheels. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.25$.

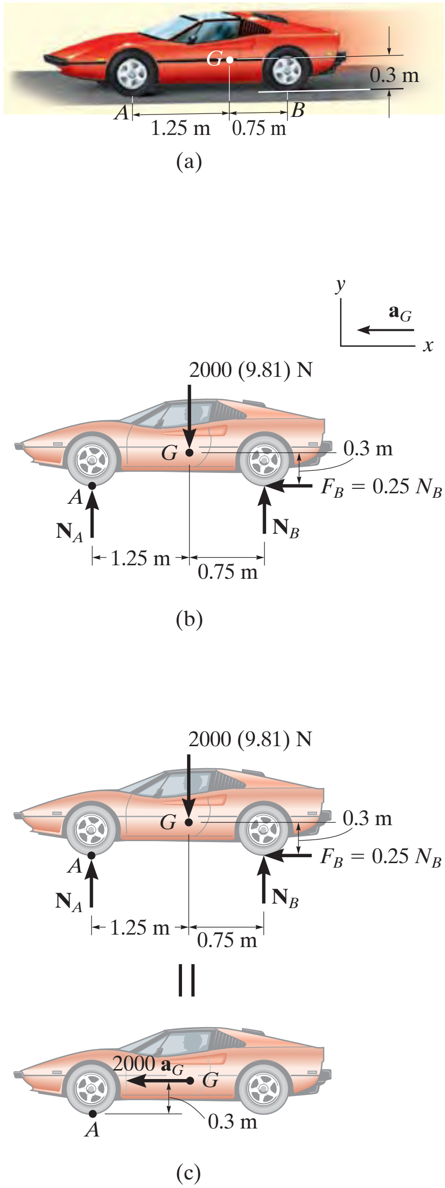


Fig. 17–10

SOLUTION I

Free-Body Diagram. As shown in Fig. 17–10*b*, the rear-wheel frictional force $\mathbf{F}_B$ pushes the car forward, and since *slipping occurs*, $F_B = 0.25N_B$. The frictional forces acting on the *front wheels* are *zero*, since these wheels have negligible mass.* There are three unknowns in the problem, N_A, N_B, and a_G. Here we will sum moments about the mass center. The car (point G) accelerates to the left, i.e., in the negative x direction, Fig. 17–10*b*.

Equations of Motion.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad -0.25N_B = -(2000\text{ kg})a_G \qquad (1)$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_A + N_B - 2000(9.81)\text{ N} = 0 \qquad (2)$$

$$\circlearrowleft+\Sigma M_G = 0; \qquad -N_A(1.25\text{ m}) - 0.25N_B(0.3\text{ m}) + N_B(0.75\text{ m}) = 0 \qquad (3)$$

Solving,

$$a_G = 1.59\text{ m/s}^2 \leftarrow \qquad \textit{Ans.}$$

$$N_A = 6.88\text{ kN}$$

$$N_B = 12.7\text{ kN}$$

SOLUTION II

Free-Body and Kinetic Diagrams. If the "moment" equation is applied about point A, then the unknown N_A will be eliminated from the equation. To "visualize" the moment of $m\mathbf{a}_G$ about A, we will include the kinetic diagram as part of the analysis, Fig. 17–10*c*.

Equation of Motion.

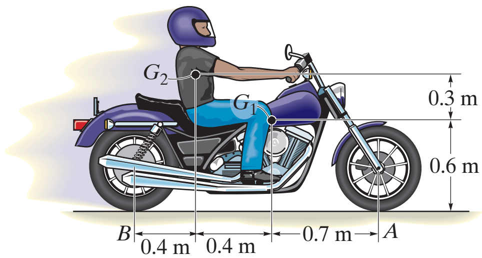
$$\circlearrowleft+\Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad N_B(2\text{ m}) - [2000(9.81)\text{ N}](1.25\text{ m}) = (2000\text{ kg})a_G(0.3\text{ m})$$

Solving this and Eq. 1 for a_G leads to a simpler solution than that obtained from Eqs. 1 to 3.

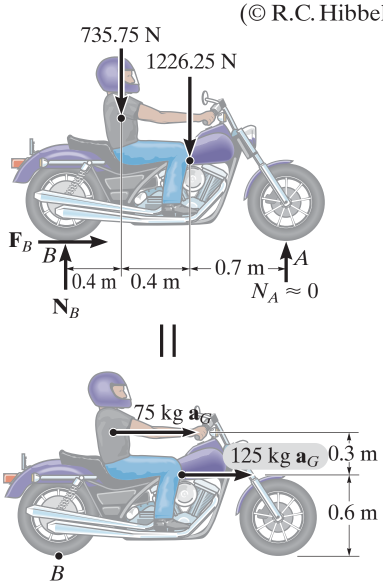
*With negligible wheel mass, $I\alpha = 0$ and the frictional force at A required to turn the wheel is zero. If the wheels' mass were included, then the solution would be more involved, since a general-plane-motion analysis of the wheels would have to be considered (see Sec. 17.5).

(© R.C. Hibbeler)

The motorcycle shown in Fig. 17–11*a* has a mass of 125 kg and a center of mass at G_1, while the rider has a mass of 75 kg and a center of mass at G_2. Determine the minimum coefficient of static friction between the wheels and the pavement in order for the rider to do a "wheely," i.e., lift the front wheel off the ground as shown in the photo. What acceleration is necessary to do this? Neglect the mass of the wheels and assume that the front wheel is free to roll.



(a)



(b)

Fig. 17–11

SOLUTION

Free-Body and Kinetic Diagrams. In this problem we will consider both the motorcycle and the rider as a single *system*. It is possible first to determine the location of the center of mass for this "system" by using the equations $\bar{x} = \Sigma\tilde{x}m/\Sigma m$ and $\bar{y} = \Sigma\tilde{y}m/\Sigma m$. Here, however, we will consider the weight and mass of the motorcycle and rider separately as shown on the free-body and kinetic diagrams, Fig. 17–11*b*. Both of these parts move with the *same* acceleration. We have assumed that the front wheel is *about* to leave the ground, so that the normal reaction $N_A \approx 0$. The three unknowns in the problem are N_B, F_B, and a_G.

Equations of Motion.

$$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad F_B = (75\text{ kg} + 125\text{ kg})a_G \qquad (1)$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_B - 735.75\text{ N} - 1226.25\text{ N} = 0$$

$$\circlearrowleft+\Sigma M_B = \Sigma(\mathcal{M}_k)_B; \; -(735.75\text{ N})(0.4\text{ m}) - (1226.25\text{ N})(0.8\text{ m}) =$$

$$-(75\text{ kg }a_G)(0.9\text{ m}) - (125\text{ kg }a_G)(0.6\text{ m}) \qquad (2)$$

Solving,

$$a_G = 8.95\text{ m/s}^2 \rightarrow \qquad \textit{Ans.}$$

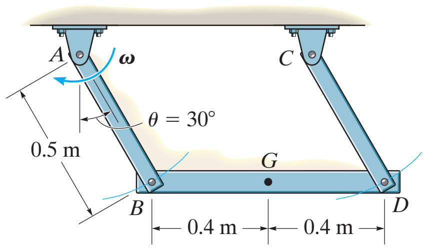
$$N_B = 1962\text{ N}$$

$$F_B = 1790\text{ N}$$

Thus the minimum coefficient of static friction is

$$(\mu_s)_{\min} = \frac{F_B}{N_B} = \frac{1790\text{ N}}{1962\text{ N}} = 0.912 \qquad \textit{Ans.}$$

The 100-kg beam BD shown in Fig. 17–12a is supported by two rods having negligible mass. Determine the force developed in each rod if at the instant $\theta = 30^\circ$, $\omega = 6$ rad/s.



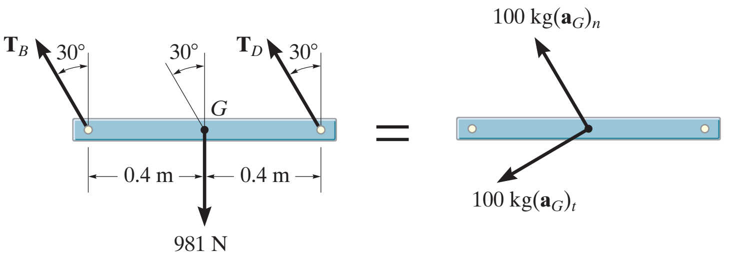
(a)

SOLUTION

Free-Body and Kinetic Diagrams. The beam moves with *curvilinear translation* since all points on the beam move along circular paths, each path having the same radius of 0.5 m, but different centers of curvature. Using normal and tangential coordinates, the free-body and kinetic diagrams for the beam are shown in Fig. 17–12b. Because of the *translation*, G has the *same* motion as the pin at B, which is connected to both the rod and the beam. Note that the tangential component of acceleration acts downward to the left due to the clockwise direction of $\boldsymbol{\alpha}$, Fig. 17–12c. Furthermore, the normal component of acceleration is *always* directed toward the center of curvature (toward point A for rod AB). Since the angular velocity of AB is 6 rad/s when $\theta = 30^\circ$, then

$$(a_G)_n = \omega^2 r = (6 \text{ rad/s})^2(0.5 \text{ m}) = 18 \text{ m/s}^2$$

The three unknowns are T_B, T_D, and $(a_G)_t$.



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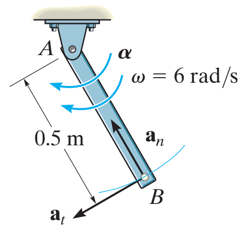


(c)

Fig. 17–12

Equations of Motion.

$$+\nwarrow\Sigma F_n = m(a_G)_n; \quad T_B + T_D - 981 \cos 30^\circ \text{ N} = 100 \text{ kg}(18 \text{ m/s}^2) \qquad (1)$$

$$+\swarrow\Sigma F_t = m(a_G)_t; \quad 981 \sin 30^\circ = 100 \text{ kg}(a_G)_t \qquad (2)$$

$$\circlearrowleft +\Sigma M_G = 0; \quad -(T_B \cos 30^\circ)(0.4 \text{ m}) + (T_D \cos 30^\circ)(0.4 \text{ m}) = 0 \qquad (3)$$

Simultaneous solution of these three equations gives

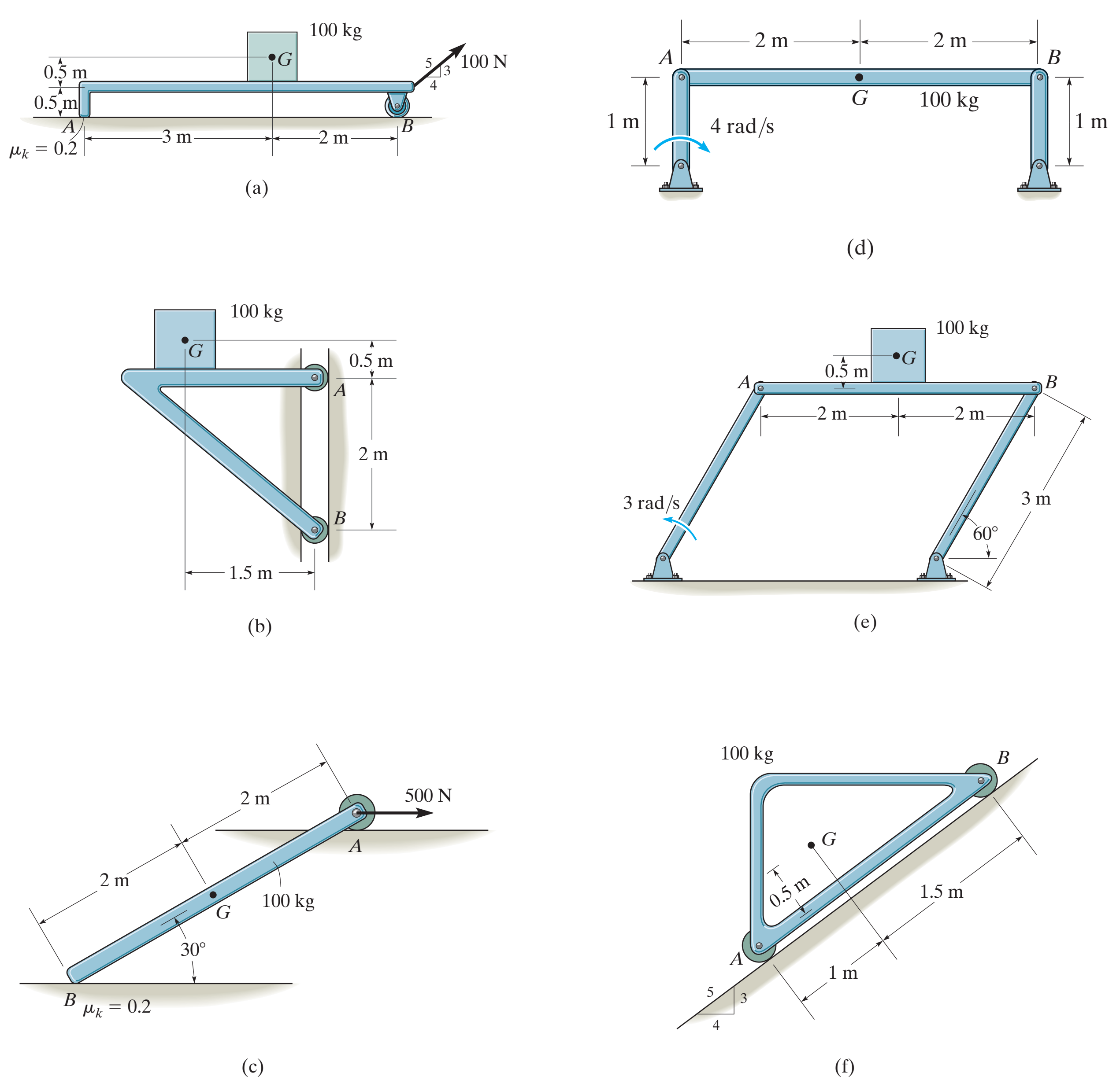
$$T_B = T_D = 1.32 \text{ kN} \qquad \textit{Ans.}$$

$$(a_G)_t = 4.905 \text{ m/s}^2$$

NOTE: It is also possible to apply the equations of motion along horizontal and vertical x, y axes, but the solution becomes more involved.

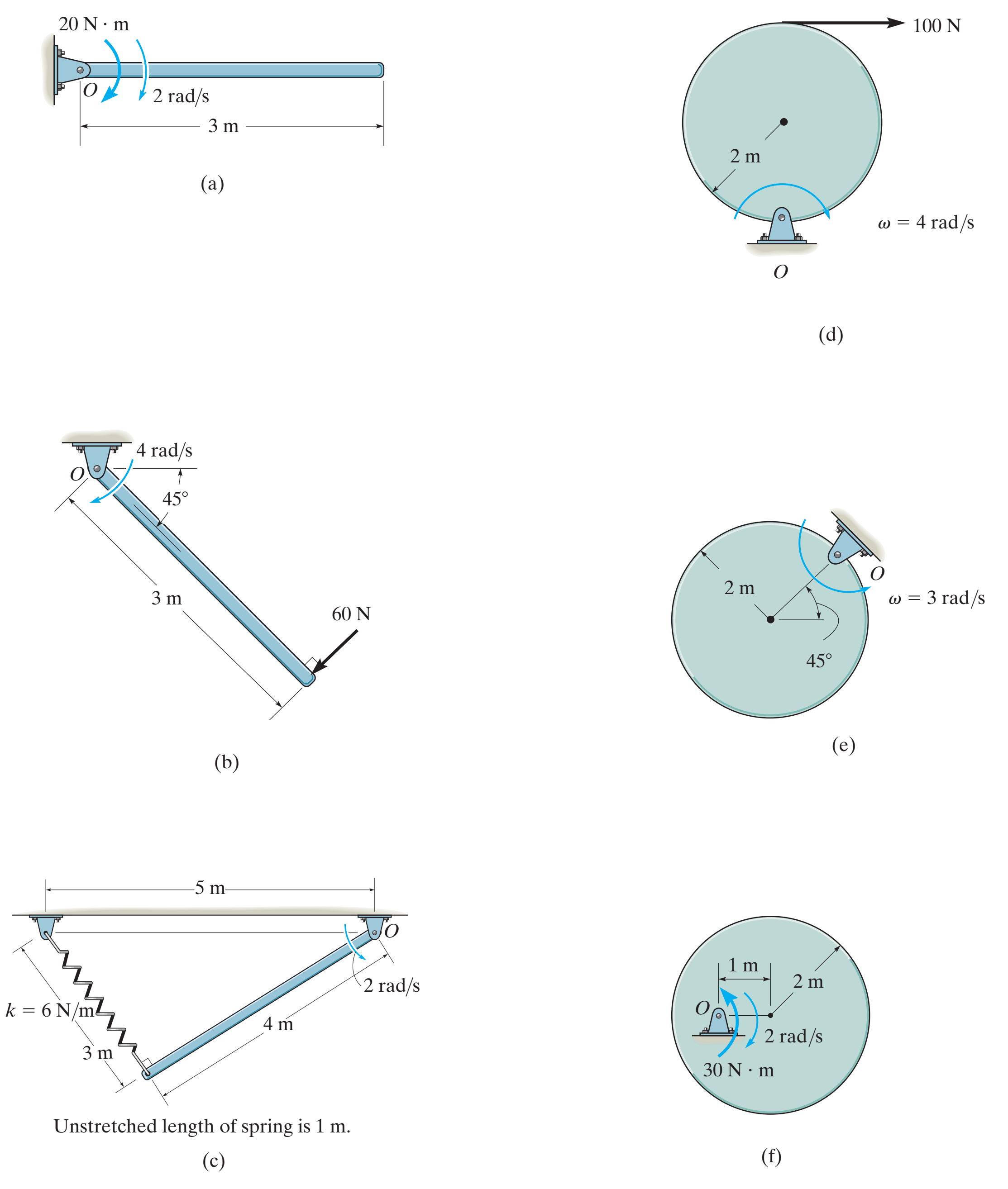
PRELIMINARY PROBLEMS

P17–1. Draw the free-body and kinetic diagrams of the object AB.



Prob. P17–1

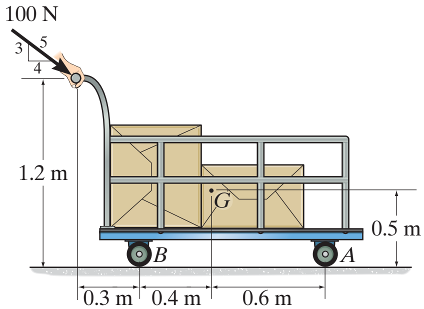
P17–2. Draw the free-body and kinetic diagrams of the 100-kg object.



Prob. P17–2

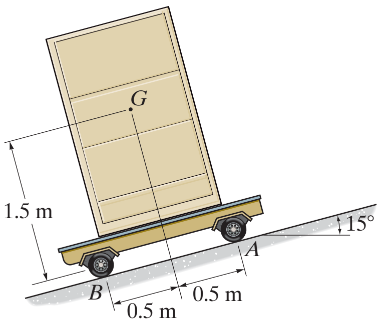
FUNDAMENTAL PROBLEMS

F17–1. The cart and its load have a total mass of 100 kg. Determine the acceleration of the cart and the normal reactions on the pair of wheels at *A* and *B*. Neglect the mass of the wheels.



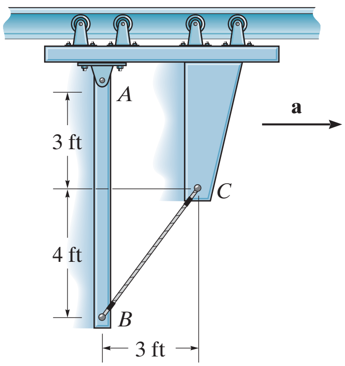
Prob. F17–1

F17–2. If the 80-kg cabinet is allowed to roll down the inclined plane, determine the acceleration of the cabinet and the normal reactions on the pair of rollers at *A* and *B* that have negligible mass.



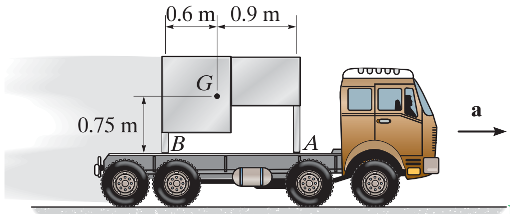
Prob. F17–2

F17–3. The 20-lb link *AB* is pinned to a moving frame at *A* and held in a vertical position by means of a string *BC* which can support a maximum tension of 10 lb. Determine the maximum acceleration of the frame without breaking the string. What are the corresponding components of reaction at the pin *A*?



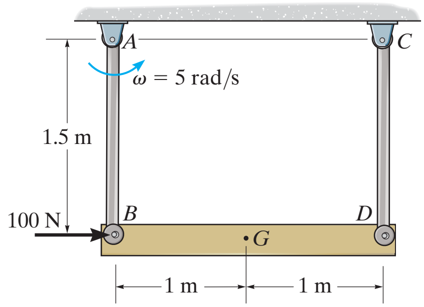
Prob. F17–3

F17–4. Determine the maximum acceleration of the truck without causing the assembly to move relative to the truck. Also what is the corresponding normal reaction on legs *A* and *B*? The 100-kg table has a mass center at *G* and the coefficient of static friction between the legs of the table and the bed of the truck is $\mu_s = 0.2$.



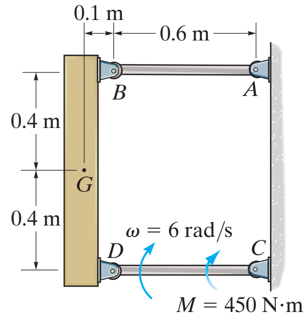
Prob. F17–4

F17–5. At the instant shown both rods of negligible mass swing with a counterclockwise angular velocity of $\omega = 5$ rad/s, while the 50-kg bar is subjected to the 100-N horizontal force. Determine the tension developed in the rods and the angular acceleration of the rods at this instant.



Prob. F17–5

F17–6. At the instant shown, link *CD* rotates with an angular velocity of $\omega = 6$ rad/s. If it is subjected to a couple moment $M = 450$ N·m, determine the force developed in link *AB*, the horizontal and vertical component of reaction on pin *D*, and the angular acceleration of link *CD* at this instant. The block has a mass of 50 kg and center of mass at *G*. Neglect the mass of links *AB* and *CD*.

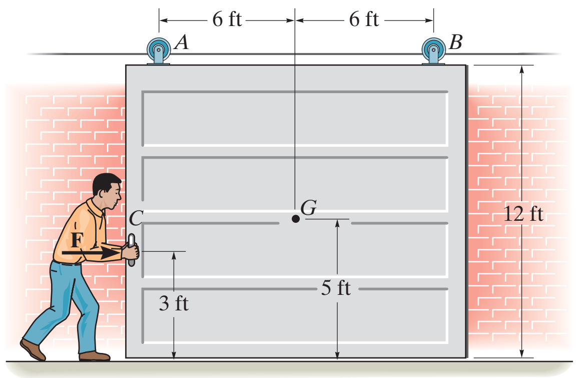


Prob. F17–6

PROBLEMS

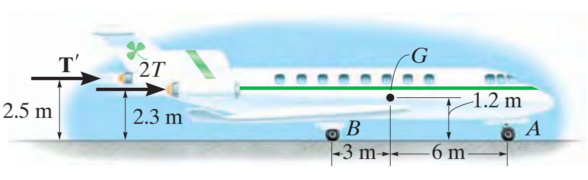
***17–24.** The door has a weight of 200 lb and a center of gravity at G. Determine how far the door moves in 2 s, starting from rest, if a man pushes on it at C with a horizontal force $F = 30$ lb. Also, find the vertical reactions at the rollers A and B.

17–25. The door has a weight of 200 lb and a center of gravity at G. Determine the constant force F that must be applied to the door to push it open 12 ft to the right in 5 s, starting from rest. Also, find the vertical reactions at the rollers A and B.



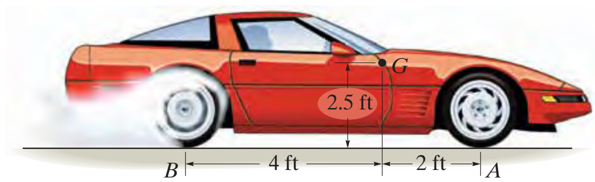
Probs. 17–24/25

17–26. The jet aircraft has a total mass of 22 Mg and a center of mass at G. Initially at take-off the engines provide a thrust $2T = 4$ kN and $T' = 1.5$ kN. Determine the acceleration of the plane and the normal reactions on the nose wheel at A and each of the *two* wing wheels located at B. Neglect the mass of the wheels and, due to low velocity, neglect any lift caused by the wings.



Prob. 17–26

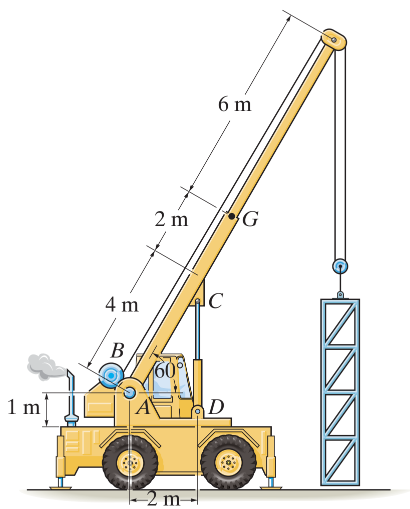
17–27. The sports car has a weight of 4500 lb and center of gravity at G. If it starts from rest it causes the rear wheels to slip as it accelerates. Determine how long it takes for it to reach a speed of 10 ft/s. Also, what are the normal reactions at each of the four wheels on the road? The coefficients of static and kinetic friction at the road are $\mu_s = 0.5$ and $\mu_k = 0.3$, respectively. Neglect the mass of the wheels.



Prob. 17–27

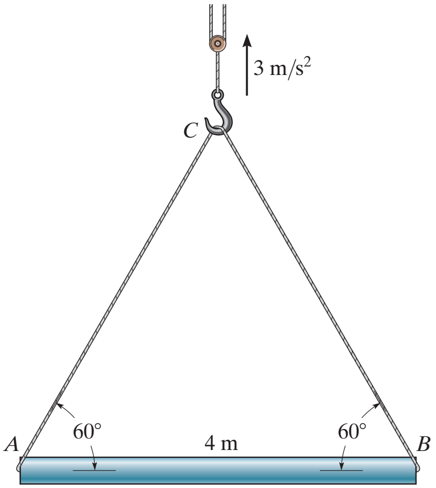
***17–28.** The assembly has a mass of 8 Mg and is hoisted using the boom and pulley system. If the winch at B draws in the cable with an acceleration of 2 m/s^2, determine the compressive force in the hydraulic cylinder needed to support the boom. The boom has a mass of 2 Mg and mass center at G.

17–29. The assembly has a mass of 4 Mg and is hoisted using the winch at B. Determine the greatest acceleration of the assembly so that the compressive force in the hydraulic cylinder supporting the boom does not exceed 180 kN. What is the tension in the supporting cable? The boom has a mass of 2 Mg and mass center at G.



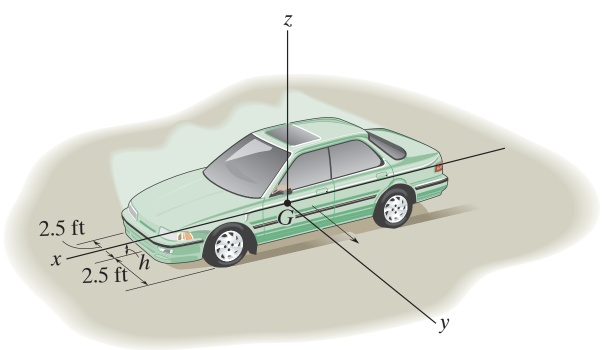
Probs. 17–28/29

17–30. The uniform girder AB has a mass of 8 Mg. Determine the internal axial, shear, and bending-moment loadings at the center of the girder if a crane gives it an upward acceleration of 3 m/s^2.



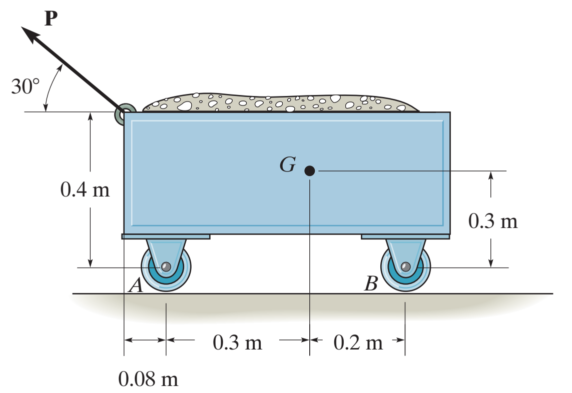
Prob. 17–30

17–31. A car having a weight of 4000 lb begins to skid and turn with the brakes applied to all four wheels. If the coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.8$, determine the maximum critical height h of the center of gravity G such that the car does not overturn. Tipping will begin to occur after the car rotates 90° from its original direction of motion and, as shown in the figure, undergoes *translation* while skidding. *Hint*: Draw a free-body diagram of the car viewed from the front. When tipping occurs, the normal reactions of the wheels on the right side (or passenger side) are zero.



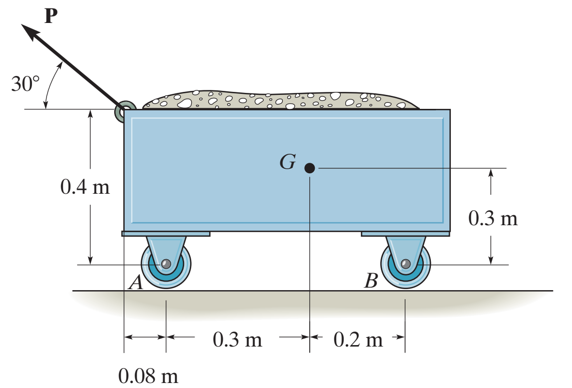
Prob. 17–31

***17–32.** A force of $P = 300$ N is applied to the 60-kg cart. Determine the reactions at both the wheels at A and both the wheels at B. Also, what is the acceleration of the cart? The mass center of the cart is at G.



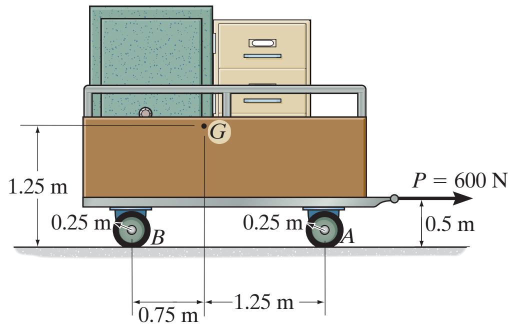
Prob. 17–32

17–33. Determine the largest force **P** that can be applied to the 60-kg cart, without causing one of the wheel reactions, either at A or at B, to be zero. Also, what is the acceleration of the cart? The mass center of the cart is at G.



Prob. 17–33

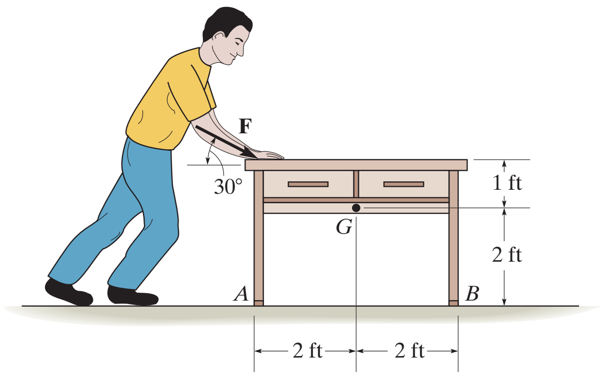
17–34. The trailer with its load has a mass of 150-kg and a center of mass at G. If it is subjected to a horizontal force of $P = 600$ N, determine the trailer's acceleration and the normal force on the pair of wheels at A and at B. The wheels are free to roll and have negligible mass.



Prob. 17–34

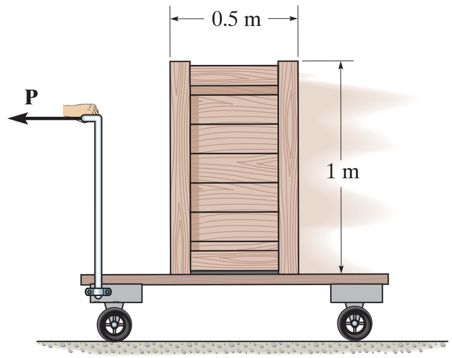
17–35. The desk has a weight of 75 lb and a center of gravity at G. Determine its initial acceleration if a man pushes on it with a force $F = 60$ lb. The coefficient of kinetic friction at A and B is $\mu_k = 0.2$.

***17–36.** The desk has a weight of 75 lb and a center of gravity at G. Determine the initial acceleration of a desk when the man applies enough force F to overcome the static friction at A and B. Also, find the vertical reactions on each of the two legs at A and at B. The coefficients of static and kinetic friction at A and B are $\mu_s = 0.5$ and $\mu_k = 0.2$, respectively.



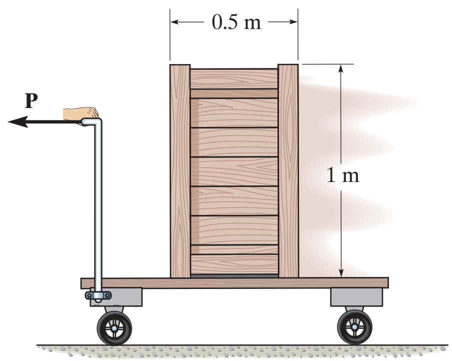
Probs. 17–35/36

17–37. The 150-kg uniform crate rests on the 10-kg cart. Determine the maximum force P that can be applied to the handle without causing the crate to tip on the cart. Slipping does not occur.



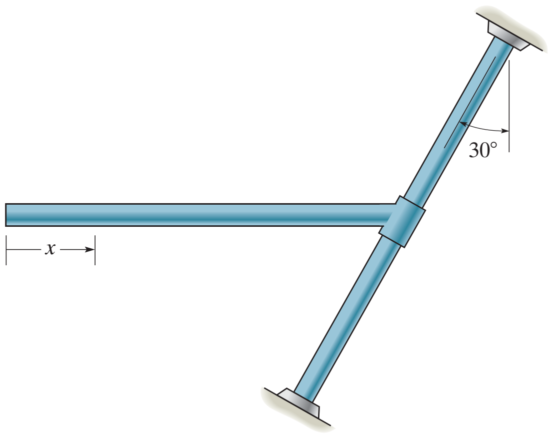
Prob. 17–37

17–38. The 150-kg uniform crate rests on the 10-kg cart. Determine the maximum force P that can be applied to the handle without causing the crate to slip or tip on the cart. The coefficient of static friction between the crate and cart is $\mu_s = 0.2$.



Prob. 17–38

17–39. The bar has a weight per length w and is supported by the smooth collar. If it is released from rest, determine the internal normal force, shear force, and bending moment in the bar as a function of x.

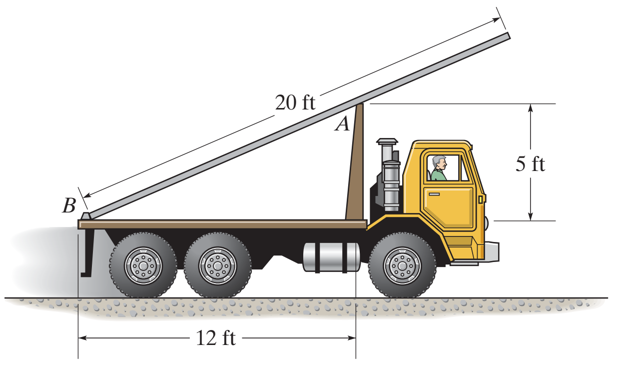


Prob. 17–39

17

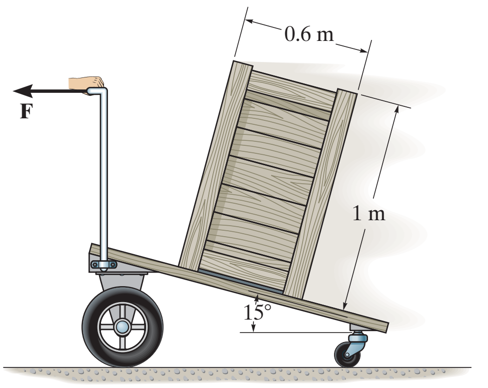
***17–40.** The smooth 180-lb pipe has a length of 20 ft and a negligible diameter. It is carried on a truck as shown. Determine the maximum acceleration which the truck can have without causing the normal reaction at A to be zero. Also determine the horizontal and vertical components of force which the truck exerts on the pipe at B.

17–41. The smooth 180-lb pipe has a length of 20 ft and a negligible diameter. It is carried on a truck as shown. If the truck accelerates at $a = 5\ \text{ft/s}^2$, determine the normal reaction at A and the horizontal and vertical components of force which the truck exerts on the pipe at B.



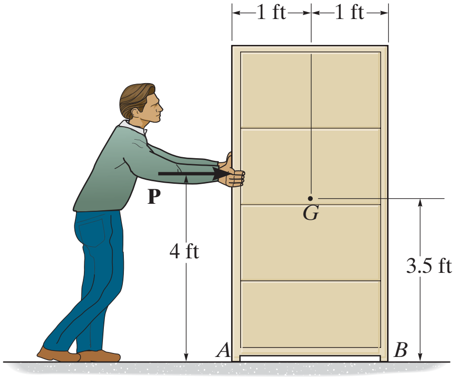
Probs. 17–40/41

17–42. The uniform crate has a mass of 50 kg and rests on the cart having an inclined surface. Determine the smallest acceleration that will cause the crate either to tip or slip relative to the cart. What is the magnitude of this acceleration? The coefficient of static friction between the crate and cart is $\mu_s = 0.5$.



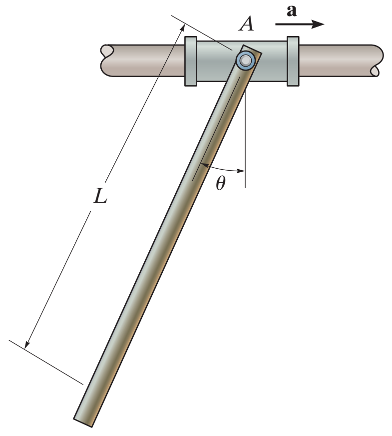
Prob. 17–42

17–43. Determine the acceleration of the 150-lb cabinet and the normal reaction under the legs A and B if $P = 35$ lb. The coefficients of static and kinetic friction between the cabinet and the plane are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively. The cabinet's center of gravity is located at G.



Prob. 17–43

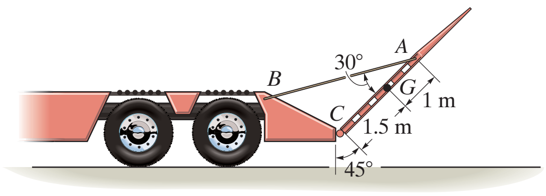
***17–44.** The uniform bar of mass m is pin connected to the collar, which slides along the smooth horizontal rod. If the collar is given a constant acceleration of $\mathbf{a}$, determine the bar's inclination angle θ. Neglect the collar's mass.



Prob. 17–44

17–45. The drop gate at the end of the trailer has a mass of 1.25 Mg and mass center at G. If it is supported by the cable AB and hinge at C, determine the tension in the cable when the truck begins to accelerate at 5 m/s^2. Also, what are the horizontal and vertical components of reaction at the hinge C?

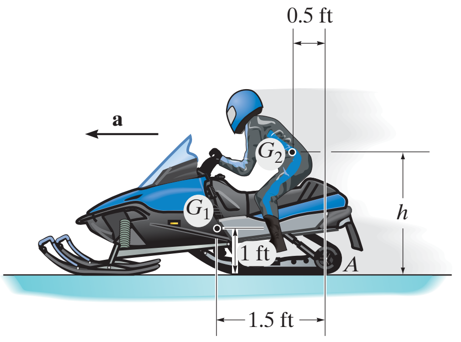
17–46. The drop gate at the end of the trailer has a mass of 1.25 Mg and mass center at G. If it is supported by the cable AB and hinge at C, determine the maximum deceleration of the truck so that the gate does not begin to rotate forward. What are the horizontal and vertical components of reaction at the hinge C?



Probs. 17–45/46

17–47. The snowmobile has a weight of 250 lb, centered at G_1, while the rider has a weight of 150 lb, centered at G_2. If the acceleration is $a = 20\ \text{ft/s}^2$, determine the maximum height h of G_2 of the rider so that the snowmobile's front skid does not lift off the ground. Also, what are the traction (horizontal) force and normal reaction under the rear tracks at A?

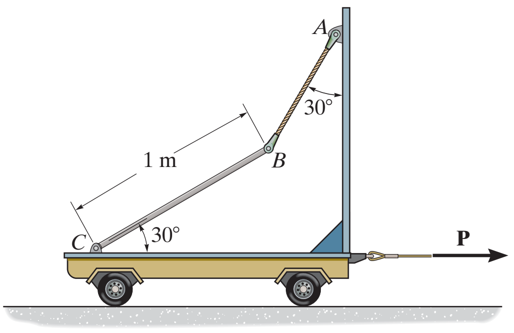
***17–48.** The snowmobile has a weight of 250 lb, centered at G_1, while the rider has a weight of 150 lb, centered at G_2. If $h = 3$ ft, determine the snowmobile's maximum permissible acceleration $\mathbf{a}$ so that its front skid does not lift off the ground. Also, find the traction (horizontal) force and the normal reaction under the rear tracks at A.



Probs. 17–47/48

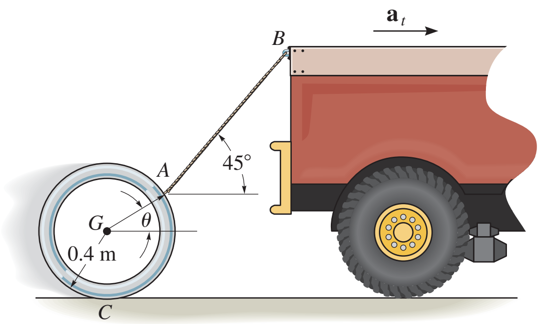
17–49. If the cart's mass is 30 kg and it is subjected to a horizontal force of $P = 90$ N, determine the tension in cord AB and the horizontal and vertical components of reaction on end C of the uniform 15-kg rod BC.

17–50. If the cart's mass is 30 kg, determine the horizontal force P that should be applied to the cart so that the cord AB just becomes slack. The uniform rod BC has a mass of 15 kg.



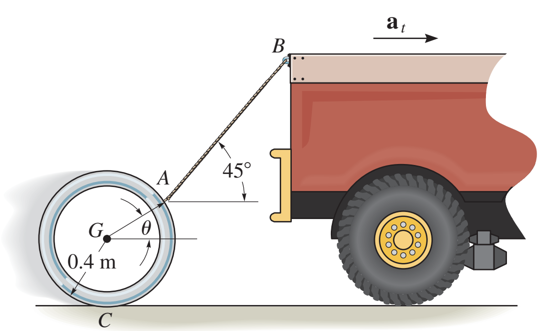
Probs. 17–49/50

17–51. The pipe has a mass of 800 kg and is being towed behind the truck. If the acceleration of the truck is $a_t = 0.5\ \text{m/s}^2$, determine the angle θ and the tension in the cable. The coefficient of kinetic friction between the pipe and the ground is $\mu_k = 0.1$.



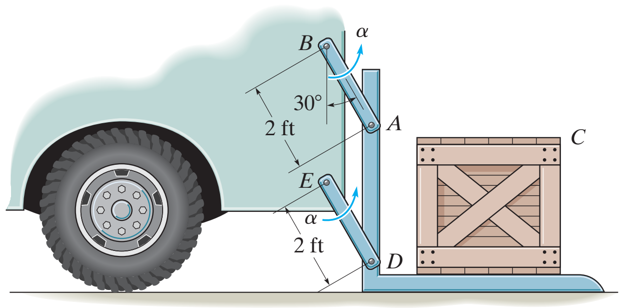
Prob. 17–51

***17–52.** The pipe has a mass of 800 kg and is being towed behind a truck. If the angle $\theta = 30°$, determine the acceleration of the truck and the tension in the cable. The coefficient of kinetic friction between the pipe and the ground is $\mu_k = 0.1$.



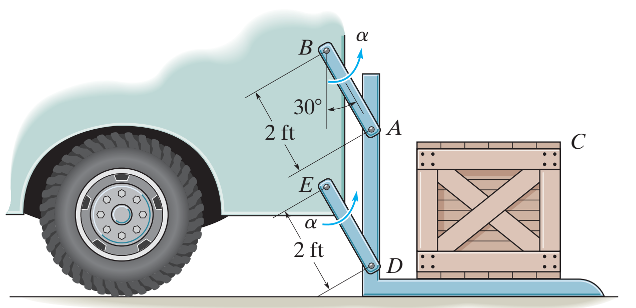
Prob. 17–52

17–53. The crate C has a weight of 150 lb and rests on the truck elevator for which the coefficient of static friction is $\mu_s = 0.4$. Determine the largest initial angular acceleration α, starting from rest, which the parallel links AB and DE can have without causing the crate to slip. No tipping occurs.



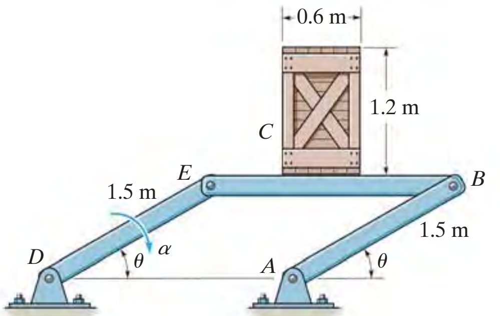
Prob. 17–53

17–54. The crate C has a weight of 150 lb and rests on the truck elevator. Determine the initial friction and normal force of the elevator on the crate if the parallel links are given an angular acceleration $\alpha = 2\ \text{rad/s}^2$ starting from rest.



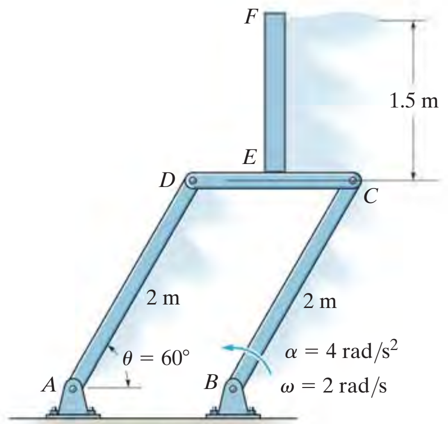
Prob. 17–54

17–55. The 100-kg uniform crate C rests on the elevator floor where the coefficient of static friction is $\mu_s = 0.4$. Determine the largest initial angular acceleration α, starting from rest at $\theta = 90°$, without causing the crate to slip. No tipping occurs.



Prob. 17–55

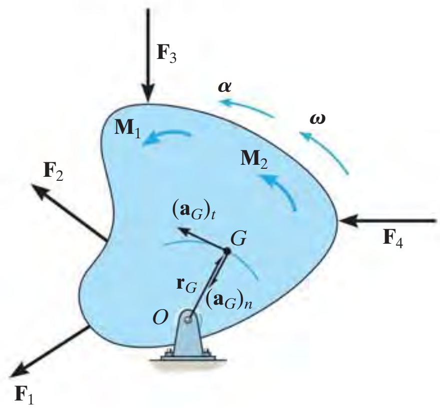
***17–56.** The two uniform 4-kg bars DC and EF are fixed (welded) together at E. Determine the normal force N_E, shear force V_E, and moment M_E, which DC exerts on EF at E if at the instant $\theta = 60°$ BC has an angular velocity $\omega = 2$ rad/s and an angular acceleration $\alpha = 4$ rad/s^2 as shown.



Prob. 17–56

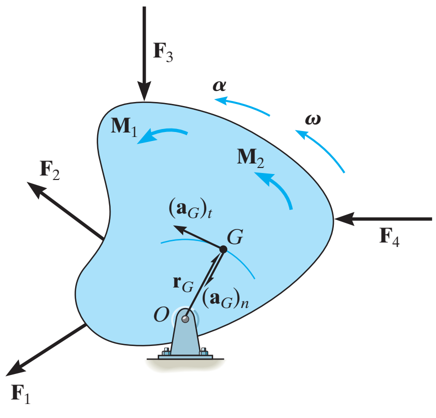
17.4 Equations of Motion: Rotation about a Fixed Axis

Consider the rigid body (or slab) shown in Fig. 17–13*a*, which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at O. The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body. Because the body's center of mass G moves around a *circular path*, the acceleration of this point is best represented by its tangential and normal components. The *tangential component of acceleration* has a *magnitude* of $(a_G)_t = \alpha r_G$ and must act in a *direction* which is *consistent* with the body's angular acceleration $\boldsymbol{\alpha}$. The *magnitude* of the *normal component of acceleration* is $(a_G)_n = \omega^2 r_G$. This component is *always directed* from point G to O, regardless of the rotational sense of $\boldsymbol{\omega}$.



(a)

Fig. 17–13



(a)

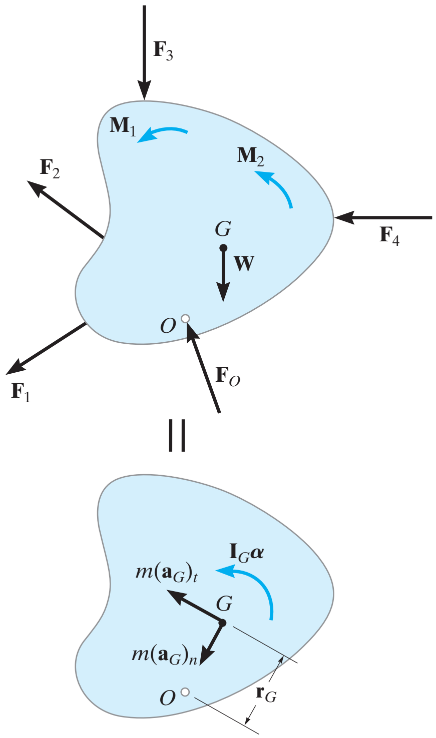
The free-body and kinetic diagrams for the body are shown in Fig. 17–13*b*. The two components $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$, shown on the kinetic diagram, are associated with the tangential and normal components of acceleration of the body's mass center. The $I_G\boldsymbol{\alpha}$ vector acts in the same *direction* as $\boldsymbol{\alpha}$ and has a *magnitude* of $I_G\alpha$, where I_G is the body's moment of inertia calculated about an axis which is perpendicular to the page and passes through G. From the derivation given in Sec. 17.2, the equations of motion which apply to the body can be written in the form

$$\begin{aligned}\Sigma F_n &= m(a_G)_n = m\omega^2 r_G\\ \Sigma F_t &= m(a_G)_t = m\alpha r_G\\ \Sigma M_G &= I_G\alpha\end{aligned} \tag{17–14}$$

The moment equation can be replaced by a moment summation about any arbitrary point P on or off the body provided one accounts for the moments $\Sigma(\mathcal{M}_k)_P$ produced by $I_G\boldsymbol{\alpha}$, $m(\mathbf{a}_G)_t$, and $m(\mathbf{a}_G)_n$ about the point.

Moment Equation About Point *O*. Often it is convenient to sum moments about the pin at O in order to eliminate the *unknown* force $\mathbf{F}_O$. From the kinetic diagram, Fig. 17–13*b*, this requires

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad \Sigma M_O = r_G m(a_G)_t + I_G\alpha \tag{17–15}$$



(b)

Fig. 17–13 (cont.)

Note that the moment of $m(\mathbf{a}_G)_n$ is not included here since the line of action of this vector passes through O. Substituting $(a_G)_t = r_G\alpha$, we may rewrite the above equation as $\circlearrowleft + \Sigma M_O = (I_G + mr_G^2)\alpha$. From the parallel-axis theorem, $I_O = I_G + md^2$, and therefore the term in parentheses represents the *moment of inertia of the body about the fixed axis of rotation passing through O*.* Consequently, we can write the three equations of motion for the body as

$$\begin{aligned}\Sigma F_n &= m(a_G)_n = m\omega^2 r_G\\ \Sigma F_t &= m(a_G)_t = m\alpha r_G\\ \Sigma M_O &= I_O\alpha\end{aligned} \tag{17–16}$$

When using these equations, remember that "$I_O\alpha$" accounts for the "moment" of *both* $m(\mathbf{a}_G)_t$ *and* $I_G\boldsymbol{\alpha}$ about point O, Fig. 17–13*b*. In other words, $\Sigma M_O = \Sigma(\mathcal{M}_k)_O = I_O\alpha$, as indicated by Eqs. 17–15 and 17–16.

*The result $\Sigma M_O = I_O\alpha$ can also be obtained *directly* from Eq. 17–6 by selecting point P to coincide with O, realizing that $(a_P)_x = (a_P)_y = 0$.

Procedure for Analysis

Kinetic problems which involve the rotation of a body about a fixed axis can be solved using the following procedure.

Free-Body Diagram.

- Establish the inertial *n, t* coordinate system and specify the direction and sense of the accelerations $(\mathbf{a}_G)_n$ and $(\mathbf{a}_G)_t$ and the angular acceleration $\boldsymbol{\alpha}$ of the body. Recall that $(\mathbf{a}_G)_t$ must act in a direction which is in accordance with the rotational sense of $\boldsymbol{\alpha}$, whereas $(\mathbf{a}_G)_n$ always acts toward the axis of rotation, point O.
- Draw the free-body diagram to account for all the external forces and couple moments that act on the body.
- Determine the moment of inertia I_G or I_O.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, i.e., P is a point other than G or O, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_n$, $m(\mathbf{a}_G)_t$, and $I_G\boldsymbol{\alpha}$ when writing the terms for the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- If moments are summed about the body's mass center, G, then $\Sigma M_G = I_G\alpha$, since $(m\mathbf{a}_G)_t$ and $(m\mathbf{a}_G)_n$ create no moment about G.
- If moments are summed about the pin support O on the axis of rotation, then $(m\mathbf{a}_G)_n$ creates no moment about O, and it can be shown that $\Sigma M_O = I_O\alpha$.

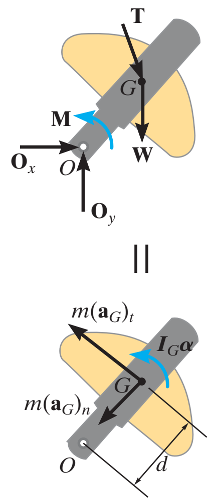
Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the angular acceleration is variable, use

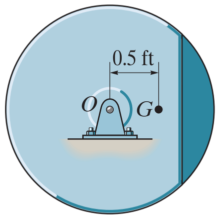
$$\alpha = \frac{d\omega}{dt} \qquad \alpha\, d\theta = \omega\, d\omega \qquad \omega = \frac{d\theta}{dt}$$

- If the angular acceleration is constant, use

$$\omega = \omega_0 + \alpha_c t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$



The crank on the oil-pumping rig undergoes rotation about a fixed axis which is caused by a driving torque **M** of the motor. The loadings shown on the free-body diagram cause the effects shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = I_G\alpha$. However, if moments are summed about point O, noting that $(a_G)_t = \alpha d$, then $\circlearrowleft\!+\Sigma M_O = I_G\alpha + m(a_G)_t d + m(a_G)_n(0) = (I_G + md^2)\alpha = I_O\alpha$. (© R.C. Hibbeler)



(a)

The unbalanced 50-lb flywheel shown in Fig. 17–14*a* has a radius of gyration of $k_G = 0.6$ ft about an axis passing through its mass center G. If it is released from rest, determine the horizontal and vertical components of reaction at the pin O.

SOLUTION

Free-Body and Kinetic Diagrams. Since G moves in a circular path, it will have both normal and tangential components of acceleration. Also, since α, which is caused by the flywheel's weight, acts clockwise, the tangential component of acceleration must act downward. Why? Since $\omega = 0$, only $m(a_G)_t = m\alpha r_G$ and $I_G\alpha$ are shown on the kinetic diagram in Fig. 17–14*b*. Here, the moment of inertia about G is

$$I_G = mk_G^2 = (50\text{ lb}/32.2\text{ ft/s}^2)(0.6\text{ ft})^2 = 0.559\text{ slug}\cdot\text{ft}^2$$

The three unknowns are O_n, O_t, and α.

Equations of Motion.

$$\overset{+}{\leftarrow}\Sigma F_n = m\omega^2 r_G; \qquad O_n = 0 \qquad \textit{Ans.}$$

$$+\downarrow\Sigma F_t = m\alpha r_G; \qquad -O_t + 50\text{ lb} = \left(\frac{50\text{ lb}}{32.2\text{ ft/s}^2}\right)(\alpha)(0.5\text{ ft}) \qquad (1)$$

$$\circlearrowright +\Sigma M_G = I_G\alpha; \qquad O_t(0.5\text{ ft}) = (0.5590\text{ slug}\cdot\text{ft}^2)\alpha$$

Solving,

$$\alpha = 26.4\text{ rad/s}^2 \quad O_t = 29.5\text{ lb} \qquad \textit{Ans.}$$

Moments can also be summed about point O in order to eliminate $\mathbf{O}_n$ and $\mathbf{O}_t$ and thereby obtain a *direct solution* for $\boldsymbol{\alpha}$, Fig. 17–14*b*. This can be done in one of *two* ways.

$$\circlearrowright +\Sigma M_O = \Sigma(\mathcal{M}_k)_O;$$

$$(50\text{ lb})(0.5\text{ ft}) = (0.5590\text{ slug}\cdot\text{ft}^2)\alpha + \left[\left(\frac{50\text{ lb}}{32.2\text{ ft/s}^2}\right)\alpha(0.5\text{ ft})\right](0.5\text{ ft})$$

$$50\text{ lb}(0.5\text{ ft}) = 0.9472\alpha \qquad (2)$$

If $\Sigma M_O = I_O\alpha$ is applied, then by the parallel-axis theorem the moment of inertia of the flywheel about O is

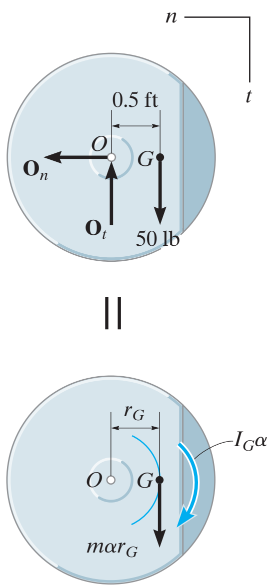
$$I_O = I_G + mr_G^2 = 0.559 + \left(\frac{50}{32.2}\right)(0.5)^2 = 0.9472\text{ slug}\cdot\text{ft}^2$$

Hence,

$$\circlearrowright +\Sigma M_O = I_O\alpha; \quad (50\text{ lb})(0.5\text{ ft}) = (0.9472\text{ slug}\cdot\text{ft}^2)\alpha$$

which is the same as Eq. 2. Solving for α and substituting into Eq. 1 yields the answer for O_t obtained previously.

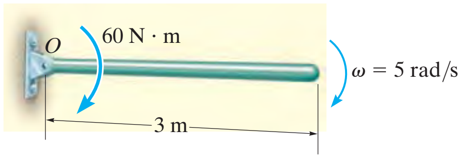
17



(b)

Fig. 17–14

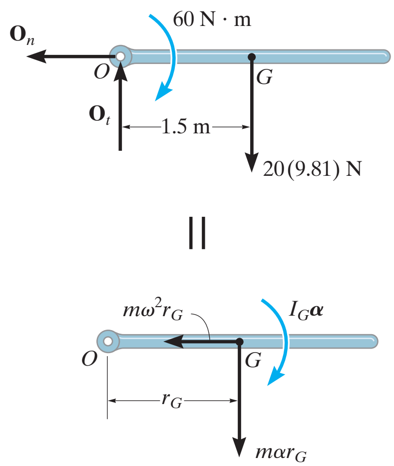
At the instant shown in Fig. 17–15*a*, the 20-kg slender rod has an angular velocity of $\omega = 5 \text{ rad/s}$. Determine the angular acceleration and the horizontal and vertical components of reaction of the pin on the rod at this instant.



(a)

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 17–15*b*. As shown on the kinetic diagram, point *G* moves around a circular path and so it has two components of acceleration. It is important that the tangential component $a_t = \alpha r_G$ act downward since it must be in accordance with the rotational sense of $\boldsymbol{\alpha}$. The three unknowns are O_n, O_t, and α.



(b)

Fig. 17–15

Equation of Motion.

$$\overset{+}{\leftarrow} \Sigma F_n = m\omega^2 r_G; \qquad O_n = (20 \text{ kg})(5 \text{ rad/s})^2(1.5 \text{ m})$$

$$+\downarrow \Sigma F_t = m\alpha r_G; \qquad -O_t + 20(9.81)\text{N} = (20 \text{ kg})(\alpha)(1.5 \text{ m})$$

$$\circlearrowright + \Sigma M_G = I_G\alpha; \qquad O_t(1.5 \text{ m}) + 60 \text{ N}\cdot\text{m} = \left[\tfrac{1}{12}(20 \text{ kg})(3 \text{ m})^2\right]\alpha$$

Solving

$$O_n = 750 \text{ N} \quad O_t = 19.05 \text{ N} \quad \alpha = 5.90 \text{ rad/s}^2 \qquad \textit{Ans.}$$

A more direct solution to this problem would be to sum moments about point *O* to eliminate $\mathbf{O}_n$ and $\mathbf{O}_t$ and obtain a *direct solution* for α. Here,

$$\circlearrowright + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \; 60 \text{ N}\cdot\text{m} + 20(9.81) \text{ N}(1.5 \text{ m}) =$$

$$\left[\tfrac{1}{12}(20 \text{ kg})(3 \text{ m})^2\right]\alpha + [20 \text{ kg}(\alpha)(1.5 \text{ m})](1.5 \text{ m})$$

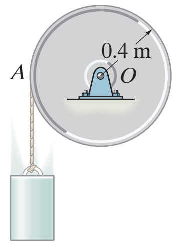
$$\alpha = 5.90 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Also, since $I_O = \frac{1}{3}ml^2$ for a slender rod, we can apply

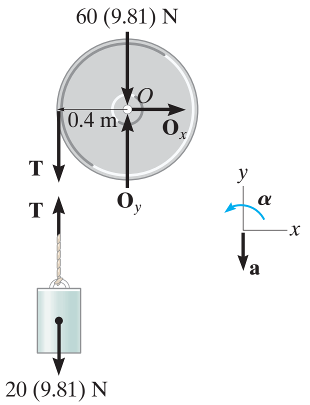
$$\circlearrowright + \Sigma M_O = I_O\alpha; \; 60 \text{ N}\cdot\text{m} + 20(9.81) \text{ N}(1.5 \text{ m}) = \left[\tfrac{1}{3}(20 \text{ kg})(3 \text{ m})^2\right]\alpha$$

$$\alpha = 5.90 \text{ rad/s}^2 \qquad \textit{Ans.}$$

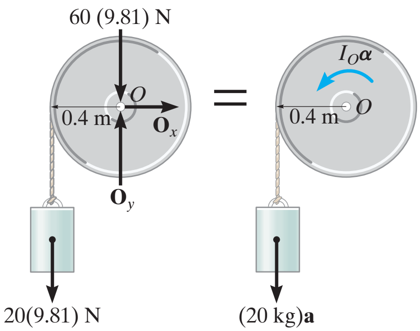
NOTE: By comparison, the last equation provides the simplest solution for α and *does not* require use of the kinetic diagram.



(a)



(b)



(c)

Fig. 17–16

The drum shown in Fig. 17–16*a* has a mass of 60 kg and a radius of gyration $k_O = 0.25$ m. A cord of negligible mass is wrapped around the periphery of the drum and attached to a block having a mass of 20 kg. If the block is released, determine the drum's angular acceleration.

SOLUTION I

Free-Body Diagram. Here we will consider the drum and block separately, Fig. 17–16*b*. Assuming the block accelerates *downward* at **a**, it creates a *counterclockwise* angular acceleration $\boldsymbol{\alpha}$ of the drum. The moment of inertia of the drum is

$$I_O = mk_O^2 = (60\text{ kg})(0.25\text{ m})^2 = 3.75\text{ kg}\cdot\text{m}^2$$

There are five unknowns, namely O_x, O_y, T, a, and α.

Equations of Motion. Applying the translational equations of motion $\Sigma F_x = m(a_G)_x$ and $\Sigma F_y = m(a_G)_y$ to the drum is of no consequence to the solution, since these equations involve the unknowns O_x and O_y. Thus, for the drum and block, respectively,

$$\circlearrowleft + \Sigma M_O = I_O\alpha; \qquad T(0.4\text{ m}) = (3.75\text{ kg}\cdot\text{m}^2)\alpha \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad -20(9.81)\text{N} + T = -(20\text{ kg})a \qquad (2)$$

Kinematics. Since the point of contact A between the cord and drum has a tangential component of acceleration **a**, Fig. 17–16*a*, then

$$\circlearrowleft + a = \alpha r; \qquad a = \alpha(0.4\text{ m}) \qquad (3)$$

Solving the above equations,

$$T = 106\text{ N} \quad a = 4.52\text{ m/s}^2$$

$$\alpha = 11.3\text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

SOLUTION II

Free-Body and Kinetic Diagrams. The cable tension T can be eliminated from the analysis by considering the drum and block as a *single system*, Fig. 17–16*c*. The kinetic diagram is shown since moments will be summed about point O.

Equations of Motion. Using Eq. 3 and applying the moment equation about O to eliminate the unknowns O_x and O_y, we have

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad [20(9.81)\text{ N}]\,(0.4\text{ m}) =$$

$$(3.75\text{ kg}\cdot\text{m}^2)\alpha + [20\text{ kg}(\alpha\, 0.4\text{ m})](0.4\text{ m})$$

$$\alpha = 11.3\text{ rad/s}^2 \qquad \textit{Ans.}$$

NOTE: If the block were *removed* and a force of 20(9.81) N were applied to the cord, show that $\alpha = 20.9\text{ rad/s}^2$. This value is larger since the block has an inertia, or resistance to acceleration.

The slender rod shown in Fig. 17–17*a* has a mass m and length l and is released from rest when $\theta = 0°$. Determine the horizontal and vertical components of force which the pin at A exerts on the rod at the instant $\theta = 90°$.

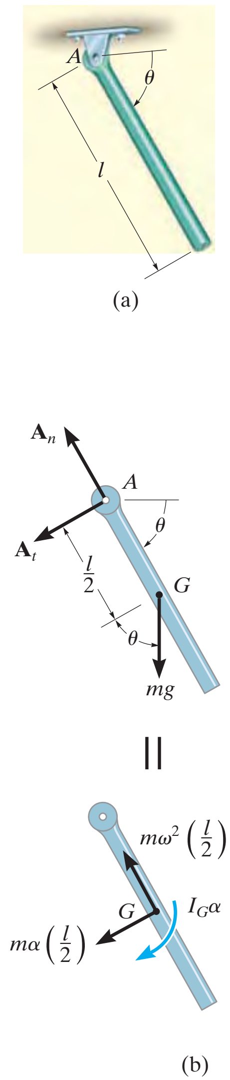


Fig. 17–17

SOLUTION

Free-Body and Kinetic Diagrams. The free-body diagram for the rod in the general position θ is shown in Fig. 17–17*b*. For convenience, the force components at A are shown acting in the n and t directions. Note that $\boldsymbol{\alpha}$ acts clockwise and so $(\mathbf{a}_G)_t$ acts in the $+t$ direction.

The moment of inertia of the rod about point A is $I_A = \frac{1}{3}ml^2$.

Equations of Motion. Moments will be summed about A in order to eliminate A_n and A_t.

$$+\nwarrow\Sigma F_n = m\omega^2 r_G; \qquad A_n - mg\sin\theta = m\omega^2(l/2) \qquad (1)$$

$$+\swarrow\Sigma F_t = m\alpha r_G; \qquad A_t + mg\cos\theta = m\alpha(l/2) \qquad (2)$$

$$\circlearrowright +\Sigma M_A = I_A\alpha; \qquad mg\cos\theta(l/2) = \left(\tfrac{1}{3}ml^2\right)\alpha \qquad (3)$$

Kinematics. For a given angle θ there are four unknowns in the above three equations: A_n, A_t, ω, and α. As shown by Eq. 3, α is *not constant;* rather, it depends on the position θ of the rod. The necessary fourth equation is obtained using kinematics, where α and ω can be related to θ by the equation

$$(\circlearrowright +) \qquad \omega\, d\omega = \alpha\, d\theta \qquad (4)$$

Note that the positive clockwise direction for this equation *agrees* with that of Eq. 3. This is important since we are seeking a simultaneous solution.

In order to solve for ω at $\theta = 90°$, eliminate α from Eqs. 3 and 4, which yields

$$\omega\, d\omega = (1.5g/l)\cos\theta\, d\theta$$

Since $\omega = 0$ at $\theta = 0°$, we have

$$\int_0^\omega \omega\, d\omega = (1.5g/l)\int_{0°}^{90°} \cos\theta\, d\theta$$

$$\omega^2 = 3g/l$$

Substituting this value into Eq. 1 with $\theta = 90°$ and solving Eqs. 1 to 3 yields

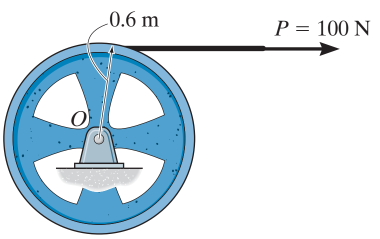
$$\alpha = 0$$

$$A_t = 0 \quad A_n = 2.5\, mg \qquad \textit{Ans.}$$

NOTE: If $\Sigma M_A = \Sigma(\mathcal{M}_k)_A$ is used, one must account for the moments of $I_G\boldsymbol{\alpha}$ and $m(\mathbf{a}_G)_t$ about A.

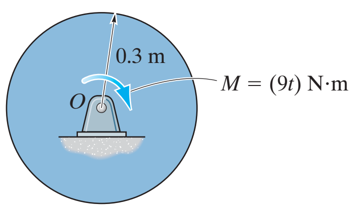
FUNDAMENTAL PROBLEMS

F17–7. The 100-kg wheel has a radius of gyration about its center O of $k_O = 500$ mm. If the wheel starts from rest, determine its angular velocity in $t = 3$ s.



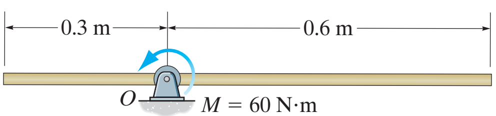
Prob. F17–7

F17–8. The 50-kg disk is subjected to the couple moment of $M = (9t)$ N·m, where t is in seconds. Determine the angular velocity of the disk when $t = 4$ s starting from rest.



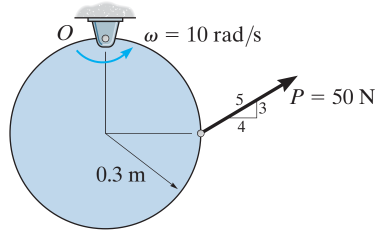
Prob. F17–8

F17–9. At the instant shown, the uniform 30-kg slender rod has a counterclockwise angular velocity of $\omega = 6$ rad/s. Determine the tangential and normal components of reaction of pin O on the rod and the angular acceleration of the rod at this instant.



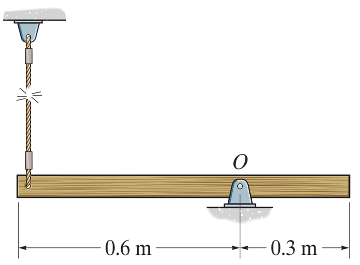
Prob. F17–9

F17–10. At the instant shown, the 30-kg disk has a counterclockwise angular velocity of $\omega = 10$ rad/s. Determine the tangential and normal components of reaction of the pin O on the disk and the angular acceleration of the disk at this instant.



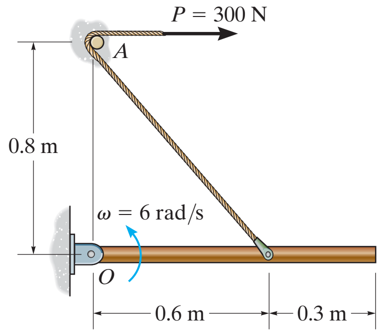
Prob. F17–10

F17–11. The uniform slender rod has a mass of 15 kg. Determine the horizontal and vertical components of reaction at the pin O, and the angular acceleration of the rod just after the cord is cut.



Prob. F17–11

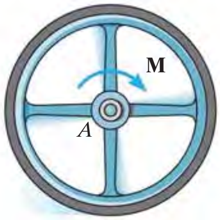
F17–12. The uniform 30-kg slender rod is being pulled by the cord that passes over the small smooth peg at A. If the rod has a counterclockwise angular velocity of $\omega = 6$ rad/s at the instant shown, determine the tangential and normal components of reaction at the pin O and the angular acceleration of the rod.



Prob. F17–12

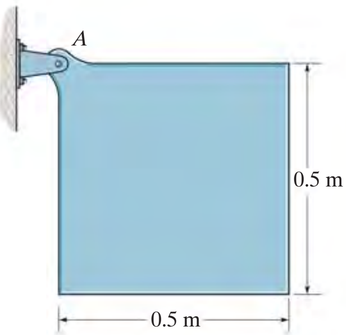
PROBLEMS

17–57. The 10-kg wheel has a radius of gyration $k_A = 200$ mm. If the wheel is subjected to a moment $M = (5t)$ N · m, where t is in seconds, determine its angular velocity when $t = 3$ s starting from rest. Also, compute the reactions which the fixed pin A exerts on the wheel during the motion.



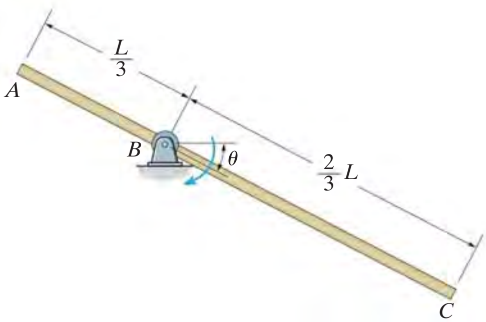
Prob. 17–57

17–58. The uniform 24-kg plate is released from rest at the position shown. Determine its initial angular acceleration and the horizontal and vertical reactions at the pin A.



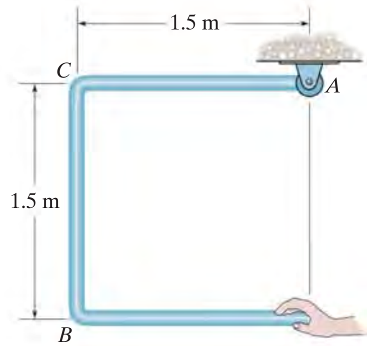
Prob. 17–58

17–59. The uniform slender rod has a mass m. If it is released from rest when $\theta = 0°$, determine the magnitude of the reactive force exerted on it by pin B when $\theta = 90°$.



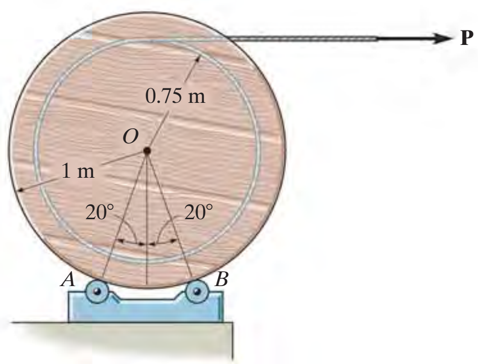
Prob. 17–59

***17–60.** The bent rod has a mass of 2 kg/m. If it is released from rest in the position shown, determine its initial angular acceleration and the horizontal and vertical components of reaction at A.



Prob. 17–60

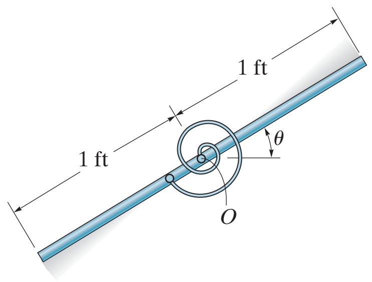
17–61. If a horizontal force of $P = 100$ N is applied to the 300-kg reel of cable, determine its initial angular acceleration. The reel rests on rollers at A and B and has a radius of gyration of $k_O = 0.6$ m.



Prob. 17–61

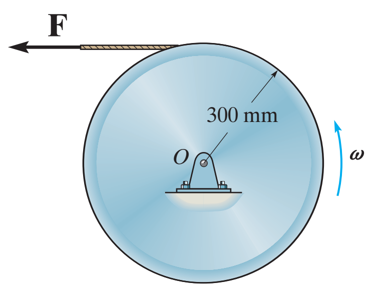
17–62. The 10-lb bar is pinned at its center O and connected to a torsional spring. The spring has a stiffness $k = 5\ \text{lb}\cdot\text{ft/rad}$, so that the torque developed is $M = (5\theta)\ \text{lb}\cdot\text{ft}$, where θ is in radians. If the bar is released from rest when it is vertical at $\theta = 90°$, determine its angular velocity at the instant $\theta = 0°$.

17–63. The 10-lb bar is pinned at its center O and connected to a torsional spring. The spring has a stiffness $k = 5\ \text{lb}\cdot\text{ft/rad}$, so that the torque developed is $M = (5\theta)\ \text{lb}\cdot\text{ft}$, where θ is in radians. If the bar is released from rest when it is vertical at $\theta = 90°$, determine its angular velocity at the instant $\theta = 45°$.



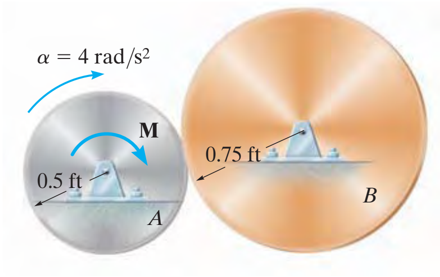
Probs. 17–62/63

***17–64.** A cord is wrapped around the outer surface of the 8-kg disk. If a force of $F = (\tfrac{1}{4}\theta^2)$ N, where θ is in radians, is applied to the cord, determine the disk's angular acceleration when it has turned 5 revolutions. The disk has an initial angular velocity of $\omega_0 = 1\ \text{rad/s}$.



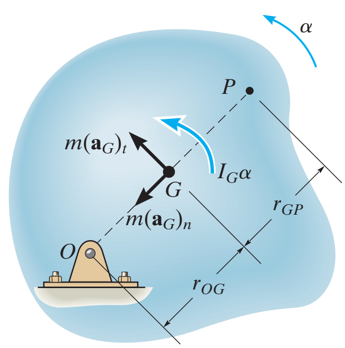
Prob. 17–64

17–65. Disk A has a weight of 5 lb and disk B has a weight of 10 lb. If no slipping occurs between them, determine the couple moment $\mathbf{M}$ which must be applied to disk A to give it an angular acceleration of $4\ \text{rad/s}^2$.



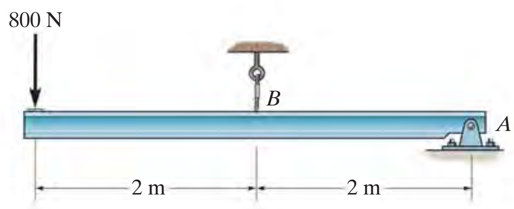
Prob. 17–65

17–66. The kinetic diagram representing the general rotational motion of a rigid body about a fixed axis passing through O is shown in the figure. Show that $I_G\boldsymbol{\alpha}$ may be eliminated by moving the vectors $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$ to point P, located a distance $r_{GP} = k_G^2/r_{OG}$ from the center of mass G of the body. Here k_G represents the radius of gyration of the body about an axis passing through G. The point P is called the *center of percussion* of the body.



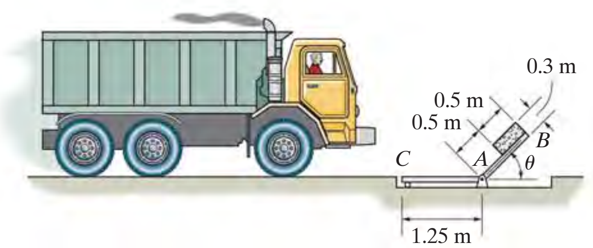
Prob. 17–66

17–67. If the cord at B suddenly fails, determine the horizontal and vertical components of the initial reaction at the pin A, and the angular acceleration of the 120-kg beam. Treat the beam as a uniform slender rod.



Prob. 17–67

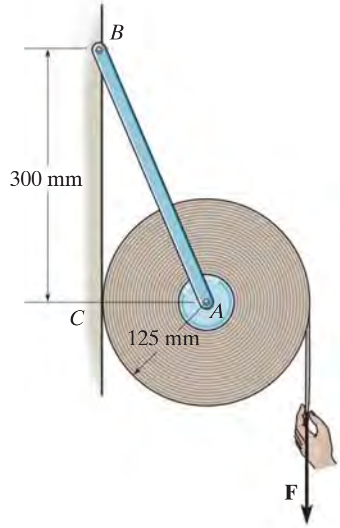
***17–68.** The device acts as a pop-up barrier to prevent the passage of a vehicle. It consists of a 100-kg steel plate AC and a 200-kg counterweight solid concrete block located as shown. Determine the moment of inertia of the plate and block about the hinged axis through A. Neglect the mass of the supporting arms AB. Also, determine the initial angular acceleration of the assembly when it is released from rest at $\theta = 45°$.



Prob. 17–68

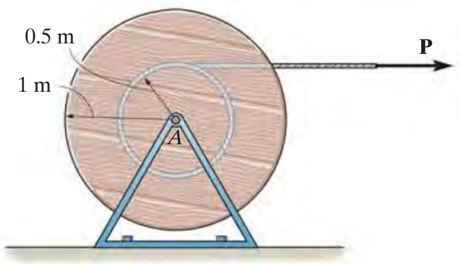
17–69. The 20-kg roll of paper has a radius of gyration $k_A = 90$ mm about an axis passing through point A. It is pin supported at both ends by two brackets AB. If the roll rests against a wall for which the coefficient of kinetic friction is $\mu_k = 0.2$ and a vertical force $F = 30$ N is applied to the end of the paper, determine the angular acceleration of the roll as the paper unrolls.

17–70. The 20-kg roll of paper has a radius of gyration $k_A = 90$ mm about an axis passing through point A. It is pin supported at both ends by two brackets AB. If the roll rests against a wall for which the coefficient of kinetic friction is $\mu_k = 0.2$, determine the constant vertical force F that must be applied to the roll to pull off 1 m of paper in $t = 3$ s starting from rest. Neglect the mass of paper that is removed.



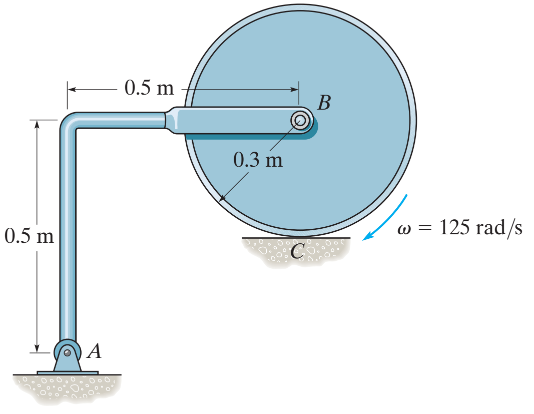
Probs. 17–69/70

17–71. The reel of cable has a mass of 400 kg and a radius of gyration of $k_A = 0.75$ m. Determine its angular velocity when $t = 2$ s, starting from rest, if the force $\mathbf{P} = (20t^2 + 80)$ N, when t is in seconds. Neglect the mass of the unwound cable, and assume it is always at a radius of 0.5 m.



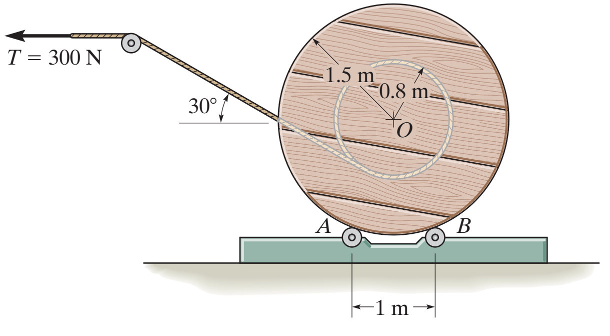
Prob. 17–71

***17–72.** The 30-kg disk is originally spinning at $\omega = 125$ rad/s. If it is placed on the ground, for which the coefficient of kinetic friction is $\mu_C = 0.5$, determine the time required for the motion to stop. What are the horizontal and vertical components of force which the member AB exerts on the pin at A during this time? Neglect the mass of AB.



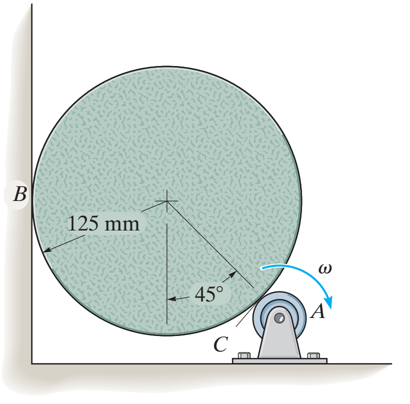
Prob. 17–72

17–73. Cable is unwound from a spool supported on small rollers at A and B by exerting a force $T = 300$ N on the cable. Compute the time needed to unravel 5 m of cable from the spool if the spool and cable have a total mass of 600 kg and a radius of gyration of $k_O = 1.2$ m. For the calculation, neglect the mass of the cable being unwound and the mass of the rollers at A and B. The rollers turn with no friction.



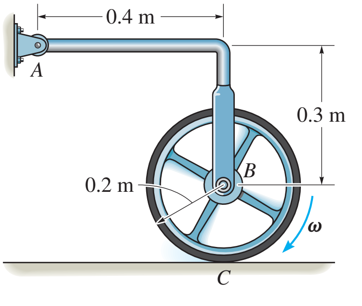
Prob. 17–73

17–74. The 5-kg cylinder is initially at rest when it is placed in contact with the wall B and the rotor at A. If the rotor always maintains a constant clockwise angular velocity $\omega = 6$ rad/s, determine the initial angular acceleration of the cylinder. The coefficient of kinetic friction at the contacting surfaces B and C is $\mu_k = 0.2$.



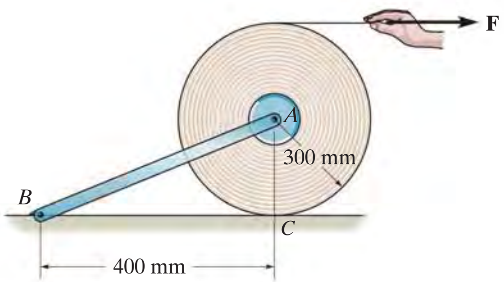
Prob. 17–74

17–75. The wheel has a mass of 25 kg and a radius of gyration $k_B = 0.15$ m. It is originally spinning at $\omega = 40$ rad/s. If it is placed on the ground, for which the coefficient of kinetic friction is $\mu_C = 0.5$, determine the time required for the motion to stop. What are the horizontal and vertical components of reaction which the pin at A exerts on AB during this time? Neglect the mass of AB.



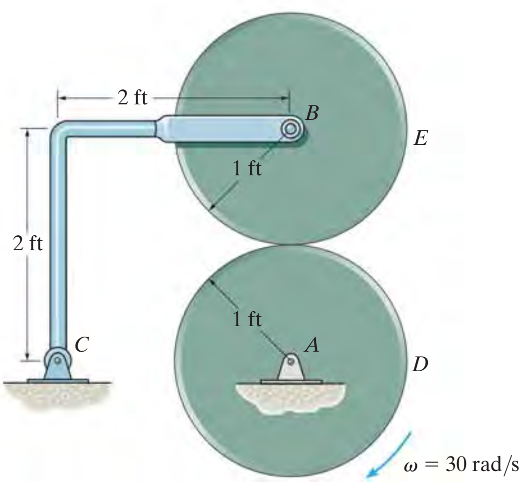
Prob. 17–75

***17–76.** The 20-kg roll of paper has a radius of gyration $k_A = 120$ mm about an axis passing through point A. It is pin supported at both ends by two brackets AB. The roll rests on the floor, for which the coefficient of kinetic friction is $\mu_k = 0.2$. If a horizontal force $F = 60$ N is applied to the end of the paper, determine the initial angular acceleration of the roll as the paper unrolls.



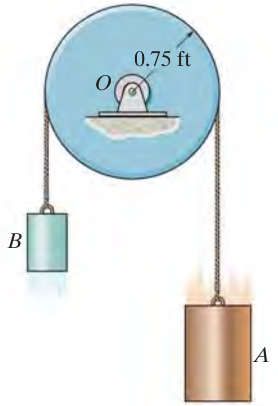
Prob. 17–76

17–77. Disk D turns with a constant clockwise angular velocity of 30 rad/s. Disk E has a weight of 60 lb and is initially at rest when it is brought into contact with D. Determine the time required for disk E to attain the same angular velocity as disk D. The coefficient of kinetic friction between the two disks is $\mu_k = 0.3$. Neglect the weight of bar BC.



Prob. 17–77

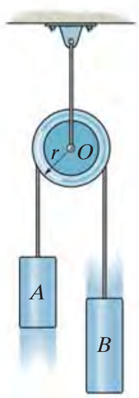
17–78. Two cylinders A and B, having a weight of 10 lb and 5 lb, respectively, are attached to the ends of a cord which passes over a 3-lb pulley (disk). If the cylinders are released from rest, determine their speed in $t = 0.5$ s. The cord does not slip on the pulley. Neglect the mass of the cord. *Suggestion*: Analyze the "system" consisting of both the cylinders and the pulley.



Prob. 17–78

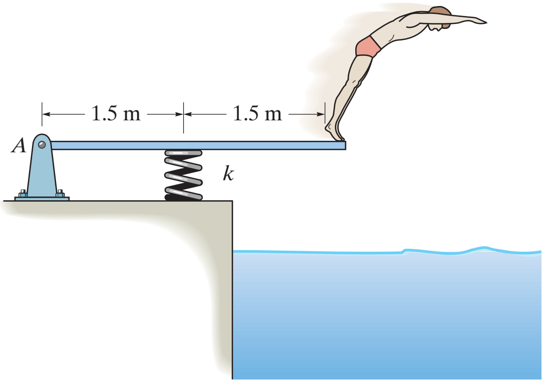
17–79. The two blocks A and B have a mass of 5 kg and 10 kg, respectively. If the pulley can be treated as a disk of mass 3 kg and radius 0.15 m, determine the acceleration of block A. Neglect the mass of the cord and any slipping on the pulley.

***17–80.** The two blocks A and B have a mass m_A and m_B, respectively, where $m_B > m_A$. If the pulley can be treated as a disk of mass M, determine the acceleration of block A. Neglect the mass of the cord and any slipping on the pulley.



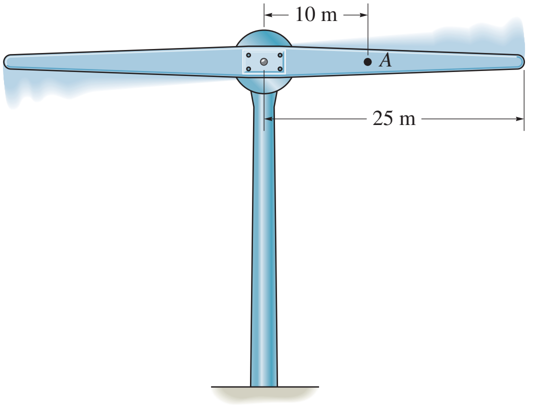
Probs. 17–79/80

17–81. Determine the angular acceleration of the 25-kg diving board and the horizontal and vertical components of reaction at the pin A the instant the man jumps off. Assume that the board is uniform and rigid, and that at the instant he jumps off the spring is compressed a maximum amount of 200 mm, $\omega = 0$, and the board is horizontal. Take $k = 7$ kN/m.



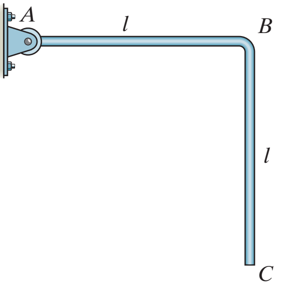
Prob. 17–81

17–82. The lightweight turbine consists of a rotor which is powered from a torque applied at its center. At the instant the rotor is horizontal it has an angular velocity of 15 rad/s and a clockwise angular acceleration of 8 rad/s^2. Determine the internal normal force, shear force, and moment at a section through A. Assume the rotor is a 50-m-long slender rod, having a mass of 3 kg/m.



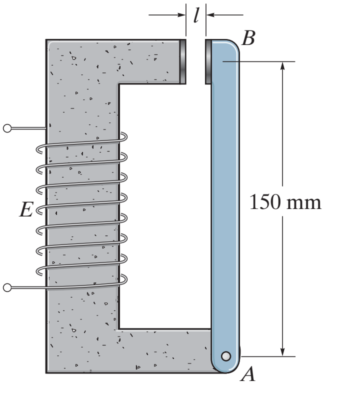
Prob. 17–82

17–83. The two-bar assembly is released from rest in the position shown. Determine the initial bending moment at the fixed joint B. Each bar has a mass m and length l.



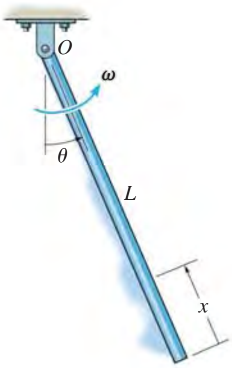
Prob. 17–83

***17–84.** The armature (slender rod) AB has a mass of 0.2 kg and can pivot about the pin at A. Movement is controlled by the electromagnet E, which exerts a horizontal attractive force on the armature at B of $F_B = (0.2(10^{-3})l^{-2})$ N, where l in meters is the gap between the armature and the magnet at any instant. If the armature lies in the horizontal plane, and is originally at rest, determine the speed of the contact at B the instant $l = 0.01$ m. Originally $l = 0.02$ m.



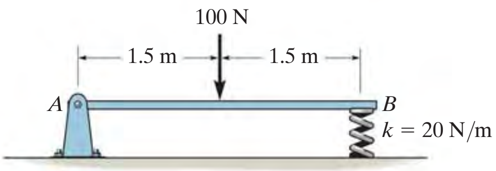
Prob. 17–84

17–85. The bar has a weight per length of w. If it is rotating in the vertical plane at a constant rate ω about point O, determine the internal normal force, shear force, and moment as a function of x and θ.



Prob. 17–85

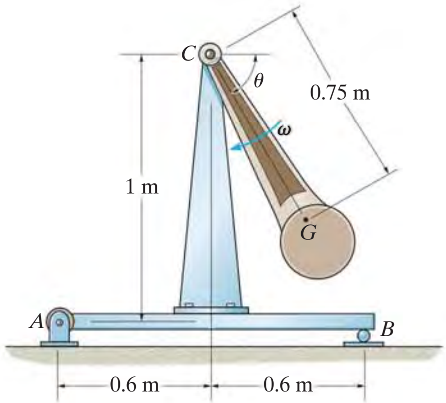
17–86. The 4-kg slender rod is initially supported horizontally by a spring at B and pin at A. Determine the angular acceleration of the rod and the acceleration of the rod's mass center at the instant the 100-N force is applied.



Prob. 17–86

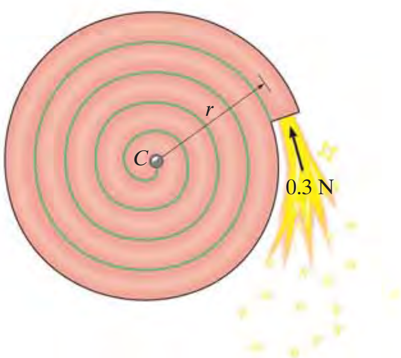
17–87. The 100-kg pendulum has a center of mass at G and a radius of gyration about G of $k_G = 250$ mm. Determine the horizontal and vertical components of reaction on the beam by the pin A and the normal reaction of the roller B at the instant $\theta = 90°$ when the pendulum is rotating at $\omega = 8$ rad/s. Neglect the weight of the beam and the support.

***17–88.** The 100-kg pendulum has a center of mass at G and a radius of gyration about G of $k_G = 250$ mm. Determine the horizontal and vertical components of reaction on the beam by the pin A and the normal reaction of the roller B at the instant $\theta = 0°$ when the pendulum is rotating at $\omega = 4$ rad/s. Neglect the weight of the beam and the support.



Probs. 17–87/88

17–89. The "Catherine wheel" is a firework that consists of a coiled tube of powder which is pinned at its center. If the powder burns at a constant rate of 20 g/s such as that the exhaust gases always exert a force having a constant magnitude of 0.3 N, directed tangent to the wheel, determine the angular velocity of the wheel when 75% of the mass is burned off. Initially, the wheel is at rest and has a mass of 100 g and a radius of $r = 75$ mm. For the calculation, consider the wheel to always be a thin disk.



Prob. 17–89

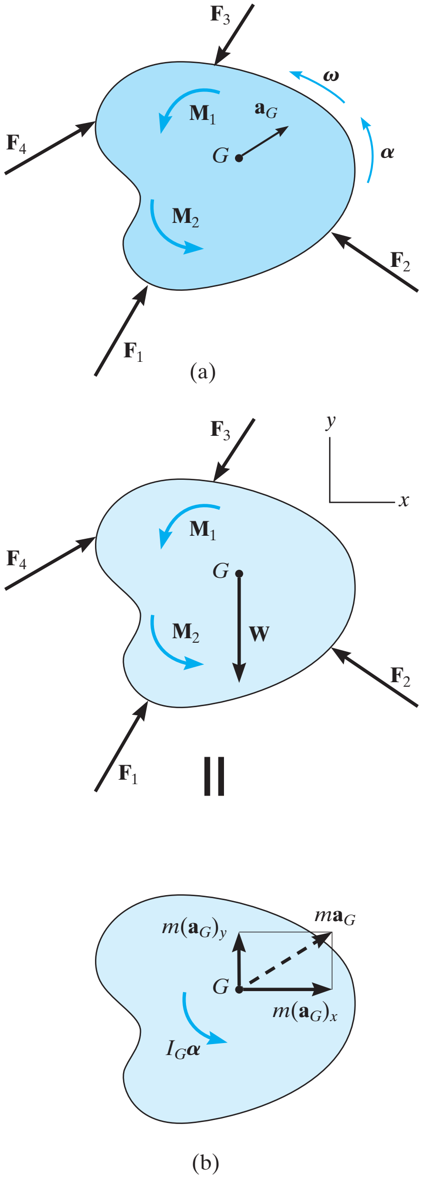


Fig. 17–18

17.5 Equations of Motion: General Plane Motion

The rigid body (or slab) shown in Fig. 17–18*a* is subjected to general plane motion caused by the externally applied force and couple-moment system. The free-body and kinetic diagrams for the body are shown in Fig. 17–18*b*. If an x and y inertial coordinate system is established as shown, the three equations of motion are

$$\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_G &= I_G\alpha \end{aligned} \tag{17–17}$$

In some problems it may be convenient to sum moments about a point P other than G in order to eliminate as many unknown forces as possible from the moment summation. When used in this more general case, the three equations of motion are

$$\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_P &= \Sigma(\mathcal{M}_k)_P \end{aligned} \tag{17–18}$$

Here $\Sigma(\mathcal{M}_k)_P$ represents the moment sum of $I_G\boldsymbol{\alpha}$ and $m\mathbf{a}_G$ (or its components) about P as determined by the data on the kinetic diagram.

Moment Equation About the *IC*. There is a particular type of problem that involves a uniform disk, or body of circular shape, that rolls on a rough surface *without slipping*, Fig. 17–19. If we sum the moments about the instantaneous center of zero velocity, then $\Sigma(\mathcal{M}_k)_{IC}$ becomes $I_{IC}\alpha$, so that

$$\Sigma M_{IC} = I_{IC}\alpha \tag{17–19}$$

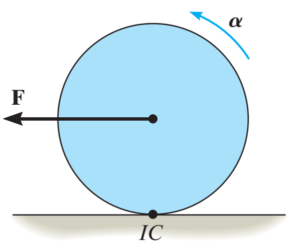


Fig. 17–19

This result compares with $\Sigma M_O = I_O\alpha$, which is used for a body pinned at point O, Eq. 17–16. See Prob. 17–90.

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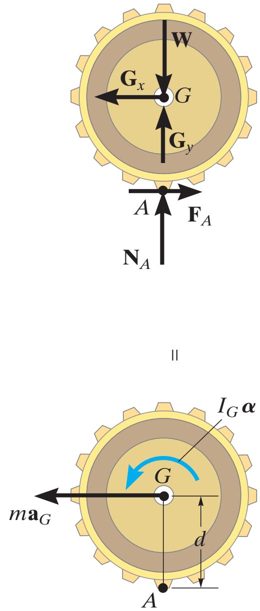


As the soil compactor, or "sheep's foot roller" moves forward, the roller has general plane motion. The forces shown on its free-body diagram cause the effects shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = I_G\alpha$. However, if moments are summed about point A (the IC) then $\zeta+\Sigma M_A = I_G\alpha + (ma_G)d = I_A\alpha$.

Procedure for Analysis

Kinetic problems involving general plane motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y inertial coordinate system and draw the free-body diagram for the body.
- Specify the direction and sense of the acceleration of the mass center, $\mathbf{a}_G$, and the angular acceleration $\boldsymbol{\alpha}$ of the body.
- Determine the moment of inertia I_G.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$ when writing the terms in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- When friction is present, there is the possibility for motion with no slipping or tipping. Each possibility for motion should be considered.

Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the body's motion is *constrained* due to its supports, additional equations may be obtained by using $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, which relates the accelerations of any two points A and B on the body.
- When a wheel, disk, cylinder, or ball *rolls without slipping*, then $a_G = \alpha r$.

Determine the angular acceleration of the spool in Fig. 17–20*a*. The spool has a mass of 8 kg and a radius of gyration of $k_G = 0.35$ m. The cords of negligible mass are wrapped around its inner hub and outer rim.

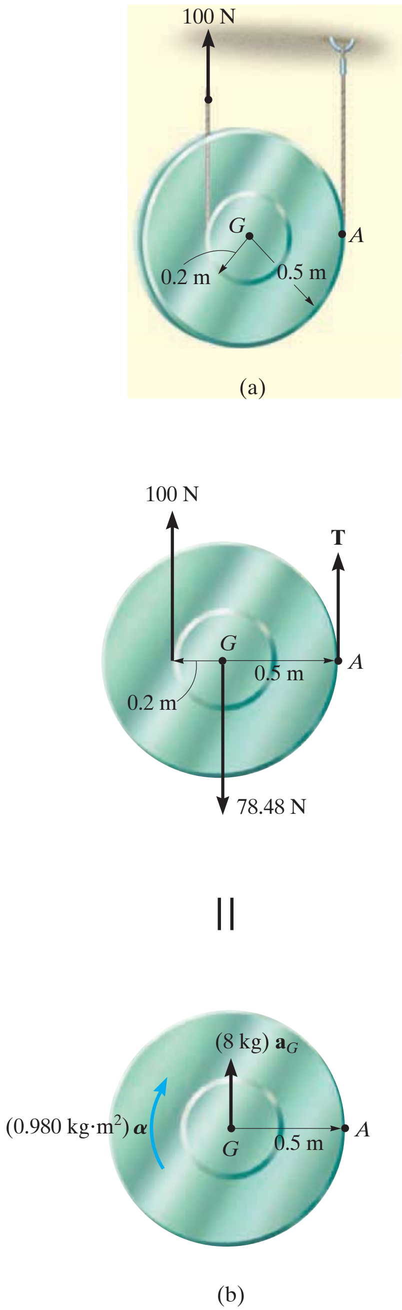


Fig. 17–20

SOLUTION I

Free-Body and Kinetic Diagrams. Fig. 17–20*b*. The 100-N force causes $\mathbf{a}_G$ to act upward. Also, $\boldsymbol{\alpha}$ acts clockwise, since the spool winds around the cord at A.

There are three unknowns T, a_G, and α. The moment of inertia of the spool about its mass center is

$$I_G = mk_G^2 = 8\text{ kg}(0.35\text{ m})^2 = 0.980\text{ kg}\cdot\text{m}^2$$

Equations of Motion.

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad T + 100\text{ N} - 78.48\text{ N} = (8\text{ kg})a_G \qquad (1)$$

$$\circlearrowright +\Sigma M_G = I_G\alpha; \quad 100\text{ N}(0.2\text{ m}) - T(0.5\text{ m}) = (0.980\text{ kg}\cdot\text{m}^2)\alpha \qquad (2)$$

Kinematics. A complete solution is obtained if kinematics is used to relate a_G to α. In this case the spool "rolls without slipping" on the cord at A. Hence, we can use the results of Example 16.4 or 16.15 so that,

$$(\circlearrowright +)\ a_G = \alpha r; \qquad a_G = \alpha\,(0.5\text{ m}) \qquad (3)$$

Solving Eqs. 1 to 3, we have

$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$
$$a_G = 5.16\text{ m/s}^2$$
$$T = 19.8\text{ N}$$

SOLUTION II

Equations of Motion. We can eliminate the unknown T by summing moments about point A. From the free-body and kinetic diagrams Figs. 17–20*b* and 17–20*c*, we have

$$\circlearrowright +\Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad 100\text{ N}(0.7\text{ m}) - 78.48\text{ N}(0.5\text{ m})$$
$$= (0.980\text{ kg}\cdot\text{m}^2)\alpha + [(8\text{ kg})a_G](0.5\text{ m})$$

Using Eq. (3),

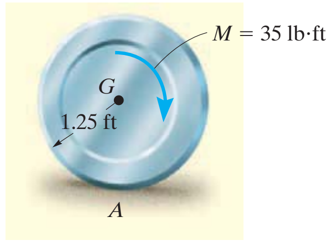
$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$

SOLUTION III

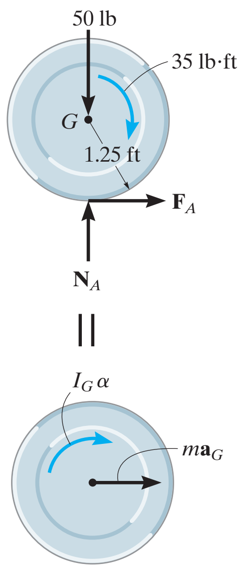
Equations of Motion. The simplest way to solve this problem is to realize that point A is the *IC* for the spool. Then Eq. 17–19 applies.

$$\circlearrowright +\Sigma M_A = I_A\alpha; \quad (100\text{ N})(0.7\text{ m}) - (78.48\text{ N})(0.5\text{ m})$$
$$= [0.980\text{ kg}\cdot\text{m}^2 + (8\text{ kg})(0.5\text{ m})^2]\alpha$$
$$\alpha = 10.3\text{ rad/s}^2$$

The 50-lb wheel shown in Fig. 17–21 has a radius of gyration $k_G = 0.70$ ft. If a 35-lb · ft couple moment is applied to the wheel, determine the acceleration of its mass center G. The coefficients of static and kinetic friction between the wheel and the plane at A are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively.



(a)



(b)

Fig. 17–21

SOLUTION

Free-Body and Kinetic Diagrams. By inspection of Fig. 17–21*b*, it is seen that the couple moment causes the wheel to have a clockwise angular acceleration of $\boldsymbol{\alpha}$. As a result, the acceleration of the mass center, $\mathbf{a}_G$, is directed to the right. The moment of inertia is

$$I_G = mk_G^2 = \frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}(0.70 \text{ ft})^2 = 0.7609 \text{ slug} \cdot \text{ft}^2$$

The unknowns are N_A, F_A, a_G, and α.

Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad F_A = \left(\frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}\right)a_G \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A - 50 \text{ lb} = 0 \qquad (2)$$

$$\circlearrowright + \Sigma M_G = I_G\alpha; \qquad 35 \text{ lb} \cdot \text{ft} - 1.25 \text{ ft}(F_A) = (0.7609 \text{ slug} \cdot \text{ft}^2)\alpha \qquad (3)$$

A fourth equation is needed for a complete solution.

Kinematics (No Slipping). If this assumption is made, then

$$(\circlearrowright +) \qquad a_G = (1.25 \text{ ft})\alpha \qquad (4)$$

Solving Eqs. 1 to 4,

$$N_A = 50.0 \text{ lb} \qquad F_A = 21.3 \text{ lb}$$
$$\alpha = 11.0 \text{ rad/s}^2 \qquad a_G = 13.7 \text{ ft/s}^2$$

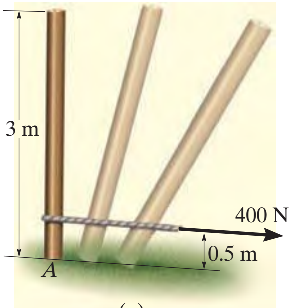
This solution requires that no slipping occurs, i.e., $F_A \leq \mu_s N_A$. However, since 21.3 lb > 0.3(50 lb) = 15 lb, the wheel slips as it rolls.

(Slipping). Equation 4 is not valid, and so $F_A = \mu_k N_A$, or

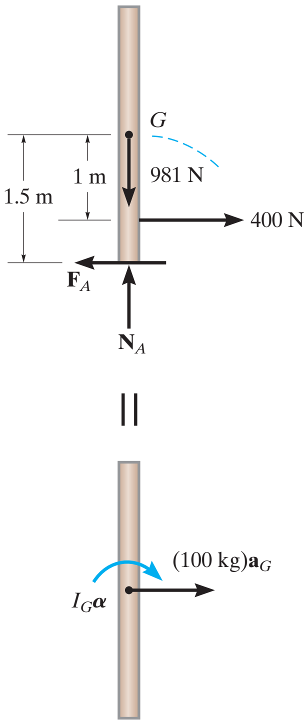
$$F_A = 0.25 N_A \qquad (5)$$

Solving Eqs. 1 to 3 and 5 yields

$$N_A = 50.0 \text{ lb} \qquad F_A = 12.5 \text{ lb}$$
$$\alpha = 25.5 \text{ rad/s}^2$$
$$a_G = 8.05 \text{ ft/s}^2 \rightarrow \qquad \textit{Ans.}$$



(a)



(b)

Fig. 17–22

The uniform slender pole shown in Fig. 17–22*a* has a mass of 100 kg. If the coefficients of static and kinetic friction between the end of the pole and the surface are $\mu_s = 0.3$, and $\mu_k = 0.25$, respectively, determine the pole's angular acceleration at the instant the 400-N horizontal force is applied. The pole is originally at rest.

SOLUTION

Free-Body and Kinetic Diagrams. Figure 17–22*b*. The path of motion of the mass center G will be along an unknown curved path having a radius of curvature ρ, which is initially on a vertical line. However, there is no normal or y component of acceleration since the pole is originally at rest, i.e., $\mathbf{v}_G = \mathbf{0}$, so that $(a_G)_y = v_G^2/\rho = 0$. We will assume the mass center accelerates to the right and that the pole has a clockwise angular acceleration of $\boldsymbol{\alpha}$. The unknowns are N_A, F_A, a_G, and α.

Equation of Motion.

$$\stackrel{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \qquad 400\text{ N} - F_A = (100\text{ kg})a_G \tag{1}$$

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad N_A - 981\text{ N} = 0 \tag{2}$$

$$\circlearrowright +\Sigma M_G = I_G\alpha; \; F_A(1.5\text{ m}) - (400\text{ N})(1\text{ m}) = [\tfrac{1}{12}(100\text{ kg})(3\text{ m})^2]\alpha \tag{3}$$

A fourth equation is needed for a complete solution.

Kinematics (No Slipping). With this assumption, point A acts as a "pivot" so that α is clockwise, then a_G is directed to the right.

$$a_G = \alpha r_{AG}; \qquad a_G = (1.5\text{ m})\,\alpha \tag{4}$$

Solving Eqs. 1 to 4 yields

$$N_A = 981\text{ N} \qquad F_A = 300\text{ N}$$

$$a_G = 1\text{ m/s}^2 \qquad \alpha = 0.667\text{ rad/s}^2$$

The assumption of no slipping requires $F_A \leq \mu_s N_A$. However, $300\text{ N} > 0.3(981\text{ N}) = 294\text{ N}$ and so the pole slips at A.

(Slipping). For this case Eq. 4 does *not* apply. Instead the frictional equation $F_A = \mu_k N_A$ must be used. Hence,

$$F_A = 0.25N_A \tag{5}$$

Solving Eqs. 1 to 3 and 5 simultaneously yields

$$N_A = 981\text{ N} \quad F_A = 245\text{ N} \quad a_G = 1.55\text{ m/s}^2$$

$$\alpha = -0.428\text{ rad/s}^2 = 0.428\text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

The uniform 50-kg bar in Fig. 17–23*a* is held in the equilibrium position by cords *AC* and *BD*. Determine the tension in *BD* and the angular acceleration of the bar immediately after *AC* is cut.

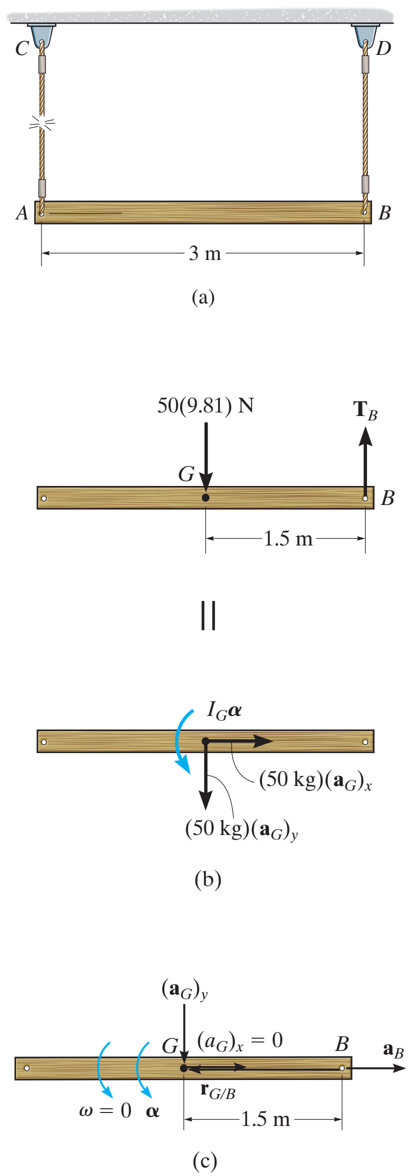


Fig. 17–23

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 17–23*b*. There are four unknowns, T_B, $(a_G)_x$, $(a_G)_y$, and α.

Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad 0 = 50 \text{ kg } (a_G)_x$$

$$(a_G)_x = 0$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \quad T_B - 50(9.81)\text{N} = -50 \text{ kg } (a_G)_y \tag{1}$$

$$\circlearrowleft + \Sigma M_G = I_G\alpha; \qquad T_B(1.5 \text{ m}) = \left[\frac{1}{12}(50 \text{ kg})(3 \text{ m})^2\right]\alpha \tag{2}$$

Kinematics. Since the bar is at rest just after the cable is cut, then its angular velocity and the velocity of point *B* at this instant are equal to zero. Thus $(a_B)_n = v_B^2/\rho_{BD} = 0$. Therefore, $\mathbf{a}_B$ only has a tangential component, which is directed along the *x* axis, Fig. 17–23*c*. Applying the relative acceleration equation to points *G* and *B*,

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} - \omega^2\mathbf{r}_{G/B}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} + (\alpha\mathbf{k}) \times (-1.5\mathbf{i}) - \mathbf{0}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} - 1.5\alpha\mathbf{j}$$

Equating the **i** and **j** components of both sides of this equation,

$$0 = a_B$$

$$(a_G)_y = 1.5\alpha \tag{3}$$

Solving Eqs. (1) through (3) yields

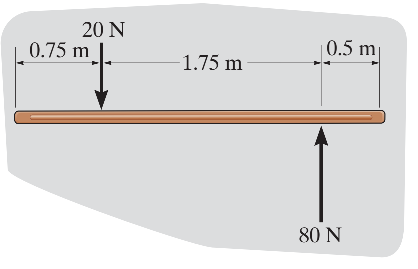
$$\alpha = 4.905 \text{ rad/s}^2 \qquad \textit{Ans.}$$

$$T_B = 123 \text{ N} \qquad \textit{Ans.}$$

$$(a_G)_y = 7.36 \text{ m/s}^2$$

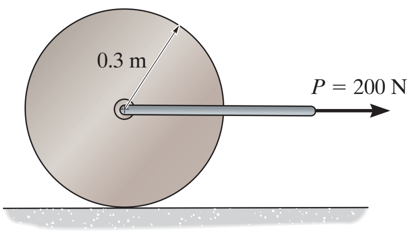
FUNDAMENTAL PROBLEMS

F17–13. The uniform 60-kg slender bar is initially at rest on a smooth *horizontal plane* when the forces are applied. Determine the acceleration of the bar's mass center and the angular acceleration of the bar at this instant.



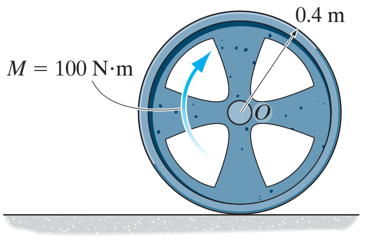
Prob. F17–13

F17–14. The 100-kg cylinder rolls without slipping on the horizontal plane. Determine the acceleration of its mass center and its angular acceleration.



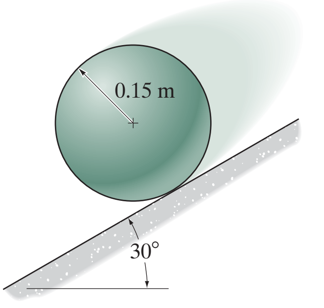
Prob. F17–14

F17–15. The 20-kg wheel has a radius of gyration about its center O of $k_O = 300$ mm. When the wheel is subjected to the couple moment, it slips as it rolls. Determine the angular acceleration of the wheel and the acceleration of the wheel's center O. The coefficient of kinetic friction between the wheel and the plane is $\mu_k = 0.5$.



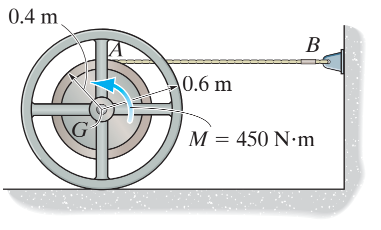
Prob. F17–15

F17–16. The 20-kg sphere rolls down the inclined plane without slipping. Determine the angular acceleration of the sphere and the acceleration of its mass center.



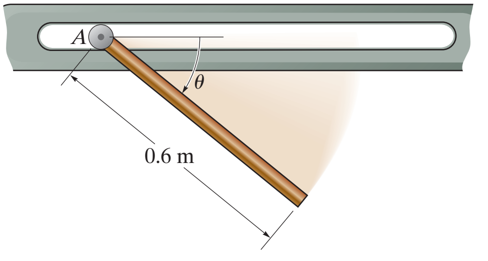
Prob. F17–16

F17–17. The 200-kg spool has a radius of gyration about its mass center of $k_G = 300$ mm. If the couple moment is applied to the spool and the coefficient of kinetic friction between the spool and the ground is $\mu_k = 0.2$, determine the angular acceleration of the spool, the acceleration of G and the tension in the cable.



Prob. F17–17

F17–18. The 12-kg slender rod is pinned to a small roller A that slides freely along the slot. If the rod is released from rest at $\theta = 0°$, determine the angular acceleration of the rod and the acceleration of the roller immediately after the release.

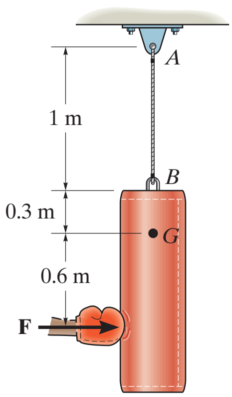


Prob. F17–18

PROBLEMS

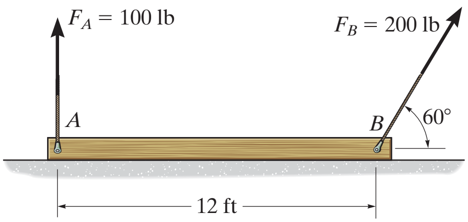
17–90. If the disk in Fig. 17–19 *rolls without slipping*, show that when moments are summed about the instantaneous center of zero velocity, IC, it is possible to use the moment equation $\Sigma M_{IC} = I_{IC}\,\alpha$, where I_{IC} represents the moment of inertia of the disk calculated about the instantaneous axis of zero velocity.

17–91. The 20-kg punching bag has a radius of gyration about its center of mass G of $k_G = 0.4$ m. If it is initially at rest and is subjected to a horizontal force $F = 30$ N, determine the initial angular acceleration of the bag and the tension in the supporting cable AB.



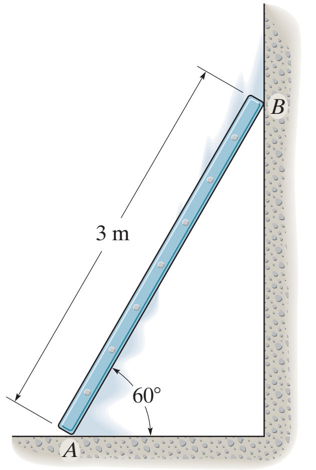
Prob. 17–91

***17–92.** The uniform 150-lb beam is initially at rest when the forces are applied to the cables. Determine the magnitude of the acceleration of the mass center and the angular acceleration of the beam at this instant.



Prob. 17–92

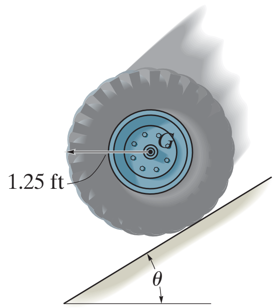
17–93. The slender 12-kg bar has a clockwise angular velocity of $\omega = 2$ rad/s when it is in the position shown. Determine its angular acceleration and the normal reactions of the smooth surface A and B at this instant.



Prob. 17–93

17–94. The tire has a weight of 30 lb and a radius of gyration of $k_G = 0.6$ ft. If the coefficients of static and kinetic friction between the tire and the plane are $\mu_s = 0.2$ and $\mu_k = 0.15$, determine the tire's angular acceleration as it rolls down the incline. Set $\theta = 12°$.

17–95. The tire has a weight of 30 lb and a radius of gyration of $k_G = 0.6$ ft. If the coefficients of static and kinetic friction between the tire and the plane are $\mu_s = 0.2$ and $\mu_k = 0.15$, determine the maximum angle θ of the inclined plane so that the tire rolls without slipping.

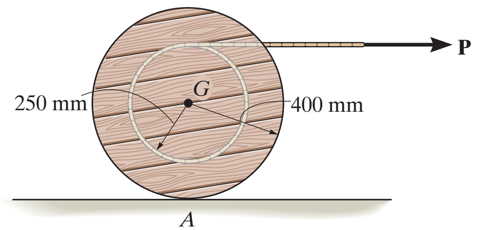


Probs. 17–94/95

***17–96.** The spool has a mass of 100 kg and a radius of gyration of $k_G = 0.3$ m. If the coefficients of static and kinetic friction at A are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively, determine the angular acceleration of the spool if $P = 50$ N.

17–97. Solve Prob. 17–96 if the cord and force $P = 50$ N are directed vertically upwards.

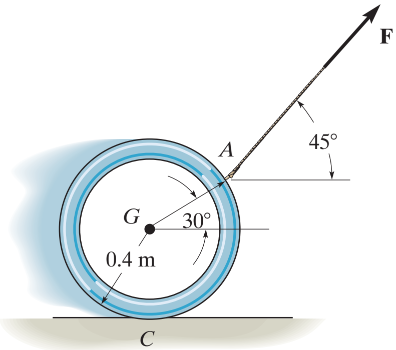
17–98. The spool has a mass of 100 kg and a radius of gyration $k_G = 0.3$ m. If the coefficients of static and kinetic friction at A are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively, determine the angular acceleration of the spool if $P = 600$ N.



Probs. 17–96/97/98

***17–100.** A force of $F = 10$ N is applied to the 10-kg ring as shown. If slipping does not occur, determine the ring's initial angular acceleration, and the acceleration of its mass center, G. Neglect the thickness of the ring.

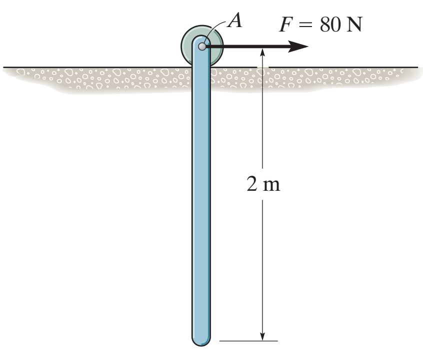
17–101. If the coefficient of static friction at C is $\mu_s = 0.3$, determine the largest force **F** that can be applied to the 5-kg ring, without causing it to slip. Neglect the thickness of the ring.



Probs. 17–100/101

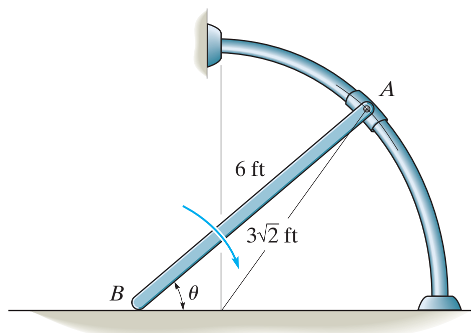
17

17–99. The 12-kg uniform bar is supported by a roller at A. If a horizontal force of $F = 80$ N is applied to the roller, determine the acceleration of the center of the roller at the instant the force is applied. Neglect the weight and the size of the roller.



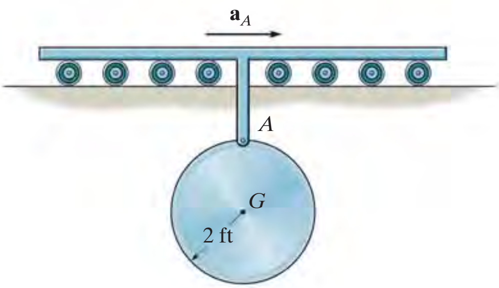
Prob. 17–99

17–102. The 25-lb slender rod has a length of 6 ft. Using a collar of negligible mass, its end A is confined to move along the smooth circular bar of radius $3\sqrt{2}$ ft. End B rests on the floor, for which the coefficient of kinetic friction is $\mu_B = 0.4$. If the bar is released from rest when $\theta = 30°$, determine the angular acceleration of the bar at this instant.



Prob. 17–102

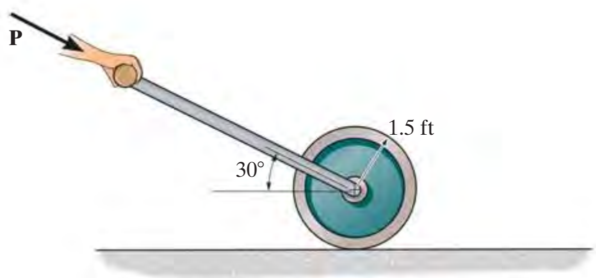
17–103. The 15-lb circular plate is suspended from a pin at A. If the pin is connected to a track which is given an acceleration $a_A = 5\ \text{ft/s}^2$, determine the horizontal and vertical components of reaction at A and the angular acceleration of the plate. The plate is originally at rest.



Prob. 17–103

***17–104.** If $P = 30$ lb, determine the angular acceleration of the 50-lb roller. Assume the roller to be a uniform cylinder and that no slipping occurs.

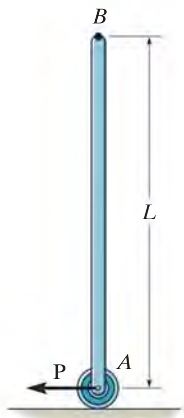
17–105. If the coefficient of static friction between the 50-lb roller and the ground is $\mu_s = 0.25$, determine the maximum force P that can be applied to the handle, so that roller rolls on the ground without slipping. Also, find the angular acceleration of the roller. Assume the roller to be a uniform cylinder.



Probs. 17–104/105

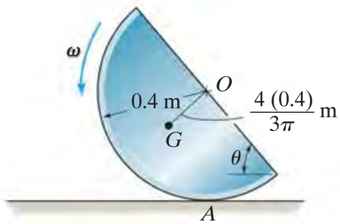
17–106. The uniform bar of mass m and length L is balanced in the vertical position when the horizontal force $\mathbf{P}$ is applied to the roller at A. Determine the bar's initial angular acceleration and the acceleration of its top point B.

17–107. Solve Prob. 17–106 if the roller is removed and the coefficient of kinetic friction at the ground is μ_k.



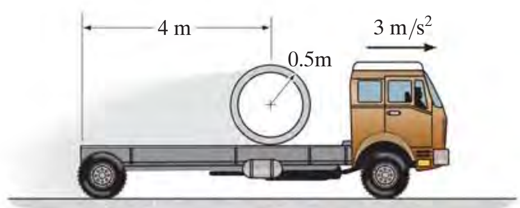
Probs. 17–106/107

***17–108.** The semicircular disk having a mass of 10 kg is rotating at $\omega = 4\ \text{rad/s}$ at the instant $\theta = 60°$. If the coefficient of static friction at A is $\mu_s = 0.5$, determine if the disk slips at this instant.



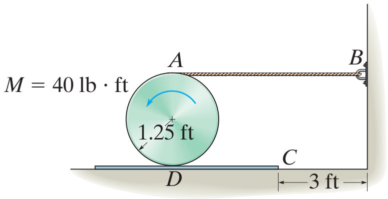
Prob. 17–108

17–109. The 500-kg concrete culvert has a mean radius of 0.5 m. If the truck has an acceleration of $3\ \text{m/s}^2$, determine the culvert's angular acceleration. Assume that the culvert does not slip on the truck bed, and neglect its thickness.



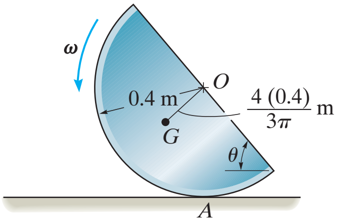
Prob. 17–109

17–110. The 15-lb disk rests on the 5-lb plate. A cord is wrapped around the periphery of the disk and attached to the wall at B. If a torque $M = 40$ lb · ft is applied to the disk, determine the angular acceleration of the disk and the time needed for the end C of the plate to travel 3 ft and strike the wall. Assume the disk does not slip on the plate and the plate rests on the surface at D having a coefficient of kinetic friction of $\mu_k = 0.2$. Neglect the mass of the cord.



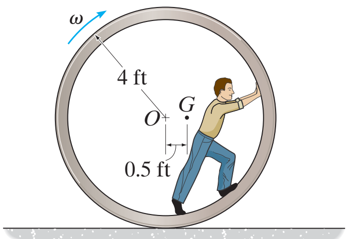
Prob. 17–110

17–111. The semicircular disk having a mass of 10 kg is rotating at $\omega = 4$ rad/s at the instant $\theta = 60°$. If the coefficient of static friction at A is $\mu_s = 0.5$, determine if the disk slips at this instant.



Prob. 17–111

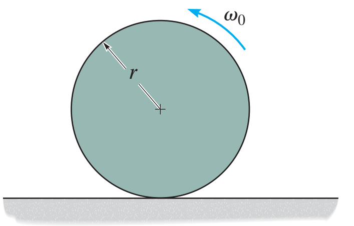
***17–112.** The circular concrete culvert rolls with an angular velocity of $\omega = 0.5$ rad/s when the man is at the position shown. At this instant the center of gravity of the culvert and the man is located at point G, and the radius of gyration about G is $k_G = 3.5$ ft. Determine the angular acceleration of the culvert. The combined weight of the culvert and the man is 500 lb. Assume that the culvert rolls without slipping, and the man does not move within the culvert.



Prob. 17–112

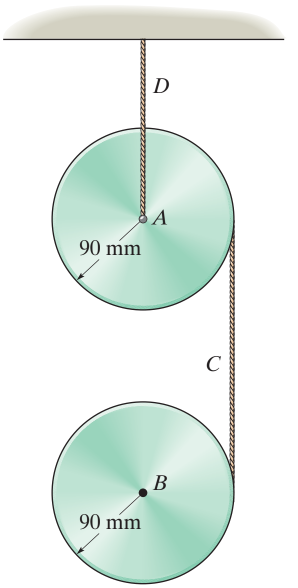
17–113. The uniform disk of mass m is rotating with an angular velocity of ω_0 when it is placed on the floor. Determine the initial angular acceleration of the disk and the acceleration of its mass center. The coefficient of kinetic friction between the disk and the floor is μ_k.

17–114. The uniform disk of mass m is rotating with an angular velocity of ω_0 when it is placed on the floor. Determine the time before it starts to roll without slipping. What is the angular velocity of the disk at this instant? The coefficient of kinetic friction between the disk and the floor is μ_k.



Probs. 17–113/114

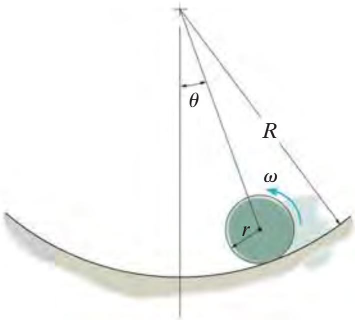
17–115. A cord is wrapped around each of the two 10-kg disks. If they are released from rest determine the angular acceleration of each disk and the tension in the cord C. Neglect the mass of the cord.



Prob. 17–115

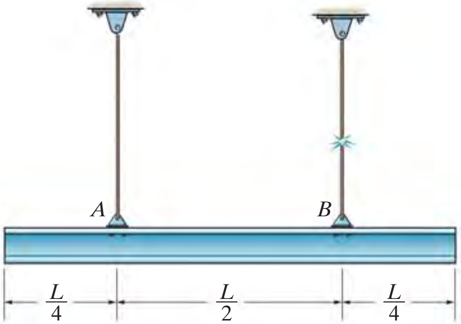
17

***17–116.** The disk of mass m and radius r rolls without slipping on the circular path. Determine the normal force which the path exerts on the disk and the disk's angular acceleration if at the instant shown the disk has an angular velocity of $\boldsymbol{\omega}$.



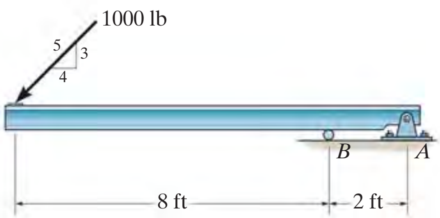
Prob. 17–116

17–117. The uniform beam has a weight W. If it is originally at rest while being supported at A and B by cables, determine the tension in cable A if cable B suddenly fails. Assume the beam is a slender rod.



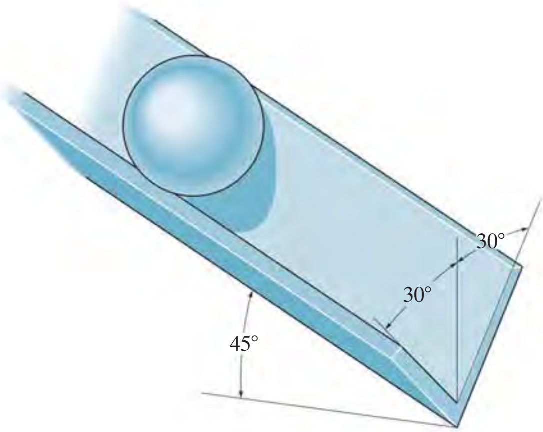
Prob. 17–117

17–118. The 500-lb beam is supported at A and B when it is subjected to a force of 1000 lb as shown. If the pin support at A suddenly fails, determine the beam's initial angular acceleration and the force of the roller support on the beam. For the calculation, assume that the beam is a slender rod so that its thickness can be neglected.



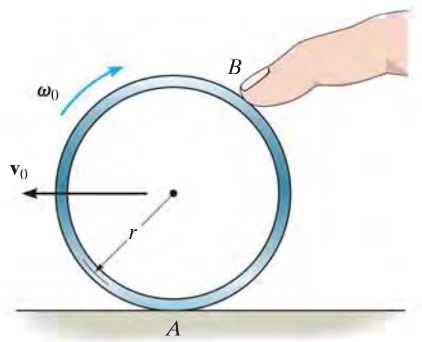
Prob. 17–118

17–119. The solid ball of radius r and mass m rolls without slipping down the 60° trough. Determine its angular acceleration.



Prob. 17–119

***17–120.** By pressing down with the finger at B, a thin ring having a mass m is given an initial velocity $\mathbf{v}_0$ and a backspin $\boldsymbol{\omega}_0$ when the finger is released. If the coefficient of kinetic friction between the table and the ring is μ_k, determine the distance the ring travels forward before backspinning stops.



Prob. 17–120

CONCEPTUAL PROBLEMS

C17–1. The truck is used to pull the heavy container. To be most effective at providing traction to the rear wheels at *A*, is it best to keep the container where it is or place it at the front of the trailer? Use appropriate numerical values to explain your answer.

Prob. C17–1 (© R.C. Hibbeler)

C17–2. The tractor is about to tow the plane to the right. Is it possible for the driver to cause the front wheel of the plane to lift off the ground as he accelerates the tractor? Draw the free-body and kinetic diagrams and explain algebraically (letters) if and how this might be possible.

Prob. C17–2 (© R.C. Hibbeler)

C17–3. How can you tell the driver is accelerating this SUV? To explain your answer, draw the free-body and kinetic diagrams. Here power is supplied to the rear wheels. Would the photo look the same if power were supplied to the front wheels? Will the accelerations be the same? Use appropriate numerical values to explain your answers.

Prob. C17–3 (© R.C. Hibbeler)

C17–4. Here is something you should not try at home, at least not without wearing a helmet! Draw the free-body and kinetic diagrams and show what the rider must do to maintain this position. Use appropriate numerical values to explain your answer.

Prob. C17–4 (© R.C. Hibbeler)

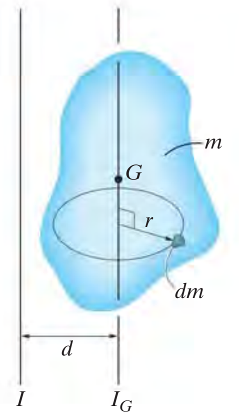
CHAPTER REVIEW

Moment of Inertia

The moment of inertia is a measure of the resistance of a body to a change in its angular velocity. It is defined by $I = \int r^2 dm$ and will be different for each axis about which it is computed.

Many bodies are composed of simple shapes. If this is the case, then tabular values of I can be used, such as the ones given on the inside back cover of this book. To obtain the moment of inertia of a composite body about any specified axis, the moment of inertia of each part is determined about the axis and the results are added together. Doing this often requires use of the parallel-axis theorem.

$$I = I_G + md^2$$



Planar Equations of Motion

The equations of motion define the translational, and rotational motion of a rigid body. In order to account for all of the terms in these equations, a free-body diagram should always accompany their application, and for some problems, it may also be convenient to draw the kinetic diagram which shows $m\mathbf{a}_G$ and $I_G\boldsymbol{\alpha}$.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = 0$$

Rectilinear translation

$$\Sigma F_n = m(a_G)_n$$
$$\Sigma F_t = m(a_G)_t$$
$$\Sigma M_G = 0$$

Curvilinear translation

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_G = I_G\alpha \text{ or } \Sigma M_O = I_O\alpha$$

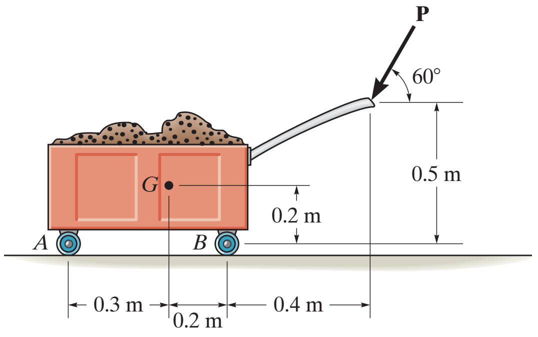
Rotation About a Fixed Axis

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = I_G\alpha \text{ or } \Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

General Plane Motion

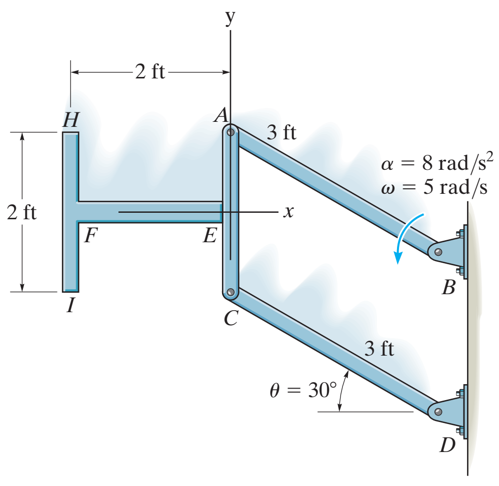
REVIEW PROBLEMS

R17–1. The handcart has a mass of 200 kg and center of mass at G. Determine the normal reactions at *each* of the wheels at A and B if a force $P = 50$ N is applied to the handle. Neglect the mass and rolling resistance of the wheels.



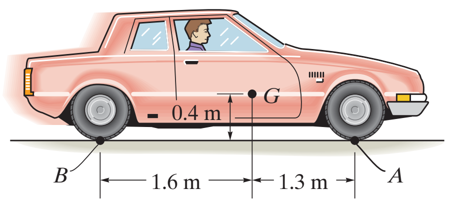
Prob. R17–1

R17–2. The two 3-lb rods EF and HI are fixed (welded) to the link AC at E. Determine the internal axial force E_x, shear force E_y, and moment M_E, which the bar AC exerts on FE at E if at the instant $\theta = 30°$ link AB has an angular velocity $\omega = 5$ rad/s and an angular acceleration $\alpha = 8$ rad/s^2 as shown.



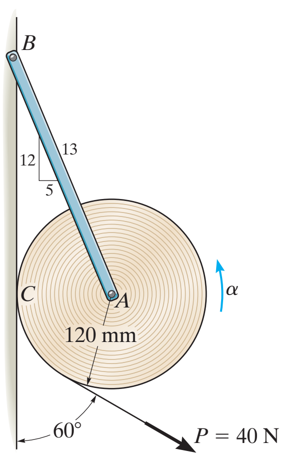
Prob. R17–2

R17–3. The car has a mass of 1.50 Mg and a mass center at G. Determine the maximum acceleration it can have if power is supplied only to the rear wheels. Neglect the mass of the wheels in the calculation, and assume that the wheels that do not receive power are free to roll. Also, assume that slipping of the powered wheels occurs, where the coefficient of kinetic friction is $\mu_k = 0.3$.



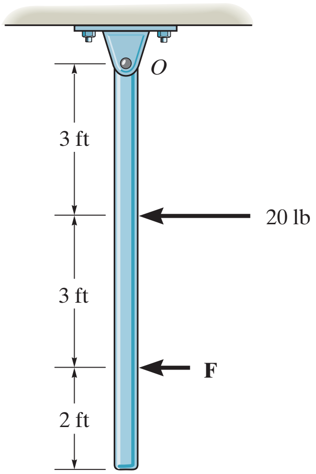
Prob. R17–3

R17–4. A 20-kg roll of paper, originally at rest, is pin-supported at its ends to bracket AB. The roll rest against a wall for which the coefficient of kinetic friction at C is $\mu_C = 0.3$. If a force of 40 N is applied uniformly to the end of the sheet, determine the initial angular acceleration of the roll and the tension in the bracket as the paper unwraps. For the calculation, treat the roll as a cylinder.



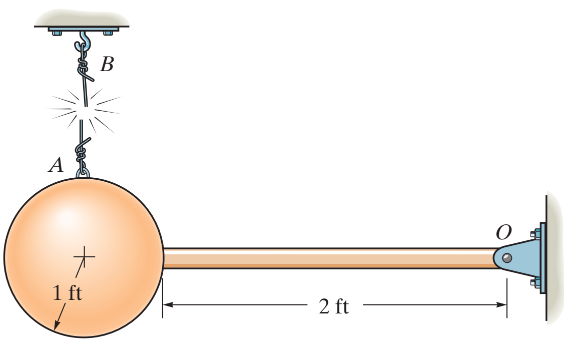
Prob. R17–4

R17–5. At the instant shown, two forces act on the 30-lb slender rod which is pinned at O. Determine the magnitude of force $\mathbf{F}$ and the initial angular acceleration of the rod so that the horizontal reaction which the *pin exerts on the rod* is 5 lb directed to the right.



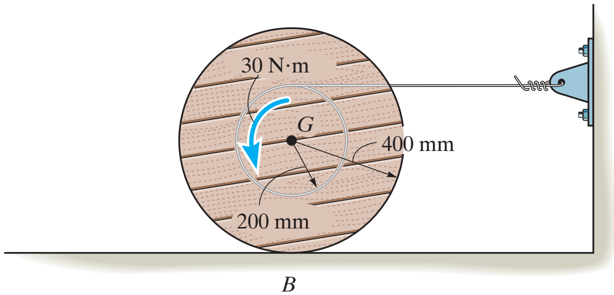
Prob. R17–5

R17–6. The pendulum consists of a 30-lb sphere and a 10-lb slender rod. Compute the reaction at the pin O just after the cord AB is cut.



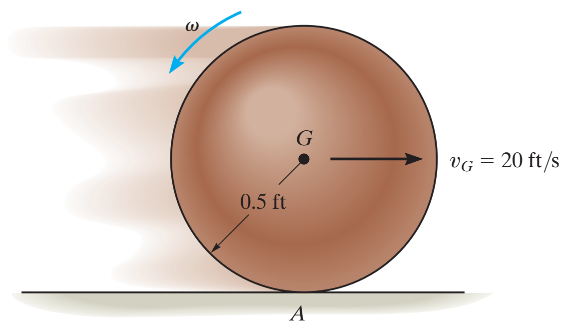
Prob. R17–6

R17–7. The spool and wire wrapped around its core have a mass of 20 kg and a centroidal radius of gyration $k_G = 250$ mm. if the coefficient of kinetic friction at the ground is $\mu_B = 0.1$, determine the angular acceleration of the spool when the 30-N · m couple moment is applied.



Prob. R17–7

R17–8. Determine the backspin $\boldsymbol{\omega}$ which should be given to the 20-lb ball so that when its center is given an initial horizontal velocity $v_G = 20$ ft/s it stops spinning and translating at the same instant. The coefficient of kinetic friction is $\mu_A = 0.3$.



Prob. R17–8

Chapter 18

(© Arinahabich/Fotolia)

Roller coasters must be able to coast over loops and through turns, and have enough energy to do so safely. Accurate calculation of this energy must account for the size of the car as it moves along the track.

Planar Kinetics of a Rigid Body: Work and Energy

CHAPTER OBJECTIVES

- To develop formulations for the kinetic energy of a body, and define the various ways a force and couple do work.
- To apply the principle of work and energy to solve rigid–body planar kinetic problems that involve force, velocity, and displacement.
- To show how the conservation of energy can be used to solve rigid–body planar kinetic problems.

18.1 Kinetic Energy

In this chapter we will apply work and energy methods to solve planar motion problems involving force, velocity, and displacement. But first it will be necessary to develop a means of obtaining the body's kinetic energy when the body is subjected to translation, rotation about a fixed axis, or general plane motion.

To do this we will consider the rigid body shown in Fig. 18–1, which is represented here by a *slab* moving in the inertial x–y reference plane. An arbitrary ith particle of the body, having a mass dm, is located a distance r from the arbitrary point P. If at the *instant* shown the particle has a velocity $\mathbf{v}_i$, then the particle's kinetic energy is $T_i = \frac{1}{2}\, dm\, v_i^2$.

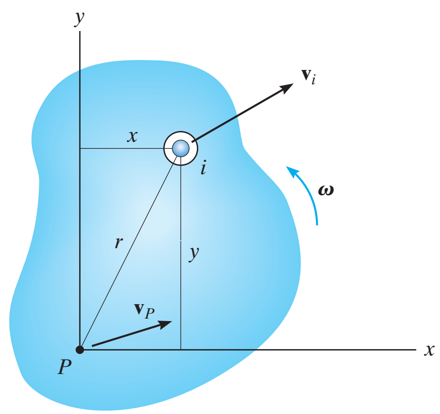


Fig. 18–1

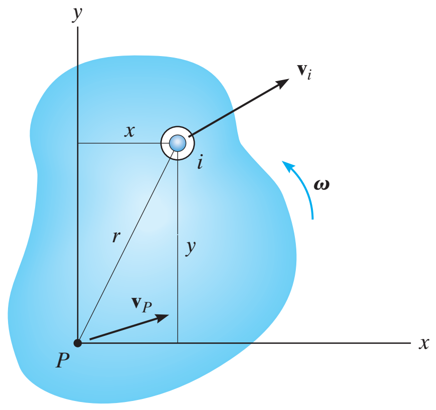


Fig. 18–1 (repeated)

The kinetic energy of the entire body is determined by writing similar expressions for each particle of the body and integrating the results, i.e.,

$$T = \frac{1}{2}\int_m dm\, v_i^2$$

This equation may also be expressed in terms of the velocity of point P. If the body has an angular velocity $\boldsymbol{\omega}$, then from Fig. 18–1 we have

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}_P + \mathbf{v}_{i/P}\\ &= (v_P)_x\mathbf{i} + (v_P)_y\mathbf{j} + \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})\\ &= [(v_P)_x - \omega y]\mathbf{i} + [(v_P)_y + \omega x]\mathbf{j}\end{aligned}$$

The square of the magnitude of $\mathbf{v}_i$ is thus

$$\begin{aligned}\mathbf{v}_i \cdot \mathbf{v}_i = v_i^2 &= [(v_P)_x - \omega y]^2 + [(v_P)_y + \omega x]^2\\ &= (v_P)_x^2 - 2(v_P)_x\omega y + \omega^2 y^2 + (v_P)_y^2 + 2(v_P)_y\omega x + \omega^2 x^2\\ &= v_P^2 - 2(v_P)_x\omega y + 2(v_P)_y\omega x + \omega^2 r^2\end{aligned}$$

Substituting this into the equation of kinetic energy yields

$$T = \frac{1}{2}\left(\int_m dm\right)v_P^2 - (v_P)_x\omega\left(\int_m y\, dm\right) + (v_P)_y\omega\left(\int_m x\, dm\right) + \frac{1}{2}\omega^2\left(\int_m r^2\, dm\right)$$

The first integral on the right represents the entire mass m of the body. Since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$, the second and third integrals locate the body's center of mass G with respect to P. The last integral represents the body's moment of inertia I_P, computed about the z axis passing through point P. Thus,

$$T = \tfrac{1}{2}mv_P^2 - (v_P)_x\omega\bar{y}m + (v_P)_y\omega\bar{x}m + \tfrac{1}{2}I_P\omega^2 \qquad (18\text{–}1)$$

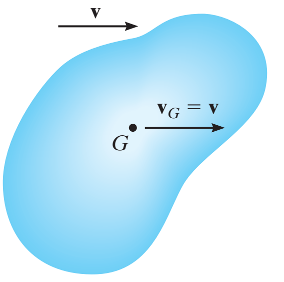
As a special case, if point P coincides with the mass center G of the body, then $\bar{y} = \bar{x} = 0$, and therefore

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2 \qquad (18\text{–}2)$$

Both terms on the right side are *always positive*, since v_G and ω are squared. The first term represents the translational kinetic energy, referenced from the mass center, and the second term represents the body's rotational kinetic energy about the mass center.

Translation. When a rigid body of mass m is subjected to either rectilinear or curvilinear *translation*, Fig. 18–2, the kinetic energy due to rotation is zero, since $\boldsymbol{\omega} = \mathbf{0}$. The kinetic energy of the body is therefore

$$T = \tfrac{1}{2}mv_G^2 \tag{18–3}$$



Translation

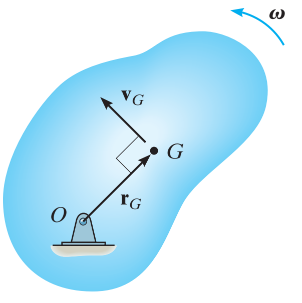
Fig. 18–2

Rotation about a Fixed Axis. When a rigid body *rotates about a fixed axis* passing through point O, Fig. 18–3, the body has both *translational* and *rotational* kinetic energy so that

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2 \tag{18–4}$$

The body's kinetic energy may also be formulated for this case by noting that $v_G = r_G\omega$, so that $T = \frac{1}{2}(I_G + mr_G^2)\omega^2$. By the parallel–axis theorem, the terms inside the parentheses represent the moment of inertia I_O of the body about an axis perpendicular to the plane of motion and passing through point O. Hence,*

$$T = \tfrac{1}{2}I_O\omega^2 \tag{18–5}$$



Rotation About a Fixed Axis

Fig. 18–3

From the derivation, this equation will give the same result as Eq. 18–4, since it accounts for *both* the translational and rotational kinetic energies of the body.

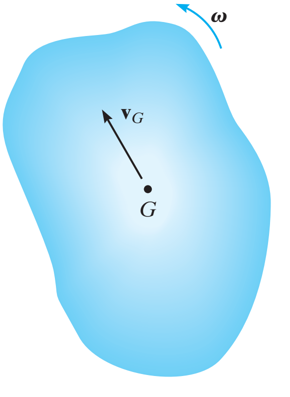
General Plane Motion. When a rigid body is subjected to general plane motion, Fig. 18–4, it has an angular velocity $\boldsymbol{\omega}$ and its mass center has a velocity $\mathbf{v}_G$. Therefore, the kinetic energy is

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2 \tag{18–6}$$

This equation can also be expressed in terms of the body's motion about its instantaneous center of zero velocity i.e.,

$$T = \tfrac{1}{2}I_{IC}\omega^2 \tag{18–7}$$

where I_{IC} is the moment of inertia of the body about its instantaneous center. The proof is similar to that of Eq. 18–5. (See Prob. 18–1.)



General Plane Motion

Fig. 18–4

*The similarity between this derivation and that of $\Sigma M_O = I_O\alpha$, should be noted. Also the same result can be obtained directly from Eq. 18–1 by selecting point P at O, realizing that $v_O = 0$.

The total kinetic energy of this soil compactor consists of the kinetic energy of the body or frame of the machine due to its translation, and the translational and rotational kinetic energies of the roller and the wheels due to their general plane motion. Here we exclude the additional kinetic energy developed by the moving parts of the engine and drive train. (© R.C. Hibbeler)

System of Bodies. Because energy is a scalar quantity, the total kinetic energy for a system of *connected* rigid bodies is the sum of the kinetic energies of all its moving parts. Depending on the type of motion, the kinetic energy of *each body* is found by applying Eq. 18–2 or the alternative forms mentioned above.

18.2 The Work of a Force

Several types of forces are often encountered in planar kinetics problems involving a rigid body. The work of each of these forces has been presented in Sec. 14.1 and is listed below as a summary.

Work of a Variable Force. If an external force **F** acts on a body, the work done by the force when the body moves along the path s, Fig. 18–5, is

$$U_F = \int \mathbf{F} \cdot d\mathbf{r} = \int_s F \cos\theta \, ds \qquad (18\text{–}8)$$

Here θ is the angle between the "tails" of the force and the differential displacement. The integration must account for the variation of the force's direction and magnitude.

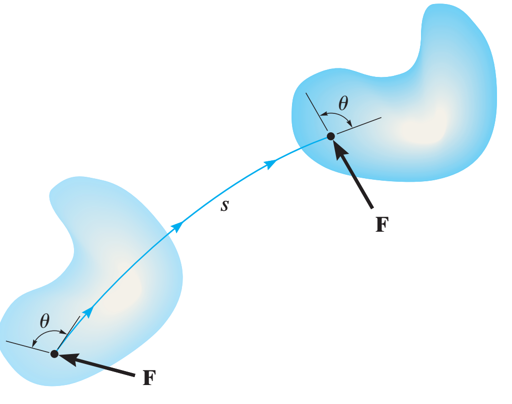


Fig. 18–5

Work of a Constant Force. If an external force $\mathbf{F}_c$ acts on a body, Fig. 18–6, and maintains a constant magnitude F_c and constant direction θ, while the body undergoes a translation s, then the above equation can be integrated, so that the work becomes

$$U_{F_c} = (F_c \cos\theta)s \qquad (18\text{–}9)$$

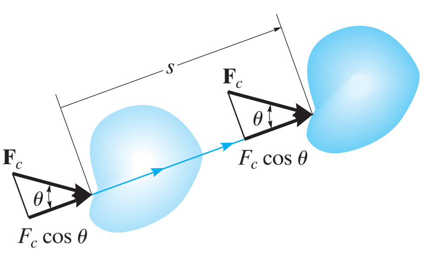


Fig. 18–6

Work of a Weight. The weight of a body does work only when the body's center of mass G undergoes a *vertical displacement* Δy. If this displacement is *upward*, Fig. 18–7, the work is negative, since the weight is opposite to the displacement.

$$U_W = -W\,\Delta y \tag{18–10}$$

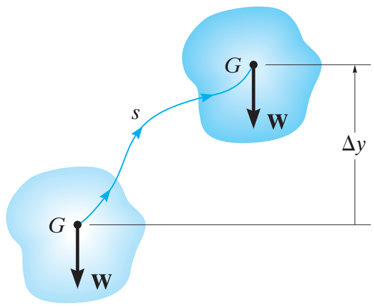


Fig. 18–7

Likewise, if the displacement is *downward* $(-\Delta y)$ the work becomes *positive*. In both cases the elevation change is considered to be small so that **W**, which is caused by gravitation, is constant.

Work of a Spring Force. If a linear elastic spring is attached to a body, the spring force $F_s = ks$ *acting on the body* does work when the spring either stretches or compresses from s_1 to a *farther* position s_2. In both cases the work will be *negative* since the *displacement of the body* is in the opposite direction to the force, Fig. 18–8. The work is

$$U_s = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right) \tag{18–11}$$

where $|s_2| > |s_1|$.

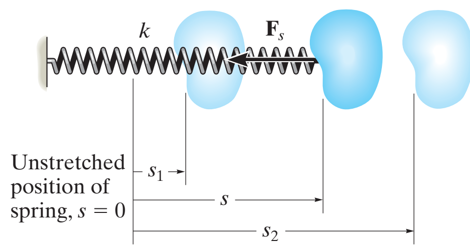


Fig. 18–8

Forces That Do No Work. There are some external forces that do no work when the body is displaced. These forces act either at *fixed points* on the body, or they have a direction *perpendicular to their displacement*. Examples include the reactions at a pin support about which a body rotates, the normal reaction acting on a body that moves along a fixed surface, and the weight of a body when the center of gravity of the body moves in a *horizontal plane*, Fig. 18–9. A frictional force $\mathbf{F}_f$ acting on a round body as it *rolls without slipping* over a rough surface also does no work.* This is because, during any *instant of time* dt, $\mathbf{F}_f$ acts at a point on the body which has *zero velocity* (instantaneous center, IC) and so the work done by the force on the point is zero. In other words, the point is not displaced in the direction of the force during this instant. Since $\mathbf{F}_f$ contacts successive points for only an instant, the work of $\mathbf{F}_f$ will be zero.

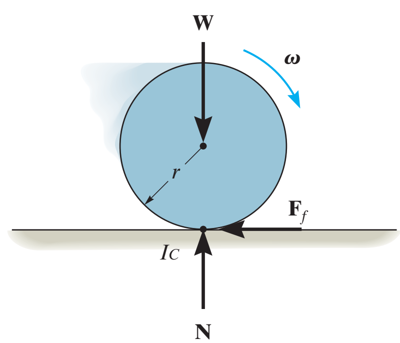
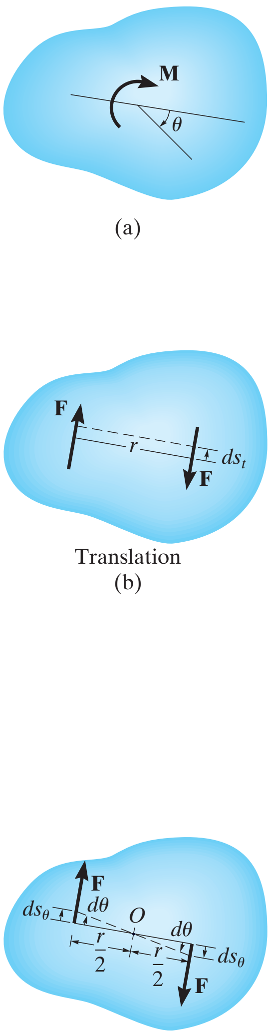


Fig. 18–9

*The work done by a frictional force *when the body slips* is discussed in Sec. 14.3.



(a)

Translation
(b)

Rotation
(c)

Fig. 18–10

18.3 The Work of a Couple Moment

Consider the body in Fig. 18–10*a*, which is subjected to a couple moment $M = Fr$. If the body undergoes a differential displacement, then the work done by the couple forces can be found by considering the displacement as the sum of a separate translation plus rotation. When the body *translates*, the work of each force is produced only by the *component of displacement* along the line of action of the forces ds_t, Fig. 18–10*b*. Clearly the "positive" work of one force *cancels* the "negative" work of the other. When the body undergoes a differential rotation $d\theta$ about the arbitrary point O, Fig. 18–10*c*, then each force undergoes a displacement $ds_\theta = (r/2)\,d\theta$ in the direction of the force. Hence, the total work done is

$$
\begin{aligned}
dU_M &= F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = (Fr)\,d\theta \\
&= M\,d\theta
\end{aligned}
$$

The work is *positive* when **M** and $d\boldsymbol{\theta}$ have the *same sense of direction* and *negative* if these vectors are in the *opposite sense*.

When the body rotates in the plane through a finite angle θ measured in radians, from θ_1 to θ_2, the work of a couple moment is therefore

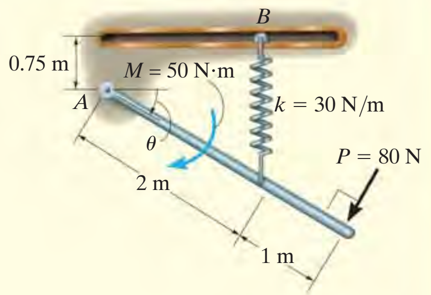
$$U_M = \int_{\theta_1}^{\theta_2} M\,d\theta \tag{18–12}$$

If the couple moment **M** has a *constant magnitude*, then

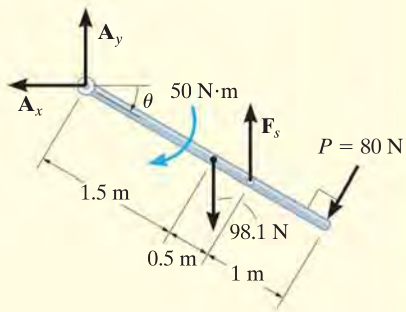
$$U_M = M(\theta_2 - \theta_1) \tag{18–13}$$

EXAMPLE 18.1

The bar shown in Fig. 18–11*a* has a mass of 10 kg and is subjected to a couple moment of $M = 50\ \text{N}\cdot\text{m}$ and a force of $P = 80$ N, which is always applied perpendicular to the end of the bar. Also, the spring has an unstretched length of 0.5 m and remains in the vertical position due to the roller guide at B. Determine the total work done by all the forces acting on the bar when it has rotated downward from $\theta = 0°$ to $\theta = 90°$.



(a)



(b)

Fig. 18–11

SOLUTION

First the free-body diagram of the bar is drawn in order to account for all the forces that act on it, Fig. 18–11*b*.

Weight *W*. Since the weight $10(9.81)\ \text{N} = 98.1\ \text{N}$ is displaced downward 1.5 m, the work is

$$U_W = 98.1\ \text{N}(1.5\ \text{m}) = 147.2\ \text{J}$$

Why is the work positive?

Couple Moment *M*. The couple moment rotates through an angle of $\theta = \pi/2$ rad. Hence,

$$U_M = 50\ \text{N}\cdot\text{m}(\pi/2) = 78.5\ \text{J}$$

Spring Force F_s. When $\theta = 0°$ the spring is stretched $(0.75\ \text{m} - 0.5\ \text{m}) = 0.25$ m, and when $\theta = 90°$, the stretch is $(2\ \text{m} + 0.75\ \text{m}) - 0.5\ \text{m} = 2.25$ m. Thus,

$$U_s = -\left[\tfrac{1}{2}(30\ \text{N/m})(2.25\ \text{m})^2 - \tfrac{1}{2}(30\ \text{N/m})(0.25\ \text{m})^2\right] = -75.0\ \text{J}$$

By inspection the spring does negative work on the bar since $\mathbf{F}_s$ acts in the opposite direction to displacement. This checks with the result.

Force *P*. As the bar moves downward, the force is displaced through a distance of $(\pi/2)(3\ \text{m}) = 4.712$ m. The work is positive. Why?

$$U_P = 80\ \text{N}(4.712\ \text{m}) = 377.0\ \text{J}$$

Pin Reactions. Forces $\mathbf{A}_x$ and $\mathbf{A}_y$ do no work since they are not displaced.

Total Work. The work of all the forces when the bar is displaced is thus

$$U = 147.2\ \text{J} + 78.5\ \text{J} - 75.0\ \text{J} + 377.0\ \text{J} = 528\ \text{J} \qquad \textit{Ans.}$$

The counterweight on this bascule bridge does positive work as the bridge is lifted and thereby cancels the negative work done by the weight of the bridge. (© R.C. Hibbeler)

18.4 Principle of Work and Energy

By applying the principle of work and energy developed in Sec. 14.2 to each of the particles of a rigid body and adding the results algebraically, since energy is a scalar, the principle of work and energy for a rigid body becomes

$$T_1 + \Sigma U_{1-2} = T_2 \tag{18–14}$$

This equation states that the body's initial translational *and* rotational kinetic energy, plus the work done by all the external forces and couple moments acting on the body as the body moves from its initial to its final position, is equal to the body's final translational *and* rotational kinetic energy. Note that the work of the body's *internal forces* does not have to be considered. These forces occur in equal but opposite collinear pairs, so that when the body moves, the work of one force cancels that of its counterpart. Furthermore, since the body is rigid, *no relative movement* between these forces occurs, so that no internal work is done.

When several rigid bodies are pin connected, connected by inextensible cables, or in mesh with one another, Eq. 18–14 can be applied to the *entire system* of connected bodies. In all these cases the internal forces, which hold the various members together, do no work and hence are eliminated from the analysis.

The work of the torque or moment developed by the driving gears on the motors is transformed into kinetic energy of rotation of the drum. (© R.C. Hibbeler)

Procedure for Analysis

The principle of work and energy is used to solve kinetic problems that involve *velocity, force*, and *displacement*, since these terms are involved in the formulation. For application, it is suggested that the following procedure be used.

Kinetic Energy (Kinematic Diagrams).

- The kinetic energy of a body is made up of two parts. Kinetic energy of translation is referenced to the velocity of the mass center, $T = \frac{1}{2}mv_G^2$, and kinetic energy of rotation is determined using the moment of inertia of the body about the mass center, $T = \frac{1}{2}I_G\omega^2$. In the special case of rotation about a fixed axis (or rotation about the *IC*), these two kinetic energies are combined and can be expressed as $T = \frac{1}{2}I_O\omega^2$, where I_O is the moment of inertia about the axis of rotation.
- *Kinematic diagrams* for velocity may be useful for determining v_G and ω or for establishing a *relationship* between v_G and ω.*

Work (Free–Body Diagram).

- Draw a free–body diagram of the body when it is located at an intermediate point along the path in order to account for all the forces and couple moments which do work on the body as it moves along the path.
- A force does work when it moves through a displacement in the direction of the force.
- Forces that are functions of displacement must be integrated to obtain the work. Graphically, the work is equal to the area under the force–displacement curve.
- The work of a weight is the product of its magnitude and the vertical displacement, $U_W = Wy$. It is positive when the weight moves downwards.
- The work of a spring is of the form $U_s = \frac{1}{2}ks^2$, where k is the spring stiffness and s is the stretch or compression of the spring.
- The work of a couple is the product of the couple moment and the angle in radians through which it rotates, $U_M = M\theta$.
- Since *algebraic addition* of the work terms is required, it is important that the proper sign of each term be specified. Specifically, work is *positive* when the force (couple moment) is in the *same direction* as its displacement (rotation); otherwise, it is negative.

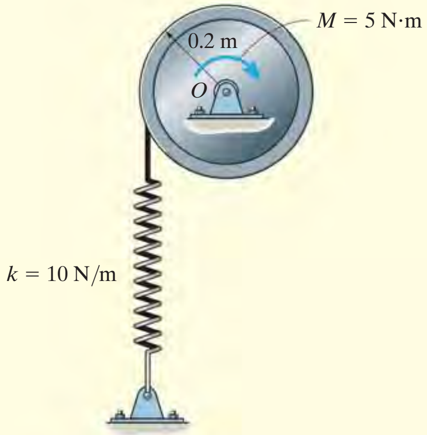
Principle of Work and Energy.

- Apply the principle of work and energy, $T_1 + \Sigma U_{1-2} = T_2$. Since this is a scalar equation, it can be used to solve for only one unknown when it is applied to a single rigid body.

*A brief review of Secs. 16.5 to 16.7 may prove helpful when solving problems, since computations for kinetic energy require a kinematic analysis of velocity.

EXAMPLE 18.2

The 30-kg disk shown in Fig. 18–12*a* is pin supported at its center. Determine the angle through which it must rotate to attain an angular velocity of 2 rad/s starting from rest. It is acted upon by a constant couple moment $M = 5\text{ N}\cdot\text{m}$. The spring is orginally unstretched and its cord wraps around the rim of the disk.



(a)

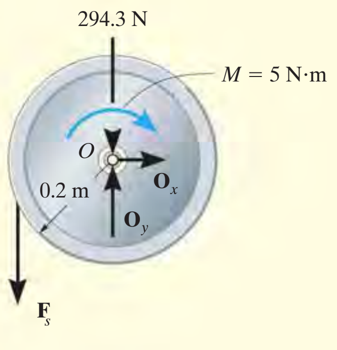
SOLUTION

Kinetic Energy. Since the disk rotates about a fixed axis, and it is initially at rest, then

$$T_1 = 0$$

$$T_2 = \tfrac{1}{2}I_O\omega_2^2 = \tfrac{1}{2}\left[\tfrac{1}{2}(30\text{ kg})(0.2\text{ m})^2\right](2\text{ rad/s})^2 = 1.2\text{ J}$$

Work (Free–Body Diagram). As shown in Fig. 18–12*b*, the pin reactions $\mathbf{O}_x$ and $\mathbf{O}_y$ and the weight (294.3 N) do no work, since they are not displaced. The *couple moment*, having a constant magnitude, does positive work $U_M = M\theta$ as the disk *rotates* through a clockwise angle of θ rad, and the spring does negative work $U_s = -\frac{1}{2}ks^2$.



(b)

Fig. 18–12

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

$$\{T_1\} + \left\{M\theta - \tfrac{1}{2}ks^2\right\} = \{T_2\}$$

$$\{0\} + \left\{(5\text{ N}\cdot\text{m})\theta - \frac{1}{2}(10\text{ N/m})[\theta(0.2\text{ m})]^2\right\} = \{1.2\text{ J}\}$$

$$-0.2\theta^2 + 5\theta - 1.2 = 0$$

Solving this quadratic equation for the smallest positive root,

$$\theta = 0.2423\text{ rad} = 0.2423\text{ rad}\left(\frac{180^\circ}{\pi\text{ rad}}\right) = 13.9^\circ \qquad \textit{Ans.}$$

The wheel shown in Fig. 18–13*a* weighs 40 lb and has a radius of gyration $k_G = 0.6$ ft about its mass center G. If it is subjected to a clockwise couple moment of 15 lb · ft and rolls from rest without slipping, determine its angular velocity after its center G moves 0.5 ft. The spring has a stiffness $k = 10$ lb/ft and is initially unstretched when the couple moment is applied.

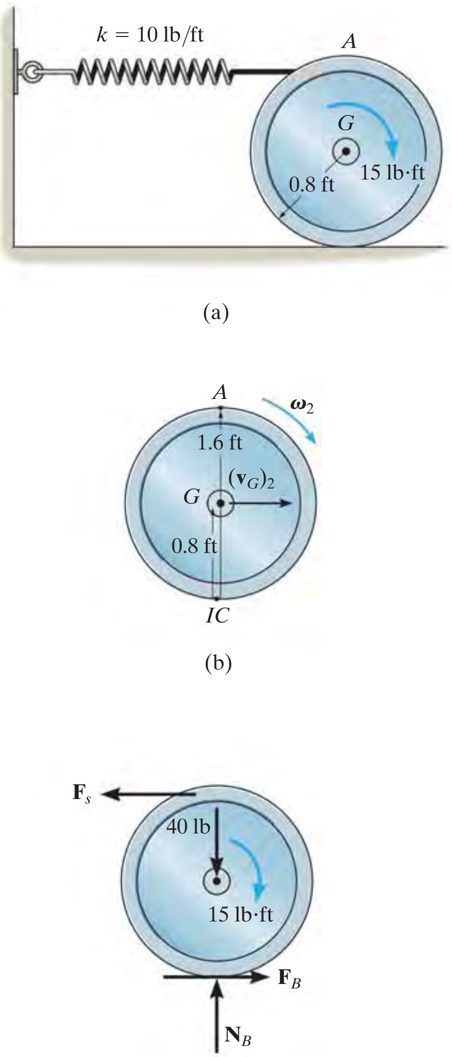


Fig. 18–13

SOLUTION

Kinetic Energy (Kinematic Diagram). Since the wheel is initially at rest,

$$T_1 = 0$$

The kinematic diagram of the wheel when it is in the final position is shown in Fig. 18–13*b*. The final kinetic energy is determined from

$$T_2 = \tfrac{1}{2} I_{IC} \omega_2^2$$

$$= \frac{1}{2}\left[\frac{40 \text{ lb}}{32.2 \text{ ft/s}^2}(0.6 \text{ ft})^2 + \left(\frac{40 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(0.8 \text{ ft})^2\right]\omega_2^2$$

$$T_2 = 0.6211\ \omega_2^2$$

Work (Free-Body Diagram). As shown in Fig. 18–13*c*, only the spring force $\mathbf{F}_s$ and the couple moment do work. The normal force does not move along its line of action and the frictional force does *no work*, since the wheel does not slip as it rolls.

The work of $\mathbf{F}_s$ is found using $U_s = -\tfrac{1}{2}ks^2$. Here the work is negative since $\mathbf{F}_s$ is in the opposite direction to displacement. Since the wheel does not slip when the center G moves 0.5 ft, then the wheel rotates $\theta = s_G/r_{G/IC} = 0.5 \text{ ft}/0.8 \text{ ft} = 0.625$ rad, Fig. 18–13*b*. Hence, the spring stretches $s = \theta r_{A/IC} = (0.625 \text{ rad})(1.6 \text{ ft}) = 1$ ft.

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

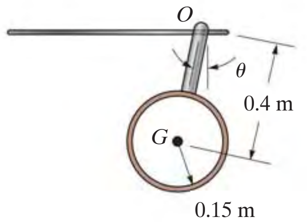
$$\{T_1\} + \left\{M\theta - \tfrac{1}{2}ks^2\right\} = \{T_2\}$$

$$\{0\} + \left\{15 \text{ lb} \cdot \text{ft}(0.625 \text{ rad}) - \frac{1}{2}(10 \text{ lb/ft})(1 \text{ ft})^2\right\} = \{0.6211\ \omega_2^2 \text{ ft} \cdot \text{lb}\}$$

$$\omega_2 = 2.65 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

(© R.C. Hibbeler)

The 700-kg pipe is equally suspended from the two tines of the fork lift shown in the photo. It is undergoing a swinging motion such that when $\theta = 30°$ it is momentarily at rest. Determine the normal and frictional forces acting on each tine which are needed to support the pipe at the instant $\theta = 0°$. Measurements of the pipe and the suspender are shown in Fig. 18–14*a*. Neglect the mass of the suspender and the thickness of the pipe.



(a)

Fig. 18–14

SOLUTION

We must use the equations of motion to find the forces on the tines since these forces do no work. Before doing this, however, we will apply the principle of work and energy to determine the angular velocity of the pipe when $\theta = 0°$.

Kinetic Energy (Kinematic Diagram). Since the pipe is originally at rest, then

$$T_1 = 0$$

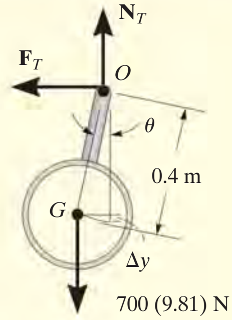
The final kinetic energy may be computed with reference to either the fixed point O or the center of mass G. For the calculation we will consider the pipe to be a thin ring so that $I_G = mr^2$. If point G is considered, we have

$$\begin{aligned} T_2 &= \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2 \\ &= \tfrac{1}{2}(700\text{ kg})[(0.4\text{ m})\omega_2]^2 + \tfrac{1}{2}[700\text{ kg}(0.15\text{ m})^2]\omega_2^2 \\ &= 63.875\omega_2^2 \end{aligned}$$

If point O is considered then the parallel-axis theorem must be used to determine I_O. Hence,

$$\begin{aligned} T_2 = \tfrac{1}{2}I_O\omega_2^2 &= \tfrac{1}{2}[700\text{ kg}(0.15\text{ m})^2 + 700\text{ kg}(0.4\text{ m})^2]\omega_2^2 \\ &= 63.875\omega_2^2 \end{aligned}$$

Work (Free-Body Diagram). Fig. 18–14*b*. The normal and frictional forces on the tines do no work since they do not move as the pipe swings. The weight does positive work since the weight moves downward through a vertical distance $\Delta y = 0.4 \text{ m} - 0.4 \cos 30° \text{ m} = 0.05359 \text{ m}$.



(b)

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

$$\{0\} + \{700(9.81) \text{ N}(0.05359 \text{ m})\} = \{63.875\omega_2^2\}$$

$$\omega_2 = 2.400 \text{ rad/s}$$

Equations of Motion. Referring to the free-body and kinetic diagrams shown in Fig. 18–14*c*, and using the result for ω_2, we have

$$\overset{+}{\leftarrow} \Sigma F_t = m(a_G)_t; \quad F_T = (700 \text{ kg})(a_G)_t$$

$$+\uparrow \Sigma F_n = m(a_G)_n; \quad N_T - 700(9.81) \text{ N} = (700 \text{ kg})(2.400 \text{ rad/s})^2(0.4 \text{ m})$$

$$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad 0 = [(700 \text{ kg})(0.15 \text{ m})^2 + (700 \text{ kg})(0.4 \text{ m})^2]\alpha$$

Since $(a_G)_t = (0.4 \text{ m})\alpha$, then

$$\alpha = 0, \quad (a_G)_t = 0$$

$$F_T = 0$$

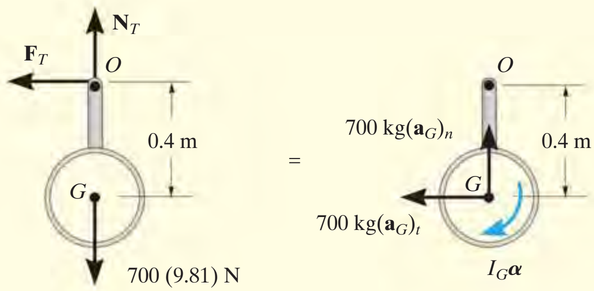
$$N_T = 8.480 \text{ kN}$$

There are two tines used to support the load, therefore

$$F'_T = 0 \qquad \textit{Ans.}$$

$$N'_T = \frac{8.480 \text{ kN}}{2} = 4.24 \text{ kN} \qquad \textit{Ans.}$$

NOTE: Due to the swinging motion the tines are subjected to a *greater* normal force than would be the case if the load were static, in which case $N'_T = 700(9.81) \text{ N}/2 = 3.43 \text{ kN}$.



(c)

Fig. 18–14

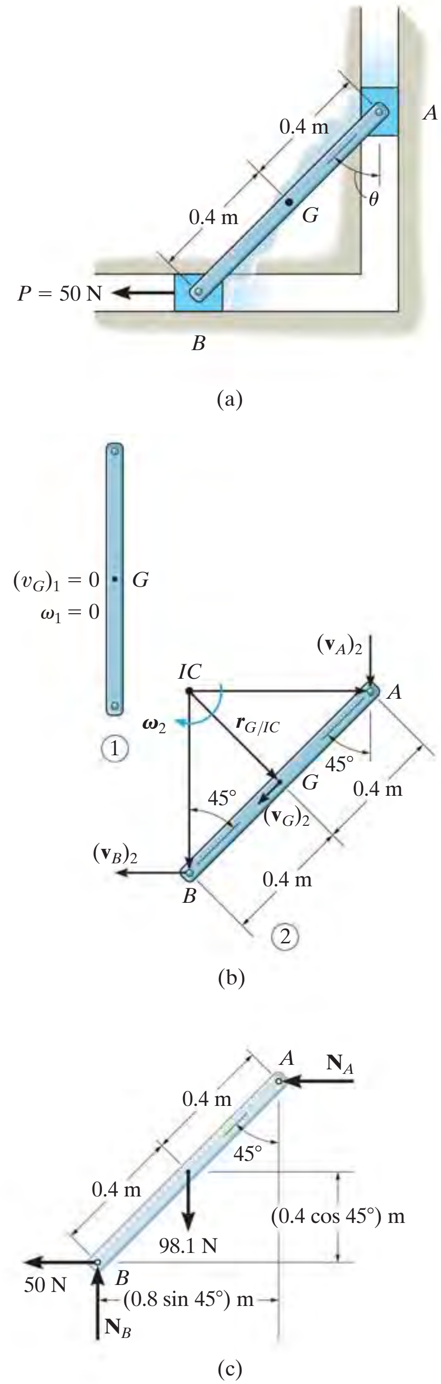


Fig. 18–15

The 10–kg rod shown in Fig. 18–15*a* is constrained so that its ends move along the grooved slots. The rod is initially at rest when $\theta = 0°$. If the slider block at B is acted upon by a horizontal force $P = 50$ N, determine the angular velocity of the rod at the instant $\theta = 45°$. Neglect friction and the mass of blocks A and B.

SOLUTION

Why can the principle of work and energy be used to solve this problem?

Kinetic Energy (Kinematic Diagrams). Two kinematic diagrams of the rod, when it is in the initial position 1 and final position 2, are shown in Fig. 18–15*b*. When the rod is in position 1, $T_1 = 0$ since $(\mathbf{v}_G)_1 = \boldsymbol{\omega}_1 = \mathbf{0}$. In position 2 the angular velocity is $\boldsymbol{\omega}_2$ and the velocity of the mass center is $(\mathbf{v}_G)_2$. Hence, the kinetic energy is

$$\begin{aligned} T_2 &= \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2 \\ &= \tfrac{1}{2}(10\text{ kg})(v_G)_2^2 + \tfrac{1}{2}\left[\tfrac{1}{12}(10\text{ kg})(0.8\text{ m})^2\right]\omega_2^2 \\ &= 5(v_G)_2^2 + 0.2667(\omega_2)^2 \end{aligned}$$

The two unknowns $(v_G)_2$ and ω_2 can be related from the instantaneous center of zero velocity for the rod. Fig. 18–15*b*. It is seen that as A moves downward with a velocity $(\mathbf{v}_A)_2$, B moves horizontally to the left with a velocity $(\mathbf{v}_B)_2$, Knowing these directions, the IC is located as shown in the figure. Hence,

$$\begin{aligned} (v_G)_2 &= r_{G/IC}\omega_2 = (0.4 \tan 45°\text{ m})\omega_2 \\ &= 0.4\omega_2 \end{aligned}$$

Therefore,

$$T_2 = 0.8\omega_2^2 + 0.2667\omega_2^2 = 1.0667\omega_2^2$$

Of course, we can also determine this result using $T_2 = \frac{1}{2}I_{IC}\omega_2^2$.

Work (Free–Body Diagram). Fig. 18–15*c*. The normal forces $\mathbf{N}_A$ and $\mathbf{N}_B$ do no work as the rod is displaced. Why? The 98.1-N weight is displaced a vertical distance of $\Delta y = (0.4 - 0.4\cos 45°)$ m; whereas the 50-N force moves a horizontal distance of $s = (0.8 \sin 45°)$ m. Both of these forces do positive work. Why?

Principle of Work and Energy.

$$\{T_1\} + \{\Sigma U_{1-2}\} = \{T_2\}$$

$$\{T_1\} + \{W\,\Delta y + Ps\} = \{T_2\}$$

$$\{0\} + \{98.1\text{ N}(0.4\text{ m} - 0.4\cos 45°\text{ m}) + 50\text{ N}(0.8 \sin 45°\text{ m})\}$$

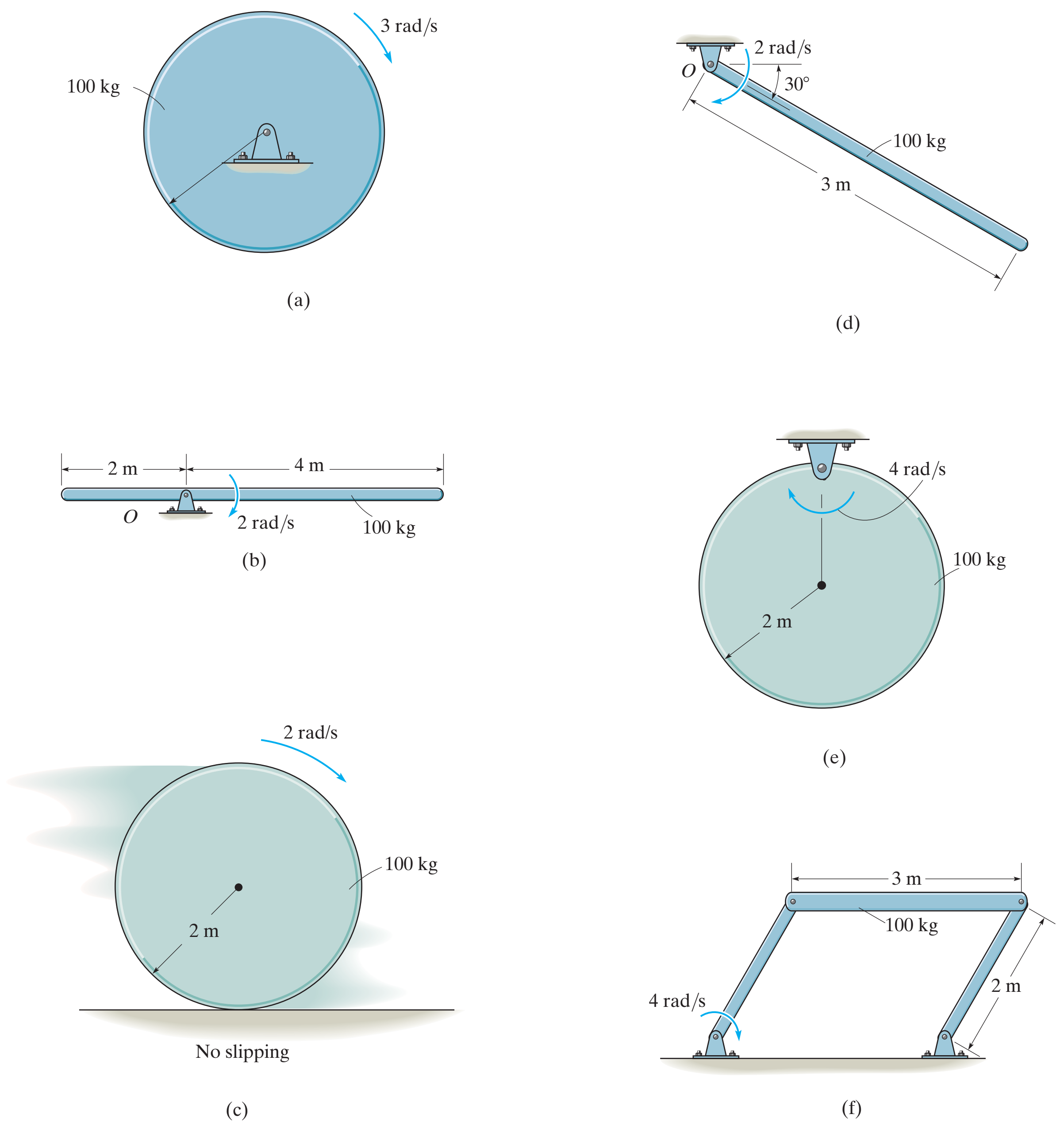
$$= \{1.0667\omega_2^2\text{ J}\}$$

Solving for ω_2 gives

$$\omega_2 = 6.11\text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

PRELIMINARY PROBLEM

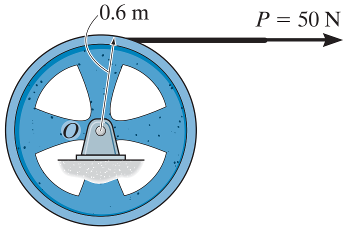
P18–1. Determine the kinetic energy of the 100-kg object.



Prob. P18–1

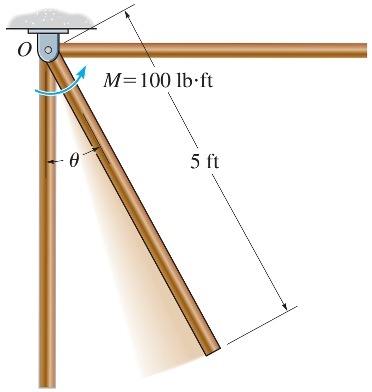
FUNDAMENTAL PROBLEMS

F18–1. The 80-kg wheel has a radius of gyration about its mass center O of $k_O = 400$ mm. Determine its angular velocity after it has rotated 20 revolutions starting from rest.



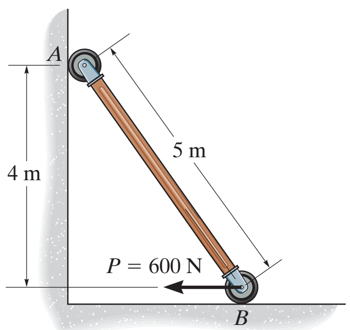
Prob. F18–1

F18–2. The uniform 50-lb slender rod is subjected to a couple moment of $M = 100$ lb · ft. If the rod is at rest when $\theta = 0°$, determine its angular velocity when $\theta = 90°$.



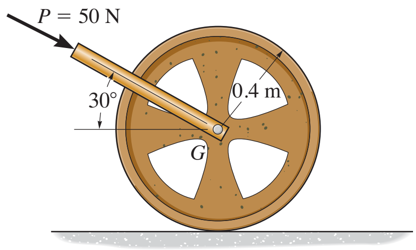
Prob. F18–2

F18–3. The uniform 50-kg slender rod is at rest in the position shown when $P = 600$ N is applied. Determine the angular velocity of the rod when the rod reaches the vertical position.



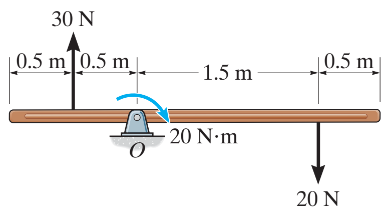
Prob. F18–3

F18–4. The 50-kg wheel is subjected to a force of 50 N. If the wheel starts from rest and rolls without slipping, determine its angular velocity after it has rotated 10 revolutions. The radius of gyration of the wheel about its mass center G is $k_G = 0.3$ m.



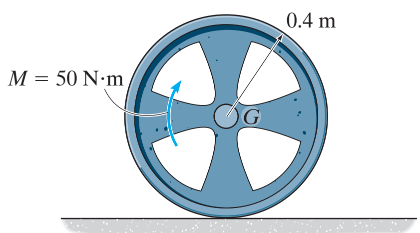
Prob. F18–4

F18–5. If the uniform 30-kg slender rod starts from rest at the position shown, determine its angular velocity after it has rotated 4 revolutions. The forces remain perpendicular to the rod.



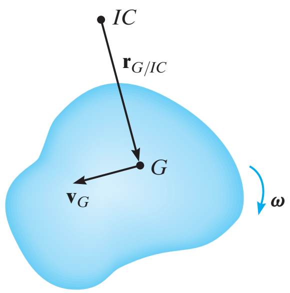
Prob. F18–5

F18–6. The 20-kg wheel has a radius of gyration about its center G of $k_G = 300$ mm. When it is subjected to a couple moment of $M = 50$ N · m, it rolls without slipping. Determine the angular velocity of the wheel after its mass center G has traveled through a distance of $s_G = 20$ m, starting from rest.



Prob. F18–6

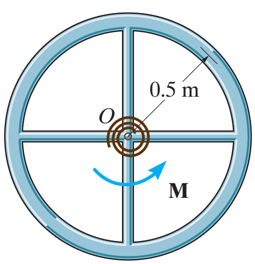
18–1. At a given instant the body of mass m has an angular velocity $\boldsymbol{\omega}$ and its mass center has a velocity $\mathbf{v}_G$. Show that its kinetic energy can be represented as $T = \frac{1}{2}I_{IC}\omega^2$, where I_{IC} is the moment of inertia of the body determined about the instantaneous axis of zero velocity, located a distance $r_{G/IC}$ from the mass center as shown.



Prob. 18–1

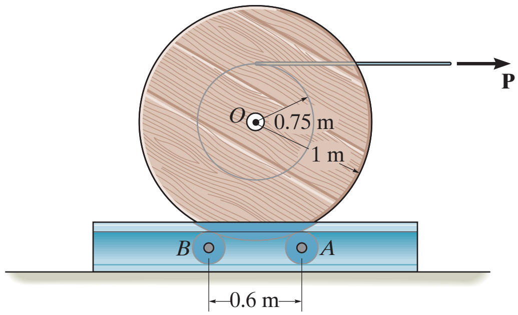
18–2. The wheel is made from a 5-kg thin ring and two 2-kg slender rods. If the torsional spring attached to the wheel's center has a stiffness $k = 2 \text{ N} \cdot \text{m/rad}$, and the wheel is rotated until the torque $M = 25 \text{ N} \cdot \text{m}$ is developed, determine the maximum angular velocity of the wheel if it is released from rest.

18–3. The wheel is made from a 5-kg thin ring and two 2-kg slender rods. If the torsional spring attached to the wheel's center has a stiffness $k = 2 \text{ N} \cdot \text{m/rad}$, so that the torque on the center of the wheel is $M = (2\theta) \text{ N} \cdot \text{m}$, where θ is in radians, determine the maximum angular velocity of the wheel if it is rotated two revolutions and then released from rest.



Probs. 18–2/3

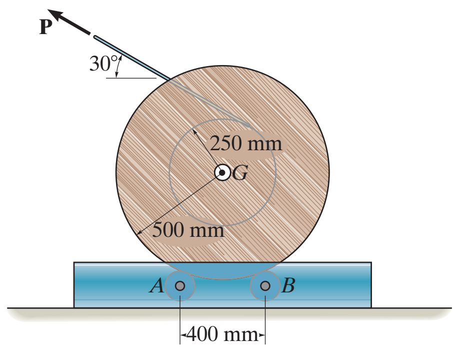
*__18–4.__ A force of $P = 60$ N is applied to the cable, which causes the 200-kg reel to turn since it is resting on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the rollers and the mass of the cable. Assume the radius of gyration of the reel about its center axis remains constant at $k_O = 0.6$ m.



Prob. 18–4

18–5. A force of $P = 20$ N is applied to the cable, which causes the 175-kg reel to turn since it is resting on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the rollers and the mass of the cable. The radius of gyration of the reel about its center axis is $k_G = 0.42$ m.

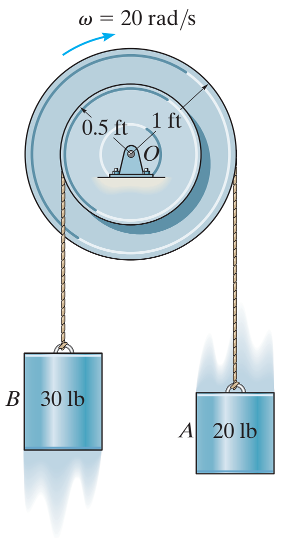
18–6. A force of $P = 20$ N is applied to the cable, which causes the 175-kg reel to turn without slipping on the two rollers A and B of the dispenser. Determine the angular velocity of the reel after it has made two revolutions starting from rest. Neglect the mass of the cable. Each roller can be considered as an 18-kg cylinder, having a radius of 0.1 m. The radius of gyration of the reel about its center axis is $k_G = 0.42$ m.



Probs. 18–5/6

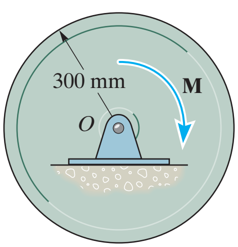
18–7. The double pulley consists of two parts that are attached to one another. It has a weight of 50 lb and a centroidal radius of gyration of $k_O = 0.6$ ft and is turning with an angular velocity of 20 rad/s clockwise. Determine the kinetic energy of the system. Assume that neither cable slips on the pulley.

***18–8.** The double pulley consists of two parts that are attached to one another. It has a weight of 50 lb and a centroidal radius of gyration of $k_O = 0.6$ ft and is turning with an angular velocity of 20 rad/s clockwise. Determine the angular velocity of the pulley at the instant the 20-lb weight moves 2 ft downward.



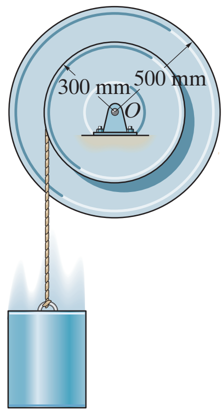
Probs. 18–7/8

18–9. The disk, which has a mass of 20 kg, is subjected to the couple moment of $M = (2\theta + 4)$ N·m, where θ is in radians. If it starts from rest, determine its angular velocity when it has made two revolutions.



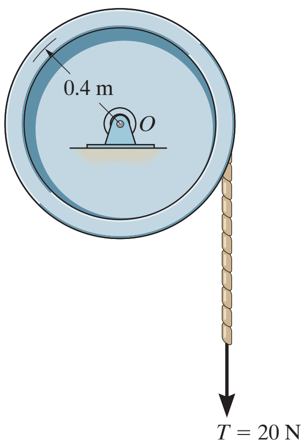
Prob. 18–9

18–10. The spool has a mass of 40 kg and a radius of gyration of $k_O = 0.3$ m. If the 10-kg block is released from rest, determine the distance the block must fall in order for the spool to have an angular velocity $\omega = 15$ rad/s. Also, what is the tension in the cord while the block is in motion? Neglect the mass of the cord.



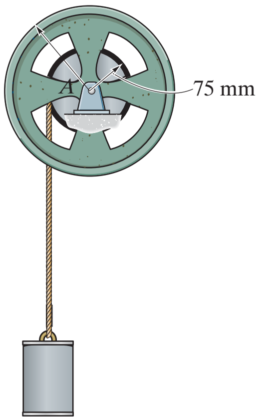
Prob. 18–10

18–11. The force of $T = 20$ N is applied to the cord of negligible mass. Determine the angular velocity of the 20-kg wheel when it has rotated 4 revolutions starting from rest. The wheel has a radius of gyration of $k_O = 0.3$ m.



Prob. 18–11

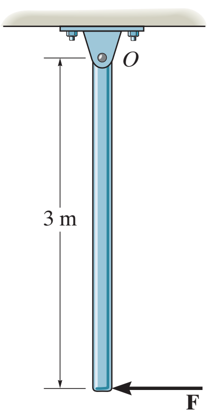
***18–12.** Determine the velocity of the 50-kg cylinder after it has descended a distance of 2 m. Initially, the system is at rest. The reel has a mass of 25 kg and a radius of gyration about its center of mass A of $k_A = 125$ mm.



Prob. 18–12

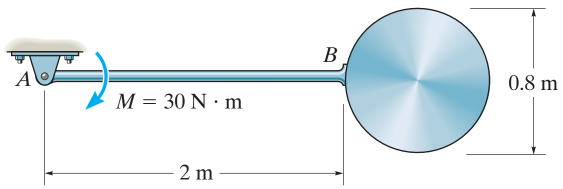
18–13. The 10-kg uniform slender rod is suspended at rest when the force of $F = 150$ N is applied to its end. Determine the angular velocity of the rod when it has rotated 90° clockwise from the position shown. The force is always perpendicular to the rod.

18–14. The 10-kg uniform slender rod is suspended at rest when the force of $F = 150$ N is applied to its end. Determine the angular velocity of the rod when it has rotated 180° clockwise from the position shown. The force is always perpendicular to the rod.



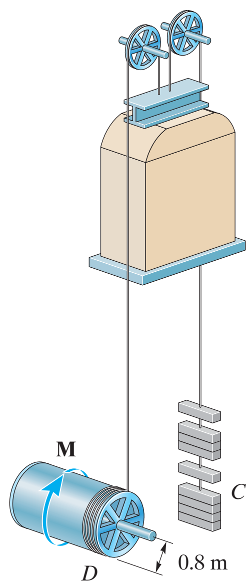
Probs. 18–13/14

18–15. The pendulum consists of a 10-kg uniform disk and a 3-kg uniform slender rod. If it is released from rest in the position shown, determine its angular velocity when it rotates clockwise 90°.



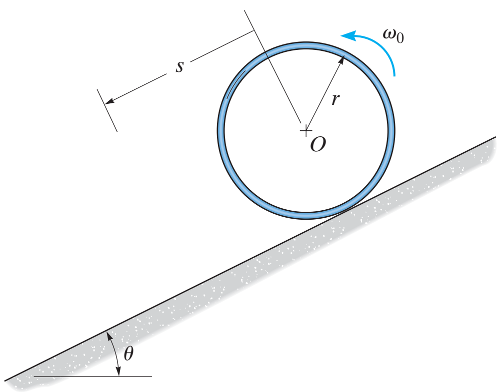
Prob. 18–15

***18–16.** A motor supplies a constant torque $M = 6$ kN · m to the winding drum that operates the elevator. If the elevator has a mass of 900 kg, the counterweight C has a mass of 200 kg, and the winding drum has a mass of 600 kg and radius of gyration about its axis of $k = 0.6$ m, determine the speed of the elevator after it rises 5 m starting from rest. Neglect the mass of the pulleys.



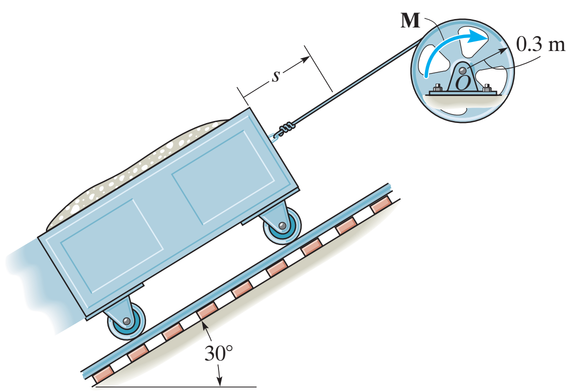
Prob. 18–16

18–17. The center O of the thin ring of mass m is given an angular velocity of ω_0. If the ring rolls without slipping, determine its angular velocity after it has traveled a distance of s down the plane. Neglect its thickness.



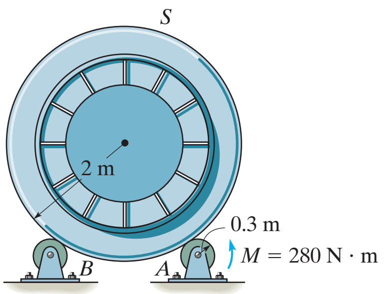
Prob. 18–17

18–18. The wheel has a mass of 100 kg and a radius of gyration of $k_O = 0.2$ m. A motor supplies a torque $M = (40\theta + 900)$ N · m, where θ is in radians, about the drive shaft at O. Determine the speed of the loading car, which has a mass of 300 kg, after it travels $s = 4$ m. Initially the car is at rest when $s = 0$ and $\theta = 0°$. Neglect the mass of the attached cable and the mass of the car's wheels.



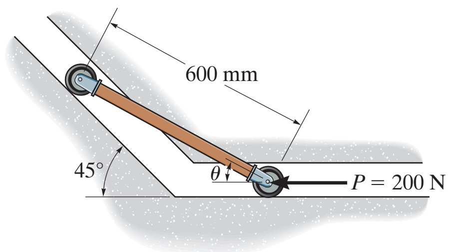
Prob. 18–18

18–19. The rotary screen S is used to wash limestone. When empty it has a mass of 800 kg and a radius of gyration of $k_G = 1.75$ m. Rotation is achieved by applying a torque of $M = 280$ N · m about the drive wheel at A. If no slipping occurs at A and the supporting wheel at B is free to roll, determine the angular velocity of the screen after it has rotated 5 revolutions. Neglect the mass of A and B.



Prob. 18–19

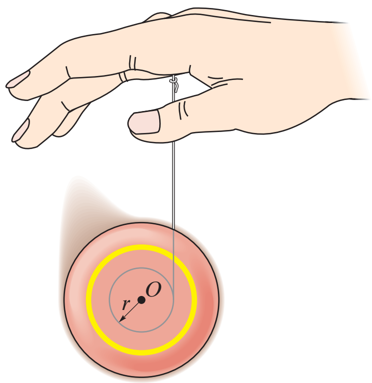
***18–20.** If $P = 200$ N and the 15-kg uniform slender rod starts from rest at $\theta = 0°$, determine the rod's angular velocity at the instant just before $\theta = 45°$.



Prob. 18–20

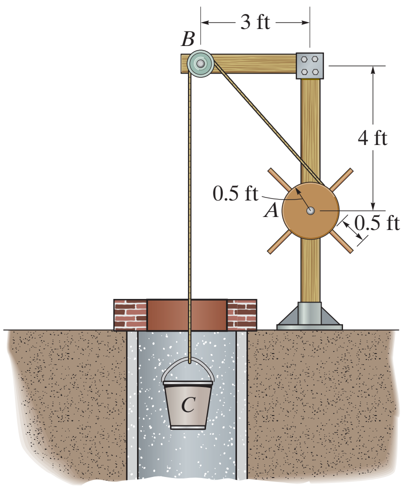
18

18–21. A yo-yo has a weight of 0.3 lb and a radius of gyration of $k_O = 0.06$ ft. If it is released from rest, determine how far it must descend in order to attain an angular velocity $\omega = 70$ rad/s. Neglect the mass of the string and assume that the string is wound around the central peg such that the mean radius at which it unravels is $r = 0.02$ ft.



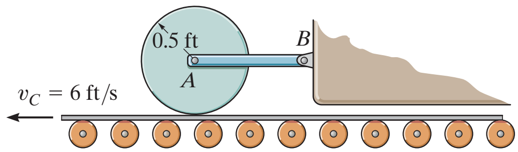
Prob. 18–21

18–22. If the 50-lb bucket, C, is released from rest, determine its velocity after it has fallen a distance of 10 ft. The windlass A can be considered as a 30-lb cylinder, while the spokes are slender rods, each having a weight of 2 lb. Neglect the pulley's weight.



Prob. 18–22

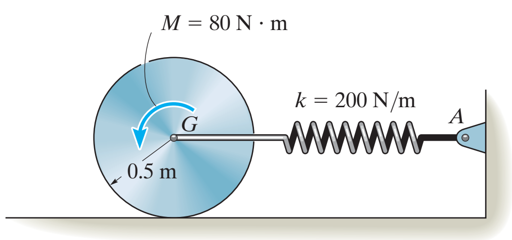
18–23. The coefficient of kinetic friction between the 100-lb disk and the surface of the conveyor belt is $\mu_A = 0.2$. If the conveyor belt is moving with a speed of $v_C = 6$ ft/s when the disk is placed in contact with it, determine the number of revolutions the disk makes before it reaches a constant angular velocity.



Prob. 18–23

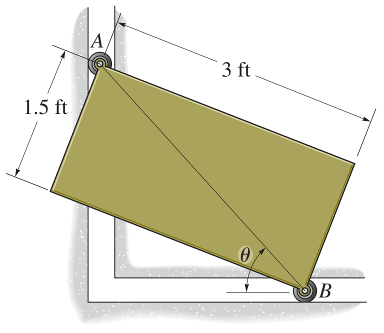
***18–24.** The 30-kg disk is originally at rest, and the spring is unstretched. A couple moment of $M = 80$ N·m is then applied to the disk as shown. Determine its angular velocity when its mass center G has moved 0.5 m along the plane. The disk rolls without slipping.

18–25. The 30-kg disk is originally at rest, and the spring is unstretched. A couple moment $M = 80$ N·m is then applied to the disk as shown. Determine how far the center of mass of the disk travels along the plane before it momentarily stops. The disk rolls without slipping.



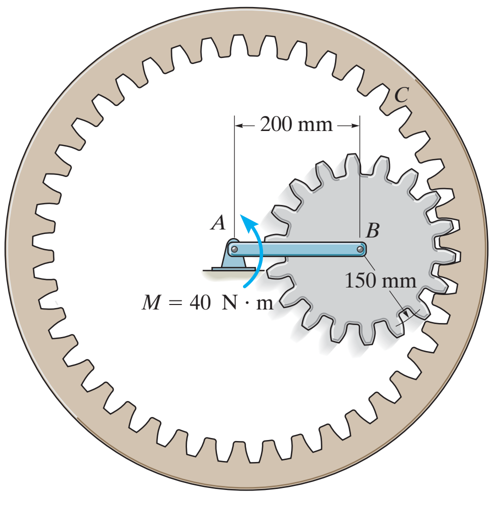
Probs. 18–24/25

18–26. Two wheels of negligible weight are mounted at corners A and B of the rectangular 75-lb plate. If the plate is released from rest at $\theta = 90°$, determine its angular velocity at the instant just before $\theta = 0°$.



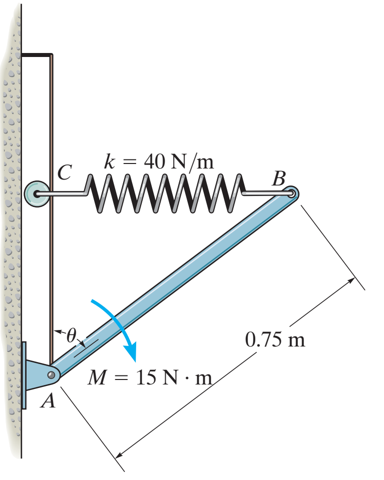
Prob. 18–26

18–27. The link AB is subjected to a couple moment of $M = 40$ N·m. If the ring gear C is fixed, determine the angular velocity of the 15-kg inner gear when the link has made two revolutions starting from rest. Neglect the mass of the link and assume the inner gear is a disk. Motion occurs in the vertical plane.



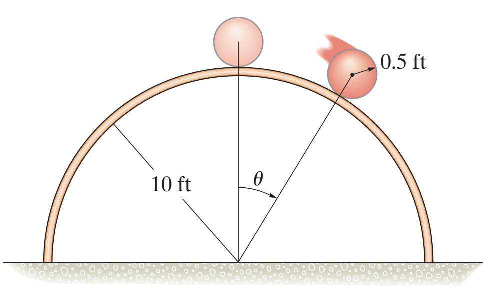
Prob. 18–27

***18–28.** The 10-kg rod AB is pin connected at A and subjected to a couple moment of $M = 15$ N·m. If the rod is released from rest when the spring is unstretched at $\theta = 30°$, determine the rod's angular velocity at the instant $\theta = 60°$. As the rod rotates, the spring always remains horizontal, because of the roller support at C.



Prob. 18–28

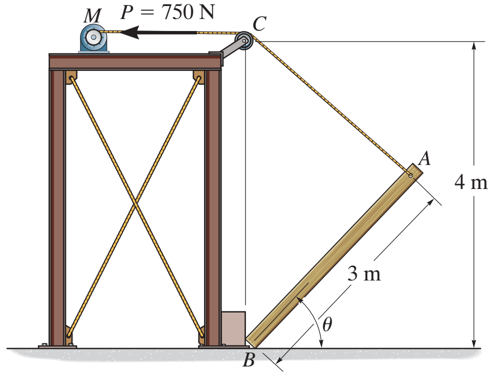
18–29. The 10-lb sphere starts from rest at $\theta = 0°$ and rolls without slipping down the cylindrical surface which has a radius of 10 ft. Determine the speed of the sphere's center of mass at the instant $\theta = 45°$.



Prob. 18–29

18

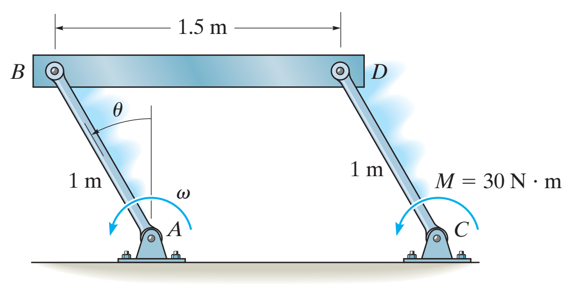
18–30. Motor M exerts a constant force of $P = 750$ N on the rope. If the 100-kg post is at rest when $\theta = 0°$, determine the angular velocity of the post at the instant $\theta = 60°$. Neglect the mass of the pulley and its size, and consider the post as a slender rod.



Prob. 18–30

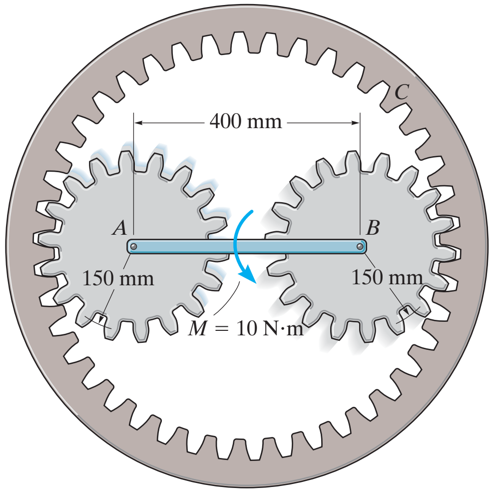
18–31. The linkage consists of two 6-kg rods AB and CD and a 20-kg bar BD. When $\theta = 0°$, rod AB is rotating with an angular velocity $\omega = 2$ rad/s. If rod CD is subjected to a couple moment of $M = 30$ N·m, determine ω_{AB} at the instant $\theta = 90°$.

***18–32.** The linkage consists of two 6-kg rods AB and CD and a 20-kg bar BD. When $\theta = 0°$, rod AB is rotating with an angular velocity $\omega = 2$ rad/s. If rod CD is subjected to a couple moment $M = 30$ N·m, determine ω at the instant $\theta = 45°$.



Probs. 18–31/32

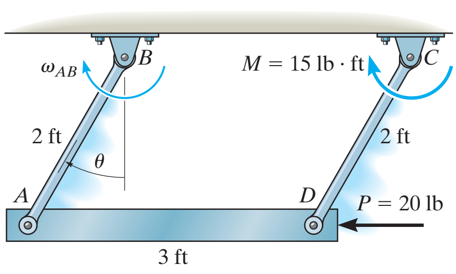
18–33. The two 2-kg gears A and B are attached to the ends of a 3-kg slender bar. The gears roll within the fixed ring gear C, which lies in the horizontal plane. If a 10-N·m torque is applied to the center of the bar as shown, determine the number of revolutions the bar must rotate starting from rest in order for it to have an angular velocity of $\omega_{AB} = 20$ rad/s. For the calculation, assume the gears can be approximated by thin disks. What is the result if the gears lie in the vertical plane?



Prob. 18–33

18–34. The linkage consists of two 8-lb rods AB and CD and a 10-lb bar AD. When $\theta = 0°$, rod AB is rotating with an angular velocity $\omega_{AB} = 2$ rad/s. If rod CD is subjected to a couple moment $M = 15$ lb·ft and bar AD is subjected to a horizontal force $P = 20$ lb as shown, determine ω_{AB} at the instant $\theta = 90°$.

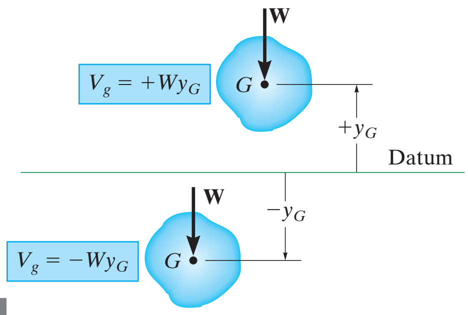
18–35. The linkage consists of two 8-lb rods AB and CD and a 10-lb bar AD. When $\theta = 0°$, rod AB is rotating with an angular velocity $\omega_{AB} = 2$ rad/s. If rod CD is subjected to a couple moment $M = 15$ lb·ft and bar AD is subjected to a horizontal force $P = 20$ lb as shown, determine ω_{AB} at the instant $\theta = 45°$.



Probs. 18–34/35

18.5 Conservation of Energy

When a force system acting on a rigid body consists only of *conservative forces*, the conservation of energy theorem can be used to solve a problem that otherwise would be solved using the principle of work and energy. This theorem is often easier to apply since the work of a conservative force is *independent of the path* and depends only on the initial and final positions of the body. It was shown in Sec. 14.5 that the work of a conservative force can be expressed as the difference in the body's potential energy measured from an arbitrarily selected reference or datum.



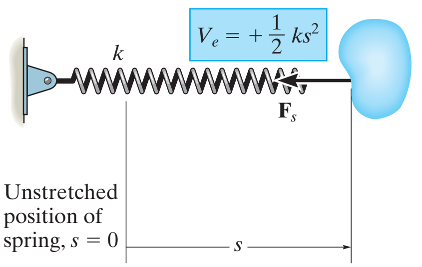
Gravitational potential energy

Fig. 18–16

Gravitational Potential Energy. Since the total weight of a body can be considered concentrated at its center of gravity, the *gravitational potential energy* of the body is determined by knowing the height of the body's center of gravity above or below a horizontal datum.

$$V_g = Wy_G \qquad (18\text{–}15)$$

Here the potential energy is *positive* when y_G is positive upward, since the weight has the ability to do *positive work* when the body moves back to the datum, Fig. 18–16. Likewise, if G is located *below* the datum $(-y_G)$, the gravitational potential energy is *negative*, since the weight does *negative work* when the body returns to the datum.



Elastic potential energy

Fig. 18–17

Elastic Potential Energy. The force developed by an elastic spring is also a conservative force. The *elastic potential energy* which a spring imparts to an attached body when the spring is stretched or compressed from an initial undeformed position ($s = 0$) to a final position s, Fig. 18–17, is

$$V_e = +\tfrac{1}{2}ks^2 \qquad (18\text{–}16)$$

In the deformed position, the spring force acting *on the body* always has the ability for doing positive work when the spring returns back to its original undeformed position (see Sec. 14.5).

Conservation of Energy. In general, if a body is subjected to both gravitational and elastic forces, the total *potential energy* can be expressed as a potential function represented as the algebraic sum

$$V = V_g + V_e \tag{18–17}$$

Here measurement of V depends upon the location of the body with respect to the selected datum.

Realizing that the work of conservative forces can be written as a difference in their potential energies, i.e., $(\Sigma U_{1-2})_{\text{cons}} = V_1 - V_2$, Eq. 14–16, we can rewrite the principle of work and energy for a rigid body as

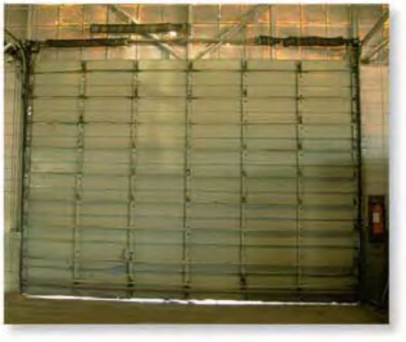
$$T_1 + V_1 + (\Sigma U_{1-2})_{\text{noncons}} = T_2 + V_2 \tag{18–18}$$

Here $(\Sigma U_{1-2})_{\text{noncons}}$ represents the work of the nonconservative forces such as friction. If this term is zero, then

$$T_1 + V_1 = T_2 + V_2 \tag{18–19}$$

This equation is referred to as the conservation of mechanical energy. It states that the *sum* of the potential and kinetic energies of the body remains *constant* when the body moves from one position to another. It also applies to a system of smooth, pin-connected rigid bodies, bodies connected by inextensible cords, and bodies in mesh with other bodies. In all these cases the forces acting at the points of contact are *eliminated* from the analysis, since they occur in equal but opposite collinear pairs and each pair of forces moves through an equal distance when the system undergoes a displacement.

It is important to remember that only problems involving conservative force systems can be solved by using Eq. 18–19. As stated in Sec. 14.5, friction or other drag-resistant forces, which depend on velocity or acceleration, are nonconservative. The work of such forces is transformed into thermal energy used to heat up the surfaces of contact, and consequently this energy is dissipated into the surroundings and may not be recovered. Therefore, problems involving frictional forces can be solved by using either the principle of work and energy written in the form of Eq. 18–18, if it applies, or the equations of motion.



The torsional springs located at the top of the garage door wind up as the door is lowered. When the door is raised, the potential energy stored in the springs is then transferred into gravitational potential energy of the door's weight, thereby making it easy to open. (© R.C. Hibbeler)

Procedure for Analysis

The conservation of energy equation is used to solve problems involving *velocity, displacement*, and *conservative force systems*. For application it is suggested that the following procedure be used.

Potential Energy.

- Draw two diagrams showing the body located at its initial and final positions along the path.
- If the center of gravity, G, is subjected to a *vertical displacement*, establish a fixed horizontal datum from which to measure the body's gravitational potential energy V_g.
- Data pertaining to the elevation y_G of the body's center of gravity from the datum and the extension or compression of any connecting springs can be determined from the problem geometry and listed on the two diagrams.
- The potential energy is determined from $V = V_g + V_e$. Here $V_g = Wy_G$, which can be positive or negative, and $V_e = \frac{1}{2}ks^2$, which is always positive.

Kinetic Energy.

- The kinetic energy of the body consists of two parts, namely translational kinetic energy, $T = \frac{1}{2}mv_G^2$, and rotational kinetic energy, $T = \frac{1}{2}I_G\omega^2$.
- Kinematic diagrams for velocity may be useful for establishing a *relationship* between v_G and ω.

Conservation of Energy.

- Apply the conservation of energy equation $T_1 + V_1 = T_2 + V_2$.

The 10-kg rod AB shown in Fig. 18–18a is confined so that its ends move in the horizontal and vertical slots. The spring has a stiffness of $k = 800$ N/m and is unstretched when $\theta = 0°$. Determine the angular velocity of AB when $\theta = 0°$, if the rod is released from rest when $\theta = 30°$. Neglect the mass of the slider blocks.

SOLUTION

Potential Energy. The two diagrams of the rod, when it is located at its initial and final positions, are shown in Fig. 18–18b. The datum, used to measure the gravitational potential energy, is placed in line with the rod when $\theta = 0°$.

When the rod is in position 1, the center of gravity G is located *below the datum* so its gravitational potential energy is *negative*. Furthermore, (positive) elastic potential energy is stored in the spring, since it is stretched a distance of $s_1 = (0.4 \sin 30°)$ m. Thus,

$$V_1 = -Wy_1 + \tfrac{1}{2}ks_1^2$$

$$= -(98.1\text{ N})(0.2 \sin 30°\text{ m}) + \tfrac{1}{2}(800\text{ N/m})(0.4 \sin 30°\text{ m})^2 = 6.19\text{ J}$$

When the rod is in position 2, the potential energy of the rod is zero, since the center of gravity G is located at the datum, and the spring is unstretched, $s_2 = 0$. Thus,

$$V_2 = 0$$

Kinetic Energy. The rod is released from rest from position 1, thus $(\mathbf{v}_G)_1 = \boldsymbol{\omega}_1 = \mathbf{0}$, and so

$$T_1 = 0$$

In position 2, the angular velocity is $\boldsymbol{\omega}_2$ and the rod's mass center has a velocity of $(\mathbf{v}_G)_2$. Thus,

$$T_2 = \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2$$

$$= \tfrac{1}{2}(10\text{ kg})(v_G)_2^2 + \tfrac{1}{2}\left[\tfrac{1}{12}(10\text{ kg})(0.4\text{ m})^2\right]\omega_2^2$$

Using *kinematics*, $(\mathbf{v}_G)_2$ can be related to $\boldsymbol{\omega}_2$ as shown in Fig. 18–18c. At the instant considered, the instantaneous center of zero velocity (IC) for the rod is at point A; hence, $(v_G)_2 = (r_{G/IC})\omega_2 = (0.2\text{ m})\omega_2$. Substituting into the above expression and simplifying (or using $\frac{1}{2}I_{IC}\omega_2^2$), we get

$$T_2 = 0.2667\omega_2^2$$

Conservation of Energy.

$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$

$$\{0\} + \{6.19\text{ J}\} = \{0.2667\omega_2^2\} + \{0\}$$

$$\omega_2 = 4.82\text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

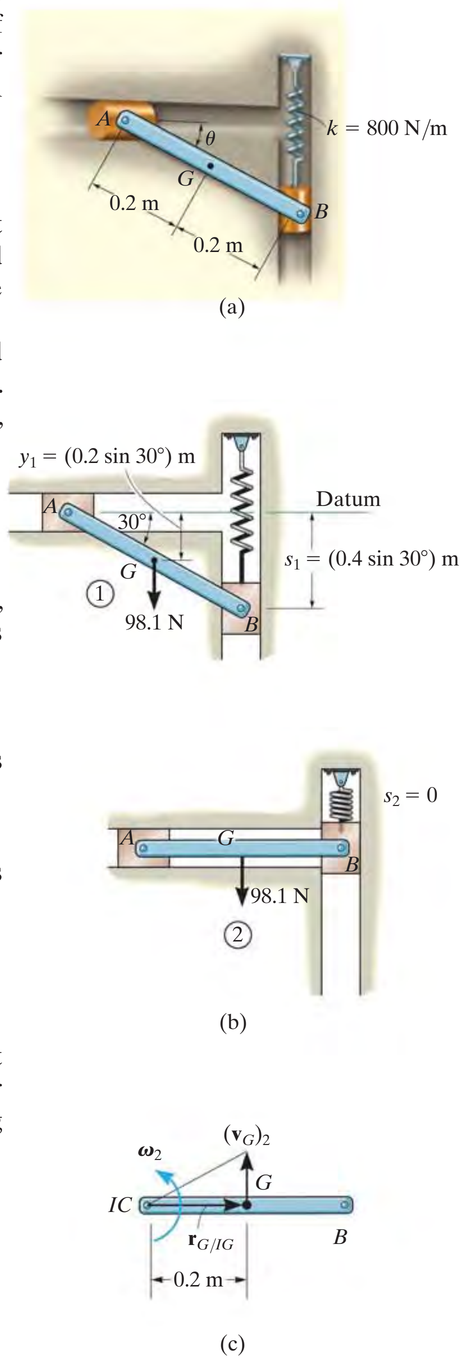


Fig. 18–18

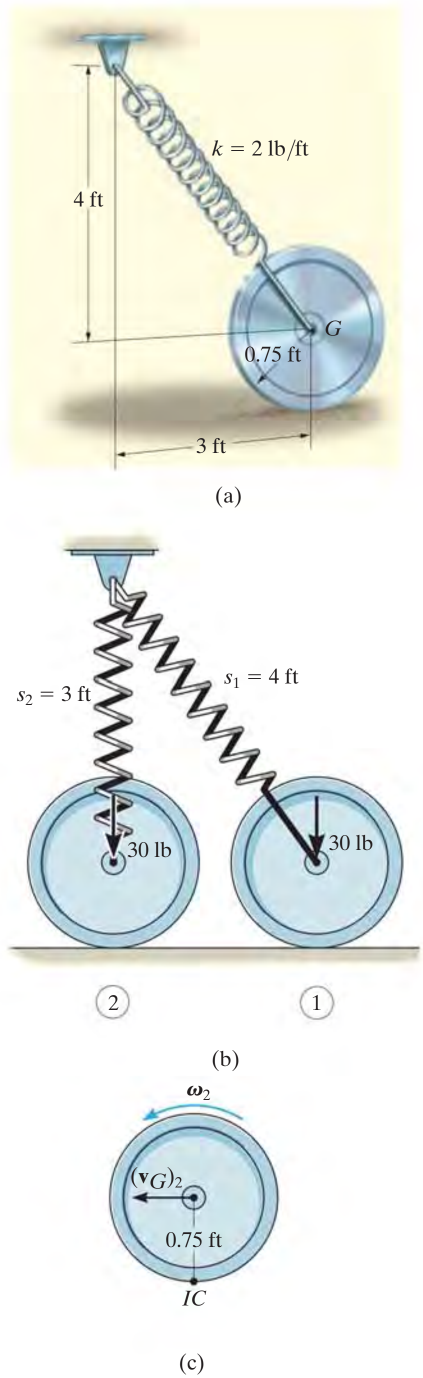


Fig. 18–19

The wheel shown in Fig. 18–19*a* has a weight of 30 lb and a radius of gyration of $k_G = 0.6$ ft. It is attached to a spring which has a stiffness $k = 2$ lb/ft and an unstretched length of 1 ft. If the disk is released from rest in the position shown and rolls without slipping, determine its angular velocity at the instant G moves 3 ft to the left.

SOLUTION

Potential Energy. Two diagrams of the wheel, when it at the initial and final positions, are shown in Fig. 18–19*b*. A gravitational datum is not needed here since the weight is not displaced vertically. From the problem geometry the spring is stretched $s_1 = \left(\sqrt{3^2 + 4^2} - 1\right) = 4$ ft in the initial position, and spring $s_2 = (4 - 1) = 3$ ft in the final position. Hence, the positive spring potential energy is

$$V_1 = \tfrac{1}{2}ks_1^2 = \tfrac{1}{2}(2\text{ lb/ft})(4\text{ ft})^2 = 16\text{ ft}\cdot\text{lb}$$
$$V_2 = \tfrac{1}{2}ks_2^2 = \tfrac{1}{2}(2\text{ lb/ft})(3\text{ ft})^2 = 9\text{ ft}\cdot\text{lb}$$

Kinetic Energy. The disk is released from rest and so $(\mathbf{v}_G)_1 = \mathbf{0}$, $\boldsymbol{\omega}_1 = \mathbf{0}$. Therefore,

$$T_1 = 0$$

Since the instantaneous center of zero velocity is at the ground, Fig. 18–19*c*, we have

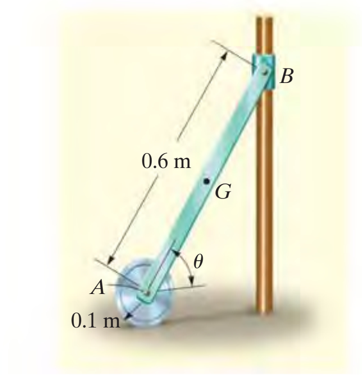
$$\begin{aligned} T_2 &= \frac{1}{2}I_{IC}\omega_2^2 \\ &= \frac{1}{2}\left[\left(\frac{30\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.6\text{ ft})^2 + \left(\frac{30\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.75\text{ ft})^2\right]\omega_2^2 \\ &= 0.4297\omega_2^2 \end{aligned}$$

Conservation of Energy.

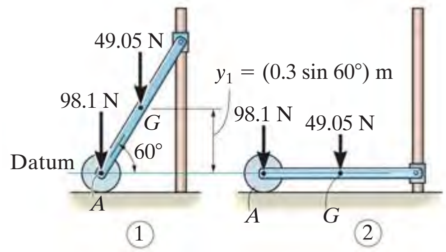
$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$
$$\{0\} + \{16\text{ ft}\cdot\text{lb}\} = \{0.4297\omega_2^2\} + \{9\text{ ft}\cdot\text{lb}\}$$
$$\omega_2 = 4.04\text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

NOTE: If the principle of work and energy were used to solve this problem, then the work of the spring would have to be determined by considering both the change in magnitude and direction of the spring force.

The 10-kg homogeneous disk shown in Fig. 18–20*a* is attached to a uniform 5-kg rod AB. If the assembly is released from rest when $\theta = 60^\circ$, determine the angular velocity of the rod when $\theta = 0^\circ$. Assume that the disk rolls without slipping. Neglect friction along the guide and the mass of the collar at B.



(a)



(b)

SOLUTION

Potential Energy. Two diagrams for the rod and disk, when they are located at their initial and final positions, are shown in Fig. 18–20*b*. For convenience the datum passes through point A.

When the system is in position 1, only the rod's weight has positive potential energy. Thus,

$$V_1 = W_r y_1 = (49.05 \text{ N})(0.3 \sin 60^\circ \text{ m}) = 12.74 \text{ J}$$

When the system is in position 2, both the weight of the rod and the weight of the disk have zero potential energy. Why? Thus,

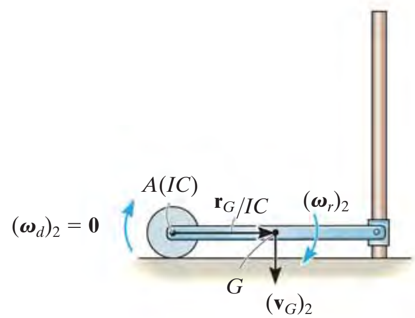
$$V_2 = 0$$

Kinetic Energy. Since the entire system is at rest at the initial position,

$$T_1 = 0$$

In the final position the rod has an angular velocity $(\boldsymbol{\omega}_r)_2$ and its mass center has a velocity $(\mathbf{v}_G)_2$, Fig. 18–20*c*. Since the rod is *fully extended* in this position, the disk is momentarily at rest, so $(\boldsymbol{\omega}_d)_2 = \mathbf{0}$ and $(\mathbf{v}_A)_2 = \mathbf{0}$. For the rod $(\mathbf{v}_G)_2$ can be related to $(\boldsymbol{\omega}_r)_2$ from the instantaneous center of zero velocity, which is located at point A, Fig. 18–20*c*. Hence, $(v_G)_2 = r_{G/IC}(\omega_r)_2$ or $(v_G)_2 = 0.3(\omega_r)_2$. Thus,

$$\begin{aligned} T_2 &= \frac{1}{2}m_r(v_G)_2^2 + \frac{1}{2}I_G(\omega_r)_2^2 + \frac{1}{2}m_d(v_A)_2^2 + \frac{1}{2}I_A(\omega_d)_2^2 \\ &= \frac{1}{2}(5 \text{ kg})[(0.3 \text{ m})(\omega_r)_2]^2 + \frac{1}{2}\left[\frac{1}{12}(5 \text{ kg})(0.6 \text{ m})^2\right](\omega_r)_2^2 + 0 + 0 \\ &= 0.3(\omega_r)_2^2 \end{aligned}$$



(c)

Fig. 18–20

Conservation of Energy.

$$\{T_1\} + \{V_1\} = \{T_2\} + \{V_2\}$$

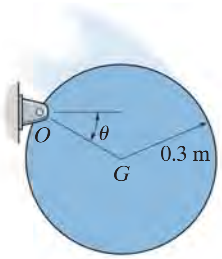
$$\{0\} + \{12.74 \text{ J}\} = \{0.3(\omega_r)_2^2\} + \{0\}$$

$$(\omega_r)_2 = 6.52 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

NOTE: We can also determine the final kinetic energy of the rod using $T_2 = \frac{1}{2}I_{IC}\omega_2^2$.

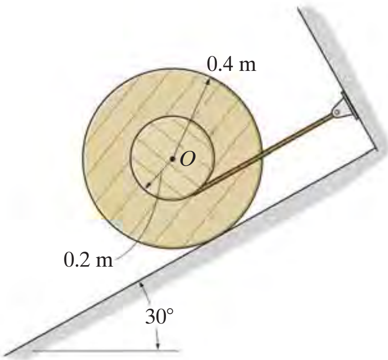
FUNDAMENTAL PROBLEMS

F18–7. If the 30-kg disk is released from rest when $\theta = 0°$, determine its angular velocity when $\theta = 90°$.



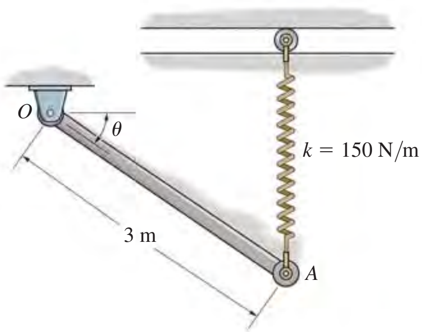
Prob. F18–7

F18–8. The 50-kg reel has a radius of gyration about its center O of $k_O = 300$ mm. If it is released from rest, determine its angular velocity when its center O has traveled 6 m down the smooth inclined plane.



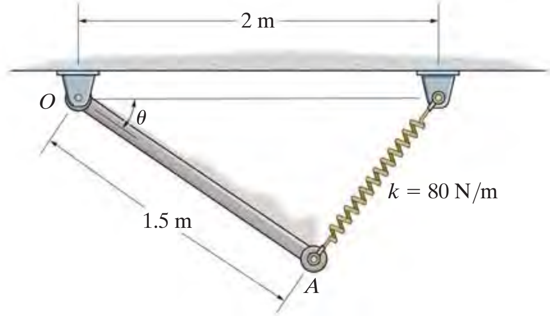
Prob. F18–8

F18–9. The 60-kg rod OA is released from rest when $\theta = 0°$. Determine its angular velocity when $\theta = 45°$. The spring remains vertical during the motion and is unstretched when $\theta = 0°$.



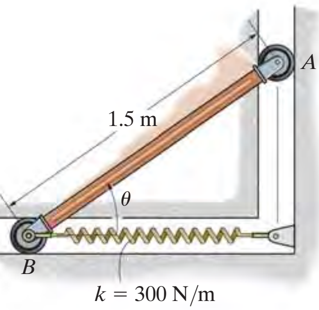
Prob. F18–9

F18–10. The 30-kg rod is released from rest when $\theta = 0°$. Determine the angular velocity of the rod when $\theta = 90°$. The spring is unstretched when $\theta = 0°$.



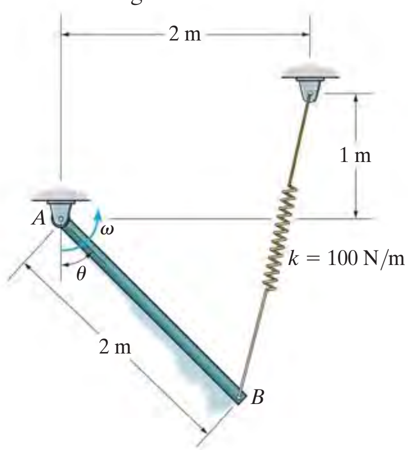
Prob. F18–10

F18–11. The 30-kg rod is released from rest when $\theta = 45°$. Determine the angular velocity of the rod when $\theta = 0°$. The spring is unstretched when $\theta = 45°$.



Prob. F18–11

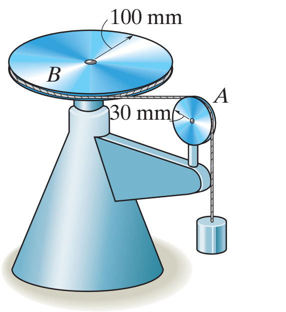
F18–12. The 20-kg rod is released from rest when $\theta = 0°$. Determine its angular velocity when $\theta = 90°$. The spring has an unstretched length of 0.5 m.



Prob. F18–12

***18–36.** The assembly consists of a 3-kg pulley A and 10-kg pulley B. If a 2-kg block is suspended from the cord, determine the block's speed after it descends 0.5 m starting from rest. Neglect the mass of the cord and treat the pulleys as thin disks. No slipping occurs.

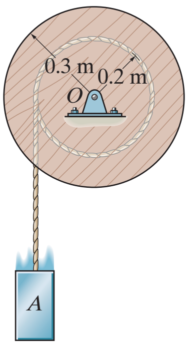
18–37. The assembly consists of a 3-kg pulley A and 10-kg pulley B. If a 2-kg block is suspended from the cord, determine the distance the block must descend, starting from rest, in order to cause B to have an angular velocity of 6 rad/s. Neglect the mass of the cord and treat the pulleys as thin disks. No slipping occurs.



Probs. 18–36/37

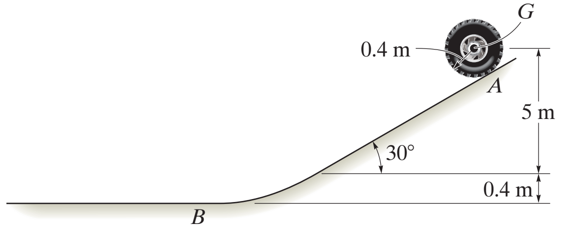
18–38. The spool has a mass of 50 kg and a radius of gyration of $k_O = 0.280$ m. If the 20-kg block A is released from rest, determine the distance the block must fall in order for the spool to have an angular velocity $\omega = 5$ rad/s. Also, what is the tension in the cord while the block is in motion? Neglect the mass of the cord.

18–39. The spool has a mass of 50 kg and a radius of gyration of $k_O = 0.280$ m. If the 20-kg block A is released from rest, determine the velocity of the block when it descends 0.5 m.



Probs. 18–38/39

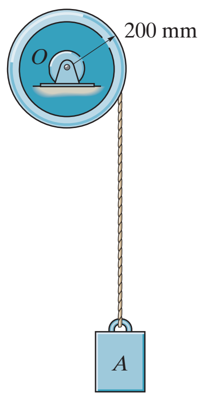
***18–40.** An automobile tire has a mass of 7 kg and radius of gyration of $k_G = 0.3$ m. If it is released from rest at A on the incline, determine its angular velocity when it reaches the horizontal plane. The tire rolls without slipping.



Prob. 18–40

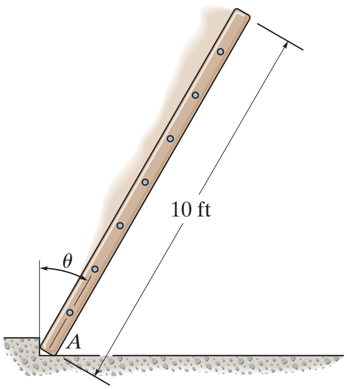
18–41. The spool has a mass of 20 kg and a radius of gyration of $k_O = 160$ mm. If the 15-kg block A is released from rest, determine the distance the block must fall in order for the spool to have an angular velocity $\omega = 8$ rad/s. Also, what is the tension in the cord while the block is in motion? Neglect the mass of the cord.

18–42. The spool has a mass of 20 kg and a radius of gyration of $k_O = 160$ mm. If the 15-kg block A is released from rest, determine the velocity of the block when it descends 600 mm.



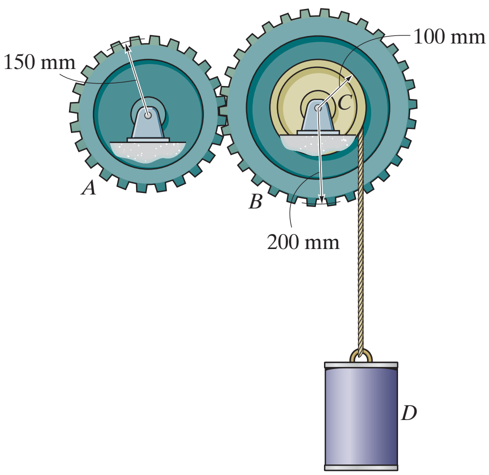
Probs. 18–41/42

18–43. A uniform ladder having a weight of 30 lb is released from rest when it is in the vertical position. If it is allowed to fall freely, determine the angle θ at which the bottom end A starts to slide to the right of A. For the calculation, assume the ladder to be a slender rod and neglect friction at A.



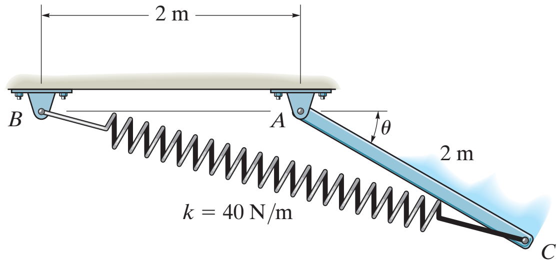
Prob. 18–43

***18–44.** Determine the speed of the 50-kg cylinder after it has descended a distance of 2 m, starting from rest. Gear A has a mass of 10 kg and a radius of gyration of 125 mm about its center of mass. Gear B and drum C have a combined mass of 30 kg and a radius of gyration about their center of mass of 150 mm.



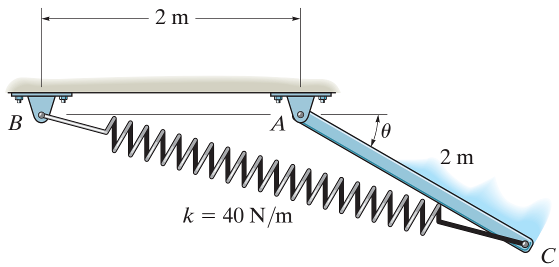
Prob. 18–44

18–45. The 12-kg slender rod is attached to a spring, which has an unstretched length of 2 m. If the rod is released from rest when $\theta = 30°$, determine its angular velocity at the instant $\theta = 90°$.



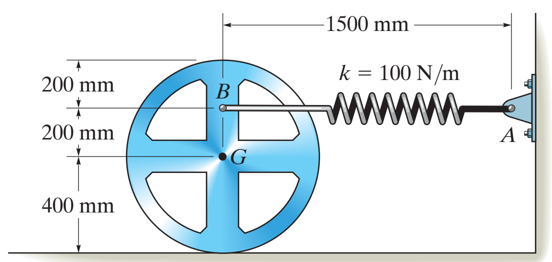
Prob. 18–45

18–46. The 12-kg slender rod is attached to a spring, which has an unstretched length of 2 m. If the rod is released from rest when $\theta = 30°$, determine the angular velocity of the rod the instant the spring becomes unstretched.



Prob. 18–46

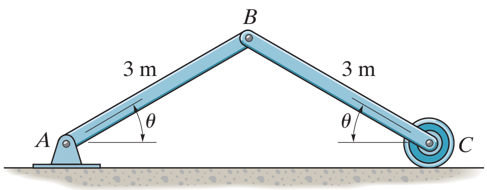
18–47. The 40-kg wheel has a radius of gyration about its center of gravity G of $k_G = 250$ mm. If it rolls without slipping, determine its angular velocity when it has rotated clockwise 90° from the position shown. The spring AB has a stiffness $k = 100$ N/m and an unstretched length of 500 mm. The wheel is released from rest.



Prob. 18–47

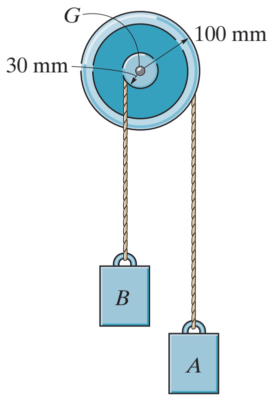
***18–48.** The assembly consists of two 10-kg bars which are pin connected. If the bars are released from rest when $\theta = 60°$, determine their angular velocities at the instant $\theta = 0°$. The 5-kg disk at C has a radius of 0.5 m and rolls without slipping.

18–49. The assembly consists of two 10-kg bars which are pin connected. If the bars are released from rest when $\theta = 60°$, determine their angular velocities at the instant $\theta = 30°$. The 5-kg disk at C has a radius of 0.5 m and rolls without slipping.



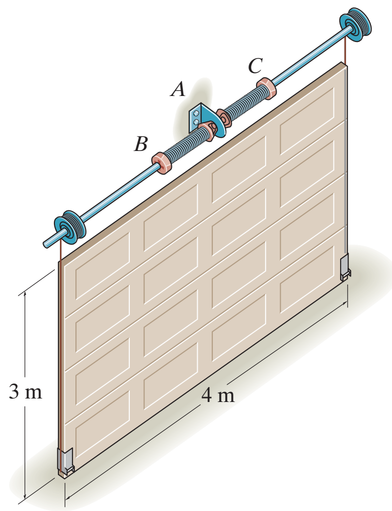
Prob. 18–48/49

18–50. The compound disk pulley consists of a hub and attached outer rim. If it has a mass of 3 kg and a radius of gyration of $k_G = 45$ mm, determine the speed of block A after A descends 0.2 m from rest. Blocks A and B each have a mass of 2 kg. Neglect the mass of the cords.



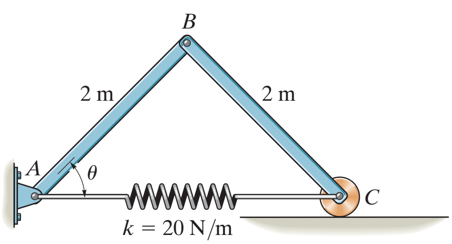
Prob. 18–50

18–51. The uniform garage door has a mass of 150 kg and is guided along smooth tracks at its ends. Lifting is done using the two springs, each of which is attached to the anchor bracket at A and to the counterbalance shaft at B and C. As the door is raised, the springs begin to unwind from the shaft, thereby assisting the lift. If each spring provides a torsional moment of $M = (0.7\theta)$ N · m, where θ is in radians, determine the angle θ_0 at which both the left-wound and right-wound spring should be attached so that the door is completely balanced by the springs, i.e., when the door is in the vertical position and is given a slight force upward, the springs will lift the door along the side tracks to the horizontal plane with no final angular velocity. *Note*: The elastic potential energy of a torsional spring is $V_e = \frac{1}{2}k\theta^2$, where $M = k\theta$ and in this case $k = 0.7$ N · m/rad.



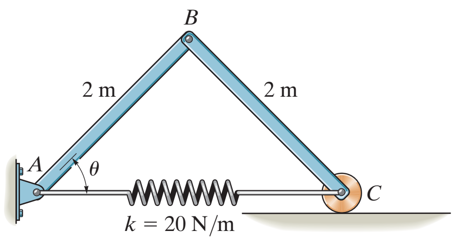
Prob. 18–51

***18–52.** The two 12-kg slender rods are pin connected and released from rest at the position $\theta = 60°$. If the spring has an unstretched length of 1.5 m, determine the angular velocity of rod BC, when the system is at the position $\theta = 0°$. Neglect the mass of the roller at C.



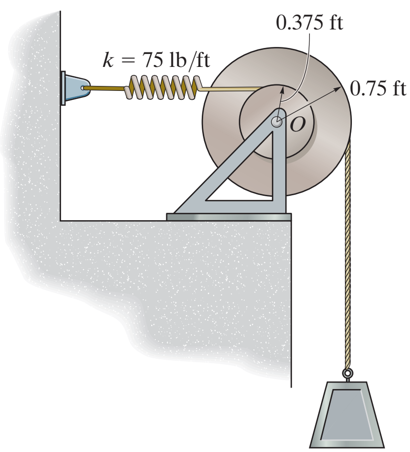
Prob. 18–52

18–53. The two 12-kg slender rods are pin connected and released from rest at the position $\theta = 60°$. If the spring has an unstretched length of 1.5 m, determine the angular velocity of rod BC, when the system is at the position $\theta = 30°$.



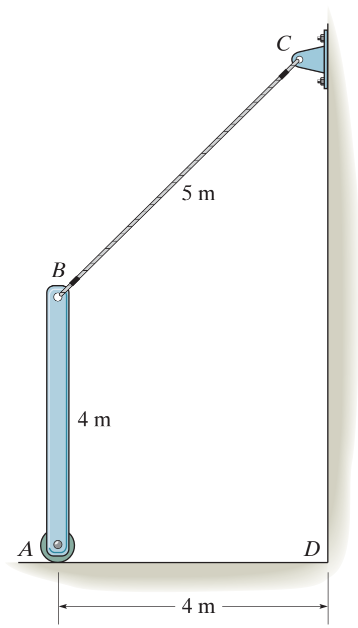
Prob. 18–53

18–54. If the 250-lb block is released from rest when the spring is unstretched, determine the velocity of the block after it has descended 5 ft. The drum has a weight of 50 lb and a radius of gyration of $k_O = 0.5$ ft about its center of mass O.



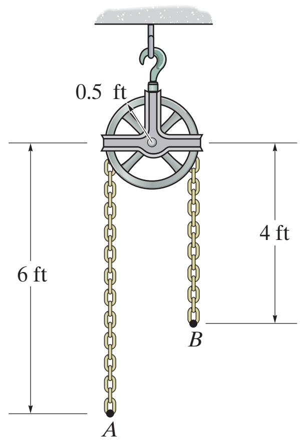
Prob. 18–54

18–55. The slender 15-kg bar is initially at rest and standing in the vertical position when the bottom end A is displaced slightly to the right. If the track in which it moves is smooth, determine the speed at which end A strikes the corner D. The bar is constrained to move in the vertical plane. Neglect the mass of the cord BC.



Prob. 18–55

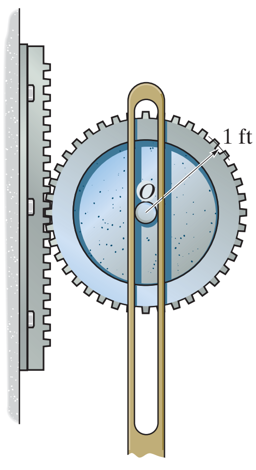
***18–56.** If the chain is released from rest from the position shown, determine the angular velocity of the pulley after the end B has risen 2 ft. The pulley has a weight of 50 lb and a radius of gyration of 0.375 ft about its axis. The chain weighs 6 lb/ft.



Prob. 18–56

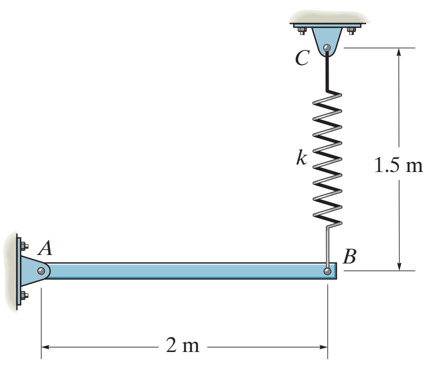
18

18–57. If the gear is released from rest, determine its angular velocity after its center of gravity O has descended a distance of 4 ft. The gear has a weight of 100 lb and a radius of gyration about its center of gravity of $k = 0.75$ ft.



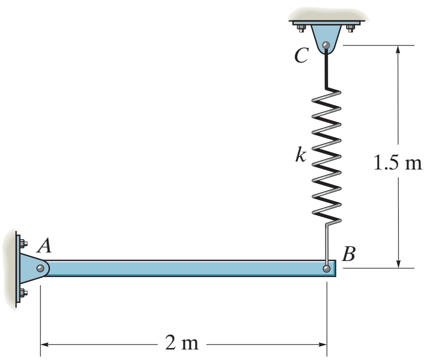
Prob. 18–57

18–58. The slender 6-kg bar AB is horizontal and at rest and the spring is unstretched. Determine the stiffness k of the spring so that the motion of the bar is momentarily stopped when it has rotated clockwise 90° after being released.



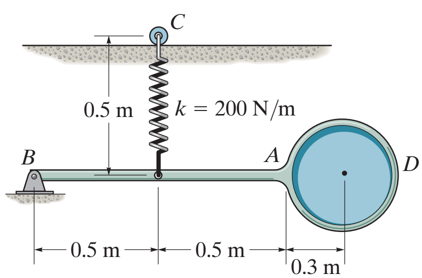
Prob. 18–58

18–59. The slender 6-kg bar AB is horizontal and at rest and the spring is unstretched. Determine the angular velocity of the bar when it has rotated clockwise 45° after being released. The spring has a stiffness of $k = 12$ N/m.



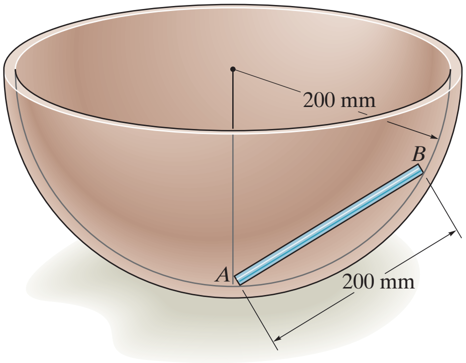
Prob. 18–59

***18–60.** The pendulum consists of a 6-kg slender rod fixed to a 15-kg disk. If the spring has an unstretched length of 0.2 m, determine the angular velocity of the pendulum when it is released from rest and rotates clockwise 90° from the position shown. The roller at C allows the spring to always remain vertical.



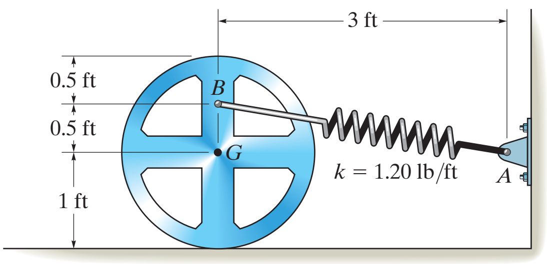
Prob. 18–60

18–61. The 500-g rod AB rests along the smooth inner surface of a hemispherical bowl. If the rod is released from rest from the position shown, determine its angular velocity at the instant it swings downward and becomes horizontal.



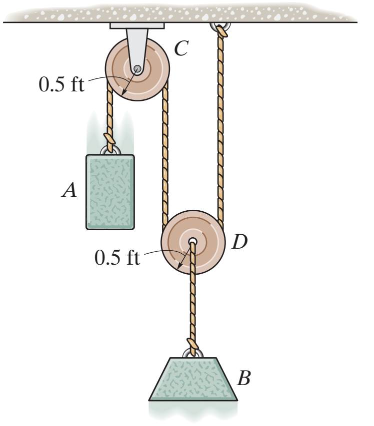
Prob. 18–61

18–62. The 50-lb wheel has a radius of gyration about its center of gravity G of $k_G = 0.7$ ft. If it rolls without slipping, determine its angular velocity when it has rotated clockwise 90° from the position shown. The spring AB has a stiffness $k = 1.20$ lb/ft and an unstretched length of 0.5 ft. The wheel is released from rest.



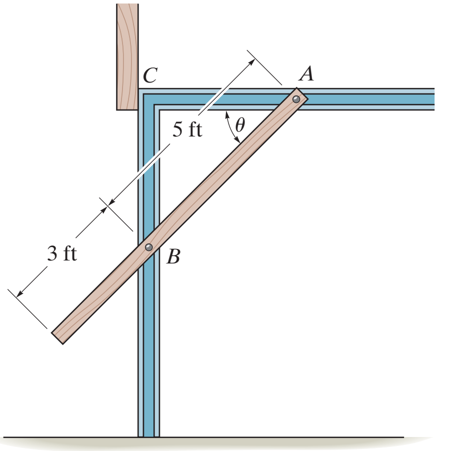
Prob. 18–62

18–63. The system consists of 60-lb and 20-lb blocks A and B, respectively, and 5-lb pulleys C and D that can be treated as thin disks. Determine the speed of block A after block B has risen 5 ft, starting from rest. Assume that the cord does not slip on the pulleys, and neglect the mass of the cord.



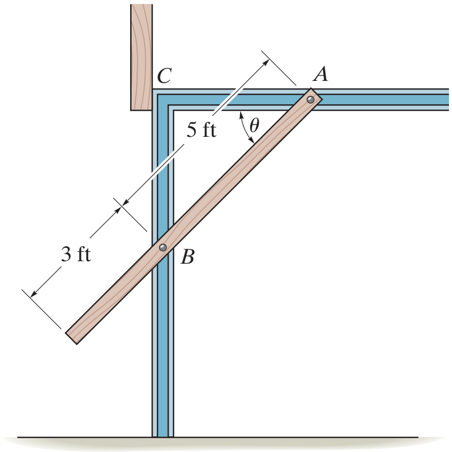
Prob. 18–63

***18–64.** The door is made from one piece, whose ends move along the horizontal and vertical tracks. If the door is in the open position, $\theta = 0°$, and then released, determine the speed at which its end A strikes the stop at C. Assume the door is a 180-lb thin plate having a width of 10 ft.



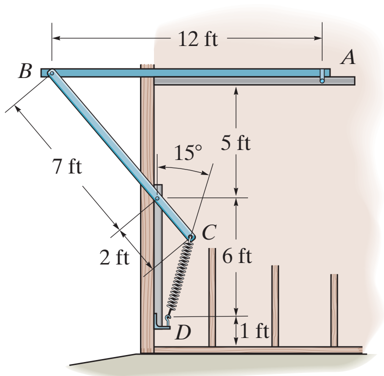
Prob. 18–64

18–65. The door is made from one piece, whose ends move along the horizontal and vertical tracks. If the door is in the open position, $\theta = 0°$, and then released, determine its angular velocity at the instant $\theta = 30°$. Assume the door is a 180-lb thin plate having a width of 10 ft.



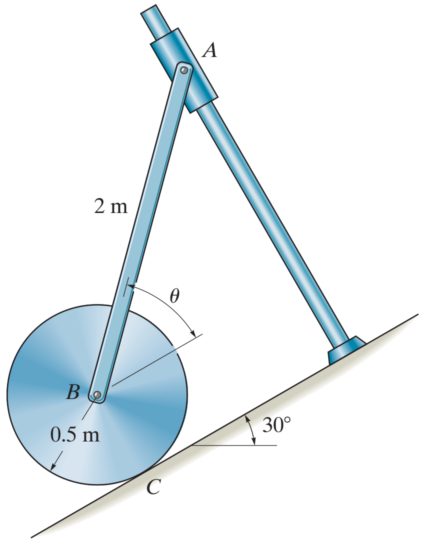
Prob. 18–65

18–66. The end A of the garage door AB travels along the horizontal track, and the end of member BC is attached to a spring at C. If the spring is originally unstretched, determine the stiffness k so that when the door falls downward from rest in the position shown, it will have zero angular velocity the moment it closes, i.e., when it and BC become vertical. Neglect the mass of member BC and assume the door is a thin plate having a weight of 200 lb and a width and height of 12 ft. There is a similar connection and spring on the other side of the door.



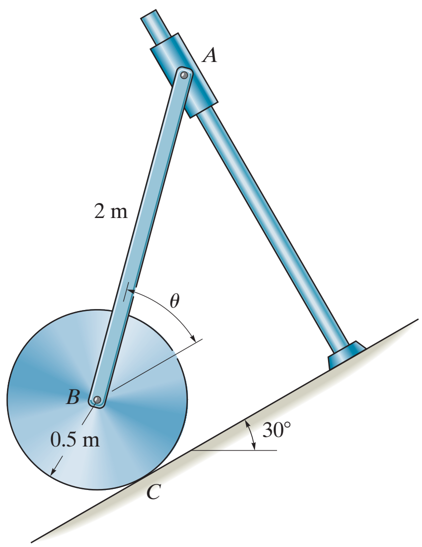
Prob. 18–66

18–67. The system consists of a 30-kg disk, 12-kg slender rod BA, and a 5-kg smooth collar A. If the disk rolls without slipping, determine the velocity of the collar at the instant $\theta = 0°$. The system is released from rest when $\theta = 45°$.



Prob. 18–67

***18–68.** The system consists of a 30-kg disk A, 12-kg slender rod BA, and a 5-kg smooth collar A. If the disk rolls without slipping, determine the velocity of the collar at the instant $\theta = 30°$. The system is released from rest when $\theta = 45°$.



Prob. 18–68

CONCEPTUAL PROBLEMS

C18–1. The bicycle and rider start from rest at the top of the hill. Show how to determine the speed of the rider when he freely coasts down the hill. Use appropriate dimensions of the wheels, and the mass of the rider, frame and wheels of the bicycle to explain your results.

Prob. C18–1 (© R.C. Hibbeler)

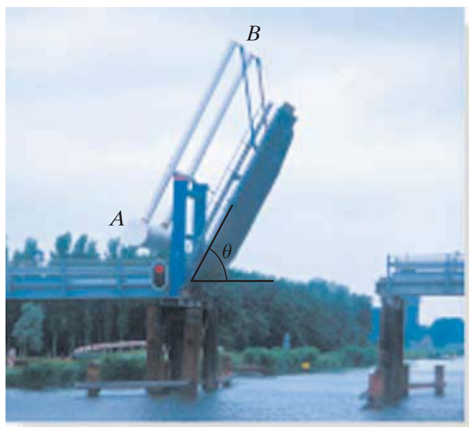
C18–2. Two torsional springs, $M = k\theta$, are used to assist in opening and closing the hood of this truck. Assuming the springs are uncoiled ($\theta = 0°$) when the hood is opened, determine the stiffness k ($\text{N}\cdot\text{m/rad}$) of each spring so that the hood can easily be lifted, i.e., practically no force applied to it, when it is closed in the unlocked position. Use appropriate numerical values to explain your result.

Prob. C18–2 (© R.C. Hibbeler)

C18–3. The operation of this garage door is assisted using two springs AB and side members BCD, which are pinned at C. Assuming the springs are unstretched when the door is in the horizontal (open) position and $ABCD$ is vertical, determine each spring stiffness k so that when the door falls to the vertical (closed) position, it will slowly come to a stop. Use appropriate numerical values to explain your result.

Prob. C18–3 (© R.C. Hibbeler)

C18–4. Determine the counterweight of A needed to balance the weight of the bridge deck when $\theta = 0°$. Show that this weight will maintain equilibrium of the deck by considering the potential energy of the system when the deck is in the arbitrary position θ. Both the deck and AB are horizontal when $\theta = 0°$. Neglect the weights of the other members. Use appropriate numerical values to explain this result.

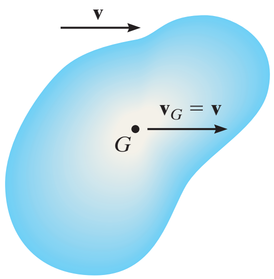


Prob. C18–4 (© R.C. Hibbeler)

CHAPTER REVIEW

Kinetic Energy

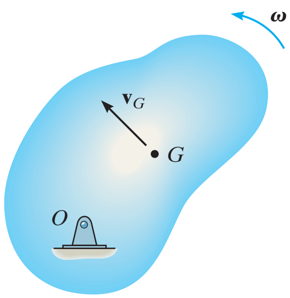
The kinetic energy of a rigid body that undergoes planar motion can be referenced to its mass center. It includes a scalar sum of its translational and rotational kinetic energies.



Translation

Translation

$$T = \frac{1}{2}mv_G^2$$



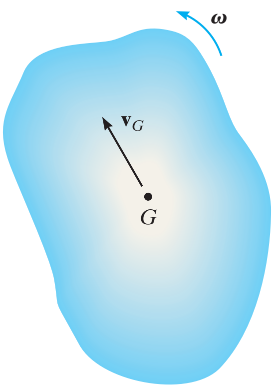
Rotation About a Fixed Axis

Rotation About a Fixed Axis

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

or

$$T = \frac{1}{2}I_O\omega^2$$



General Plane Motion

General Plane Motion

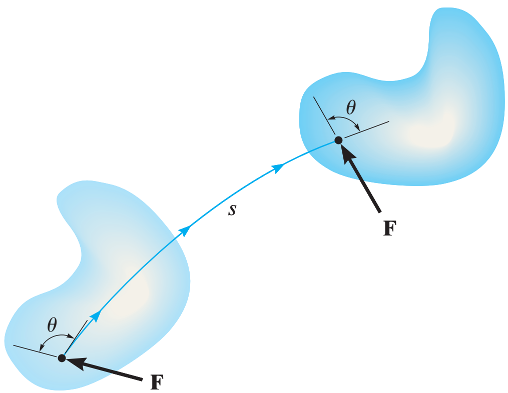
$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

or

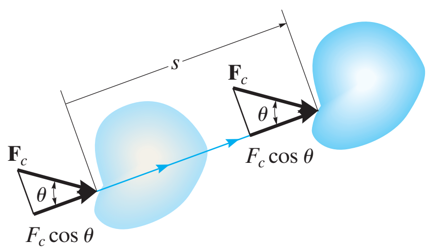
$$T = \frac{1}{2}I_{IC}\omega^2$$

Work of a Force and a Couple Moment

A force does work when it undergoes a displacement *ds* in the direction of the force. In particular, the frictional and normal forces that act on a cylinder or any circular body that rolls *without slipping* will do no work, since the normal force does not undergo a displacement and the frictional force acts on successive points on the surface of the body.

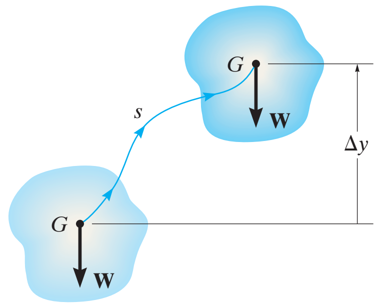


$$U_F = \int F \cos\theta \, ds$$



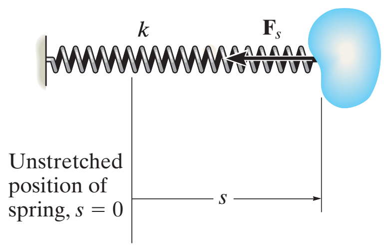
$$U_{F_c} = (F_c \cos\theta)s$$

Constant Force



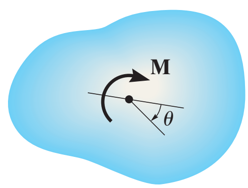
$$U_W = -W\Delta y$$

Weight



$$U = -\frac{1}{2} k s^2$$

Spring



$$U_M = \int_{\theta_1}^{\theta_2} M \, d\theta$$

$$U_M = M(\theta_2 - \theta_1)$$

Constant Magnitude

Principle of Work and Energy

Problems that involve velocity, force, and displacement can be solved using the principle of work and energy. The kinetic energy is the sum of both its rotational and translational parts. For application, a free-body diagram should be drawn in order to account for the work of all of the forces and couple moments that act on the body as it moves along the path.

$$T_1 + \Sigma U_{1-2} = T_2$$

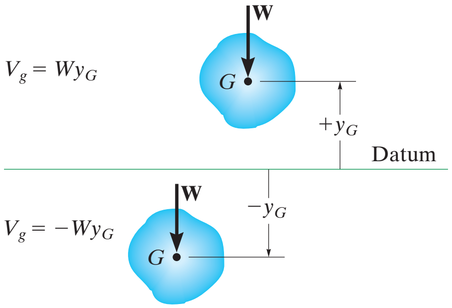
Conservation of Energy

If a rigid body is subjected only to conservative forces, then the conservation-of-energy equation can be used to solve the problem. This equation requires that the sum of the potential and kinetic energies of the body remain the same at any two points along the path.

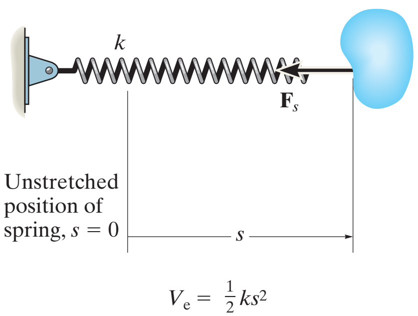
$$T_1 + V_1 = T_2 + V_2$$

where $V = V_g + V_e$

The potential energy is the sum of the body's gravitational and elastic potential energies. The gravitational potential energy will be positive if the body's center of gravity is located above a datum. If it is below the datum, then it will be negative. The elastic potential energy is always positive, regardless if the spring is stretched or compressed.



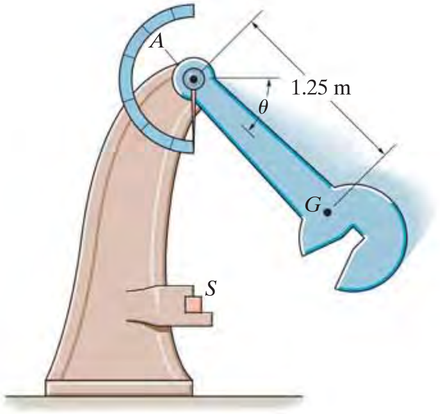
Gravitational potential energy



Elastic potential energy

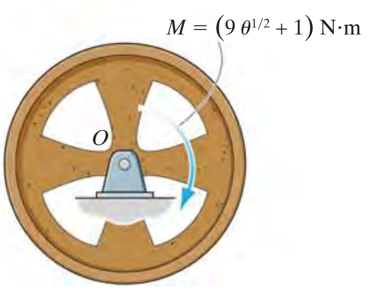
REVIEW PROBLEMS

R18–1. The pendulum of the Charpy impact machine has a mass of 50 kg and a radius of gyration of $k_A = 1.75$ m. If it is released from rest when $\theta = 0°$, determine its angular velocity just before it strikes the specimen S, $\theta = 90°$



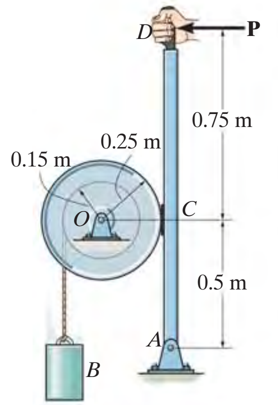
Prob. R18–1

R18–2. The 50-kg flywheel has a radius of gyration of $k_0 = 200$ mm about its center of mass. If it is subjected to a torque of $M = (9\theta^{1/2} + 1)$ N·m, where θ is in radians, determine its angular velocity when it has rotated 5 revolutions, starting from rest.



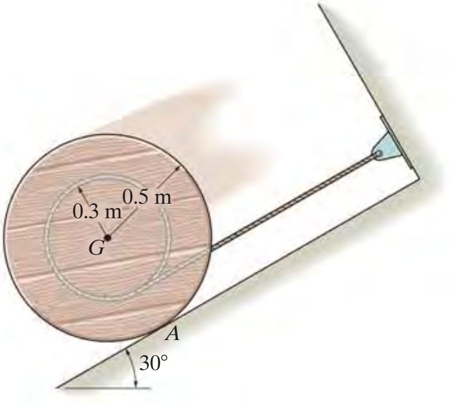
Prob. R18–2

R18–3. The drum has a mass of 50 kg and a radius of gyration about the pin at O of $k_O = 0.23$ m. Starting from rest, the suspended 15-kg block B is allowed to fall 3 m without applying the brake ACD. Determine the speed of the block at this instant. If the coefficient of kinetic friction at the brake pad C is $\mu_k = 0.5$, determine the force $\mathbf{P}$ that must be applied at the brake handle which will then stop the block after it descends *another* 3 m. Neglect the thickness of the handle.



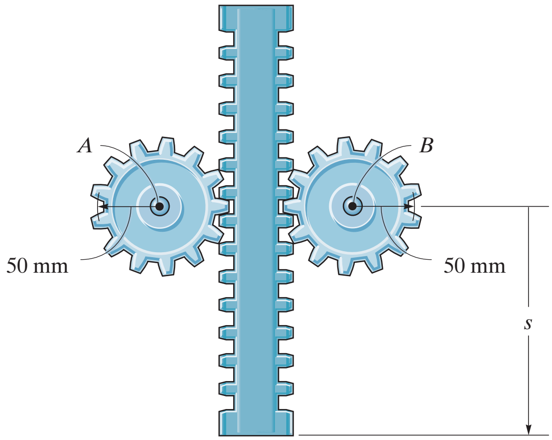
Prob. R18–3

R18–4. The spool has a mass of 60 kg and a radius of gyration of $k_G = 0.3$ m. If it is released from rest, determine how far its center descends down the smooth plane before it attains an angular velocity of $\omega = 6$ rad/s. Neglect the mass of the cord which is wound around the central core. The coefficient of kinetic friction between the spool and plane at A is $\mu_k = 0.2$.



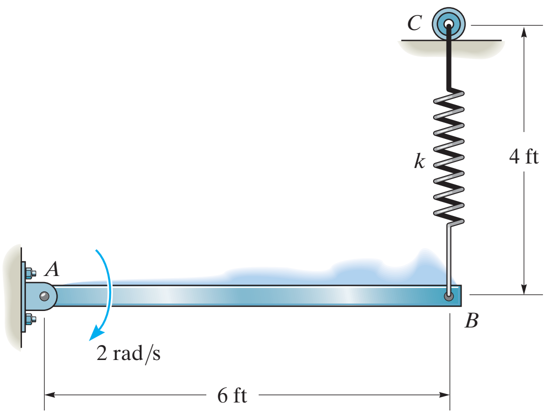
Prob. R18–4

R18–5. The gear rack has a mass of 6 kg, and the gears each have a mass of 4 kg and a radius of gyration of $k = 30$ mm at their centers. If the rack is originally moving downward at 2 m/s, when $s = 0$, determine the speed of the rack when $s = 600$ mm. The gears are free to turn about their centers A and B.



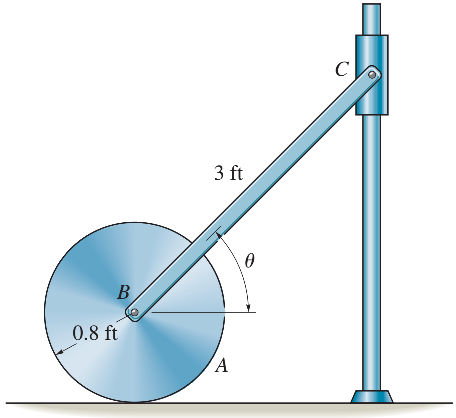
Prob. R18–5

R18-6. At the instant shown, the 50-lb bar rotates clockwise at 2 rad/s. The spring attached to its end always remains vertical due to the roller guide at C. If the spring has an unstretched length of 2 ft and a stiffness of $k = 6$ lb/ft, determine the angular velocity of the bar the instant it has rotated 30° clockwise.



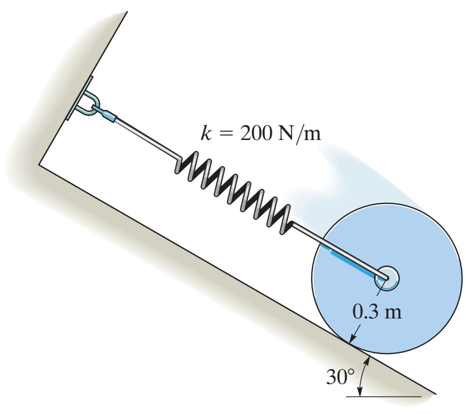
Prob. R18–6

R18–7. The system consists of a 20-lb disk A, 4-lb slender rod BC, and a 1-lb smooth collar C. If the disk rolls without slipping, determine the velocity of the collar at the instant the rod becomes horizontal, i.e., $\theta = 0°$. The system is released from rest when $\theta = 45°$.



Prob. R18–7

R18–8. At the instant the spring becomes undeformed, the center of the 40-kg disk has a speed of 4 m/s. From this point determine the distance d the disk moves down the plane before momentarily stopping. The disk rolls without slipping.



Prob. R18–8

Chapter 19

(© Hellen Sergeyeva/Fotolia)

The impulse that this tugboat imparts to this ship will cause it to turn in a manner that can be predicted by applying the principles of impulse and momentum.

Planar Kinetics of a Rigid Body: Impulse and Momentum

CHAPTER OBJECTIVES

- To develop formulations for the linear and angular momentum of a body.
- To apply the principles of linear and angular impulse and momentum to solve rigid-body planar kinetic problems that involve force, velocity, and time.
- To discuss application of the conservation of momentum.
- To analyze the mechanics of eccentric impact.

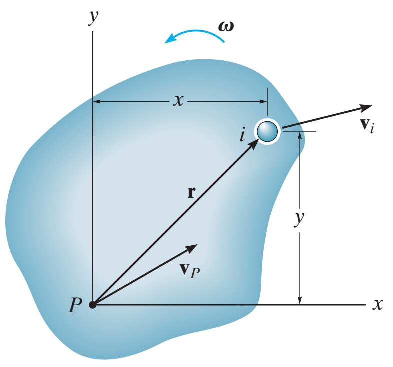
19.1 Linear and Angular Momentum

In this chapter we will use the principles of linear and angular impulse and momentum to solve problems involving force, velocity, and time as related to the planar motion of a rigid body. Before doing this, we will first formalize the methods for obtaining a body's linear and angular momentum, assuming the body is symmetric with respect to an inertial *x*–*y* reference plane.

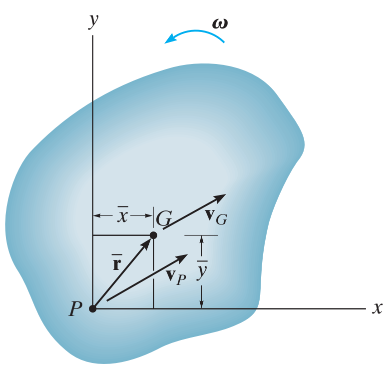
Linear Momentum. The linear momentum of a rigid body is determined by summing vectorially the linear momenta of all the particles of the body, i.e., $\mathbf{L} = \Sigma m_i \mathbf{v}_i$. Since $\Sigma m_i \mathbf{v}_i = m\mathbf{v}_G$ (see Sec. 15.2) we can also write

$$\mathbf{L} = m\mathbf{v}_G \qquad (19\text{–}1)$$

This equation states that the body's linear momentum is a vector quantity having a *magnitude* mv_G, which is commonly measured in units of $\text{kg}\cdot\text{m/s}$ or $\text{slug}\cdot\text{ft/s}$ and a *direction* defined by $\mathbf{v}_G$ the velocity of the body's mass center.



(a)



(b)

Fig. 19–1

Angular Momentum. Consider the body in Fig. 19–1*a*, which is subjected to general plane motion. At the instant shown, the arbitrary point P has a known velocity $\mathbf{v}_P$, and the body has an angular velocity $\boldsymbol{\omega}$. Therefore the velocity of the ith particle of the body is

$$\mathbf{v}_i = \mathbf{v}_P + \mathbf{v}_{i/P} = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}$$

The angular momentum of this particle about point P is equal to the "moment" of the particle's linear momentum about P, Fig. 19–1*a*. Thus,

$$(\mathbf{H}_P)_i = \mathbf{r} \times m_i \mathbf{v}_i$$

Expressing $\mathbf{v}_i$ in terms of $\mathbf{v}_P$ and using Cartesian vectors, we have

$$(H_P)_i \mathbf{k} = m_i(x\mathbf{i} + y\mathbf{j}) \times [(v_P)_x \mathbf{i} + (v_P)_y \mathbf{j} + \omega \mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]$$
$$(H_P)_i = -m_i y (v_P)_x + m_i x (v_P)_y + m_i \omega r^2$$

Letting $m_i \rightarrow dm$ and integrating over the entire mass m of the body, we obtain

$$H_P = -\left(\int_m y\, dm\right)(v_P)_x + \left(\int_m x\, dm\right)(v_P)_y + \left(\int_m r^2\, dm\right)\omega$$

Here H_P represents the angular momentum of the body about an axis (the z axis) perpendicular to the plane of motion that passes through point P. Since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$, the integrals for the first and second terms on the right are used to locate the body's center of mass G with respect to P, Fig. 19–1*b*. Also, the last integral represents the body's moment of inertia about point P. Thus,

$$H_P = -\bar{y}m(v_P)_x + \bar{x}m(v_P)_y + I_P\omega \tag{19–2}$$

This equation reduces to a simpler form if P coincides with the mass center G for the body,* in which case $\bar{x} = \bar{y} = 0$. Hence,

$$H_G = I_G\omega \tag{19–3}$$

*It also reduces to the same simple form, $H_P = I_P\omega$, if point P is a *fixed point* (see Eq. 19–9) or the velocity of P is directed along the line PG.

Here the angular momentum of the body about G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular velocity. Realize that $\mathbf{H}_G$ is a vector quantity having a *magnitude* $I_G\omega$, which is commonly measured in units of $\text{kg}\cdot\text{m}^2/\text{s}$ or $\text{slug}\cdot\text{ft}^2/\text{s}$, and a *direction* defined by $\boldsymbol{\omega}$, which is always perpendicular to the plane of motion.

Equation 19–2 can also be rewritten in terms of the x and y components of the velocity of the body's mass center, $(\mathbf{v}_G)_x$ and $(\mathbf{v}_G)_y$, and the body's moment of inertia I_G. Since G is located at coordinates $(\bar{x},\bar{y})$, then by the parallel-axis theorem, $I_P = I_G + m(\bar{x}^2 + \bar{y}^2)$. Substituting into Eq. 19–2 and rearranging terms, we have

$$H_P = \bar{y}m[-(v_P)_x + \bar{y}\omega] + \bar{x}m[(v_P)_y + \bar{x}\omega] + I_G\omega \qquad (19\text{–}4)$$

From the kinematic diagram of Fig. 19–1*b*, $\mathbf{v}_G$ can be expressed in terms of $\mathbf{v}_P$ as

$$\mathbf{v}_G = \mathbf{v}_P + \omega \times \bar{\mathbf{r}}$$

$$(v_G)_x\mathbf{i} + (v_G)_y\mathbf{j} = (v_P)_x\mathbf{i} + (v_P)_y\mathbf{j} + \omega\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

Carrying out the cross product and equating the respective $\mathbf{i}$ and $\mathbf{j}$ components yields the two scalar equations

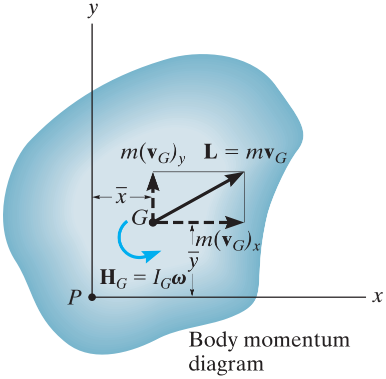
$$(v_G)_x = (v_P)_x - \bar{y}\omega$$

$$(v_G)_y = (v_P)_y + \bar{x}\omega$$

Substituting these results into Eq. 19–4 yields

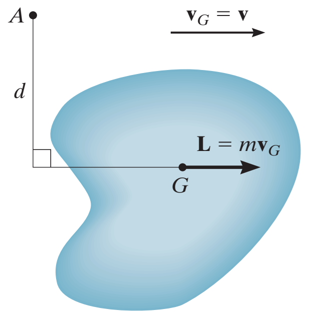
$$(\circlearrowleft +)\, H_P = -\bar{y}m(v_G)_x + \bar{x}m(v_G)_y + I_G\omega \qquad (19\text{–}5)$$

As shown in Fig. 19–1*c*, *this result indicates that when the angular momentum of the body is computed about point P, it is equivalent to the moment of the linear momentum* $m\mathbf{v}_G$, *or its components* $m(\mathbf{v}_G)_x$ *and* $m(\mathbf{v}_G)_y$, *about P plus the angular momentum* $I_G\boldsymbol{\omega}$. Using these results, we will now consider three types of motion.

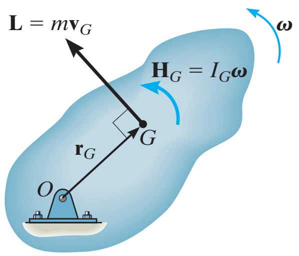


(c)

Fig. 19–1



Translation
(a)



Rotation about a fixed axis
(b)

Fig. 19–2

Translation. When a rigid body is subjected to either rectilinear or curvilinear *translation*, Fig. 19–2*a*, then $\boldsymbol{\omega} = \mathbf{0}$ and its mass center has a velocity of $\mathbf{v}_G = \mathbf{v}$. Hence, the linear momentum, and the angular momentum about G, become

$$\boxed{\begin{aligned} L &= mv_G \\ H_G &= 0 \end{aligned}} \qquad (19\text{–}6)$$

If the angular momentum is computed about some other point A, the "moment" of the linear momentum $\mathbf{L}$ must be found about the point. Since d is the "moment arm" as shown in Fig. 19–2*a*, then in accordance with Eq. 19–5, $H_A = (d)(mv_G) \circlearrowright$.

Rotation About a Fixed Axis. When a rigid body is *rotating about a fixed axis*, Fig. 19–2*b*, the linear momentum, and the angular momentum about G, are

$$\boxed{\begin{aligned} L &= mv_G \\ H_G &= I_G\omega \end{aligned}} \qquad (19\text{–}7)$$

It is sometimes convenient to compute the angular momentum about point O. Noting that $\mathbf{L}$ (or $\mathbf{v}_G$) is always *perpendicular to* $\mathbf{r}_G$, we have

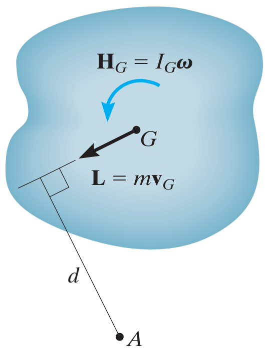
$$(\circlearrowleft +)\ H_O = I_G\omega + r_G(mv_G) \qquad (19\text{–}8)$$

Since $v_G = r_G\omega$, this equation can be written as $H_O = (I_G + mr_G^2)\omega$. Using the parallel-axis theorem,*

$$\boxed{H_O = I_O\omega} \qquad (19\text{–}9)$$

For the calculation, then, either Eq. 19–8 or 19–9 can be used.

*The similarity between this derivation and that of Eq. 17–16 ($\Sigma M_O = I_O\alpha$) and Eq. 18–5 $\left(T = \frac{1}{2}I_O\omega^2\right)$ should be noted. Also note that the same result can be obtained from Eq. 19–2 by selecting point P at O, realizing that $(v_O)_x = (v_O)_y = 0$.



General plane motion

(c)

Fig. 19–2

General Plane Motion. When a rigid body is subjected to general plane motion, Fig. 19–2*c*, the linear momentum, and the angular momentum about *G*, become

$$L = mv_G$$
$$H_G = I_G\omega \qquad (19\text{–}10)$$

If the angular momentum is computed about point *A*, Fig. 19–2*c*, it is necessary to include the moment of **L** and $\mathbf{H}_G$ about this point. In this case,

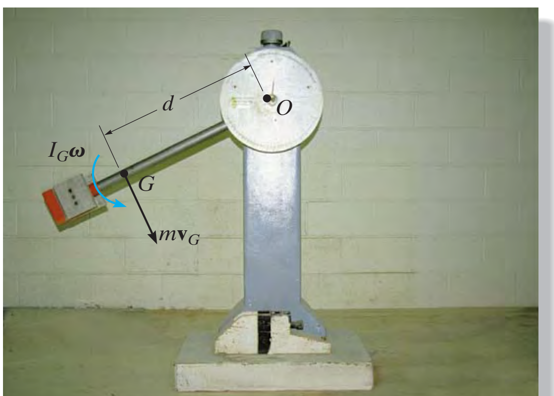
$$(\circlearrowleft+) \quad H_A = I_G\omega + (d)(mv_G)$$

Here *d* is the moment arm, as shown in the figure.

As a special case, if point *A* is the instantaneous center of zero velocity then, like Eq. 19–9, we can write the above equation in simplified form as

$$H_{IC} = I_{IC}\omega \qquad (19\text{–}11)$$

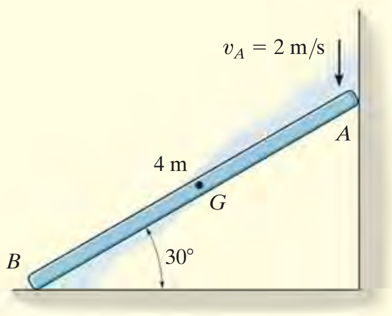
where I_{IC} is the moment of inertia of the body about the *IC*. (See Prob. 19–2.)



As the pendulum swings downward, its angular momentum about point *O* can be determined by computing the moment of $I_G\boldsymbol{\omega}$ and $m\mathbf{v}_G$ about *O*. This is $H_O = I_G\omega + (mv_G)d$. Since $v_G = \omega d$, then $H_O = I_G\omega + m(\omega d)d = (I_G + md^2)\omega = I_O\omega$. (© R.C. Hibbeler)

EXAMPLE 19.1

At a given instant the 5-kg slender bar has the motion shown in Fig. 19–3*a*. Determine its angular momentum about point *G* and about the *IC* at this instant.



(a)

SOLUTION

Bar. The bar undergoes *general plane motion*. The *IC* is established in Fig. 19–3*b*, so that

$$\omega = \frac{2 \text{ m/s}}{4 \text{ m} \cos 30^\circ} = 0.5774 \text{ rad/s}$$

$$v_G = (0.5774 \text{ rad/s})(2 \text{ m}) = 1.155 \text{ m/s}$$

Thus,

$$(\circlearrowright +) H_G = I_G\omega = \left[\tfrac{1}{12}(5 \text{ kg})(4 \text{ m})^2\right](0.5774 \text{ rad/s}) = 3.85 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright \quad \textit{Ans.}$$

Adding $I_G\omega$ and the moment of mv_G about the *IC* yields

$$(\circlearrowright +) H_{IC} = I_G\omega + d(mv_G)$$

$$= \left[\tfrac{1}{12}(5 \text{ kg})(4 \text{ m})^2\right](0.5774 \text{ rad/s}) + (2 \text{ m})(5 \text{ kg})(1.155 \text{ m/s})$$

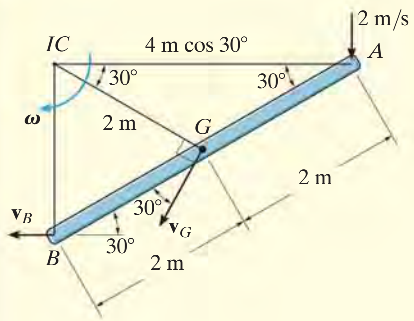
$$= 15.4 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright \quad \textit{Ans.}$$

We can also use

$$(\circlearrowright +) H_{IC} = I_{IC}\omega$$

$$= \left[\tfrac{1}{12}(5 \text{ kg})(4 \text{ m})^2 + (5 \text{ kg})(2 \text{ m})^2\right](0.5774 \text{ rad/s})$$

$$= 15.4 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright \quad \textit{Ans.}$$



(b)

Fig. 19–3

19

19.2 Principle of Impulse and Momentum

Like the case for particle motion, the principle of impulse and momentum for a rigid body can be developed by *combining* the equation of motion with kinematics. The resulting equation will yield a *direct solution to problems involving force, velocity, and time.*

Principle of Linear Impulse and Momentum. The equation of translational motion for a rigid body can be written as $\Sigma \mathbf{F} = m\mathbf{a}_G = m(d\mathbf{v}_G/dt)$. Since the mass of the body is constant,

$$\Sigma \mathbf{F} = \frac{d}{dt}(m\mathbf{v}_G)$$

Multiplying both sides by dt and integrating from $t = t_1$, $\mathbf{v}_G = (\mathbf{v}_G)_1$ to $t = t_2$, $\mathbf{v}_G = (\mathbf{v}_G)_2$ yields

$$\Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m(\mathbf{v}_G)_2 - m(\mathbf{v}_G)_1$$

This equation is referred to as the *principle of linear impulse and momentum*. It states that the sum of all the impulses created by the *external force system* which acts on the body during the time interval t_1 to t_2 is equal to the change in the linear momentum of the body during this time interval, Fig. 19–4.

Principle of Angular Impulse and Momentum. If the body has *general plane motion* then $\Sigma M_G = I_G\alpha = I_G(d\omega/dt)$. Since the moment of inertia is constant,

$$\Sigma M_G = \frac{d}{dt}(I_G\omega)$$

Multiplying both sides by dt and integrating from $t = t_1$, $\omega = \omega_1$ to $t = t_2$, $\omega = \omega_2$ gives

$$\Sigma \int_{t_1}^{t_2} M_G\, dt = I_G\omega_2 - I_G\omega_1 \qquad (19\text{–}12)$$

In a similar manner, for *rotation about a fixed axis* passing through point O, Eq. 17–16 ($\Sigma M_O = I_O\alpha$) when integrated becomes

$$\Sigma \int_{t_1}^{t_2} M_O\, dt = I_O\omega_2 - I_O\omega_1 \qquad (19\text{–}13)$$

Equations 19–12 and 19–13 are referred to as the *principle of angular impulse and momentum*. Both equations state that the sum of the angular impulses acting on the body during the time interval t_1 to t_2 is equal to the change in the body's angular momentum during this time interval.

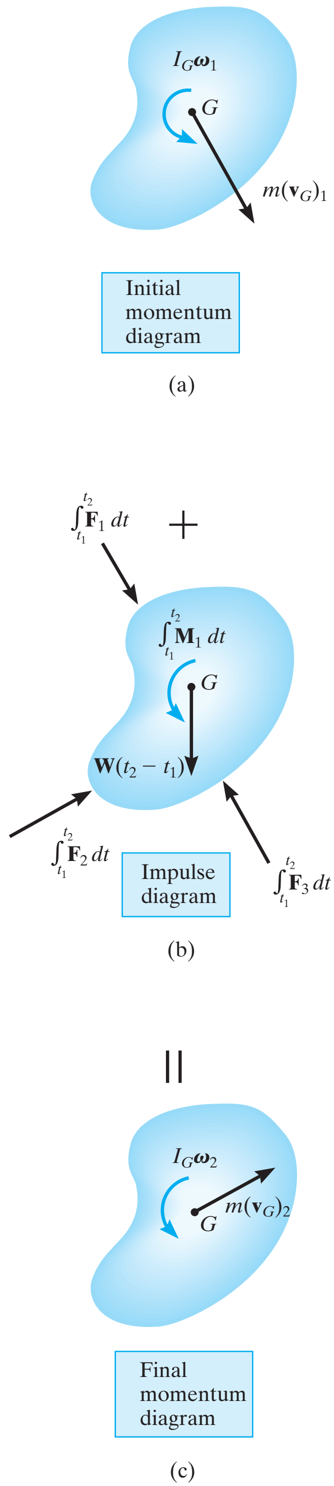


Fig. 19–4

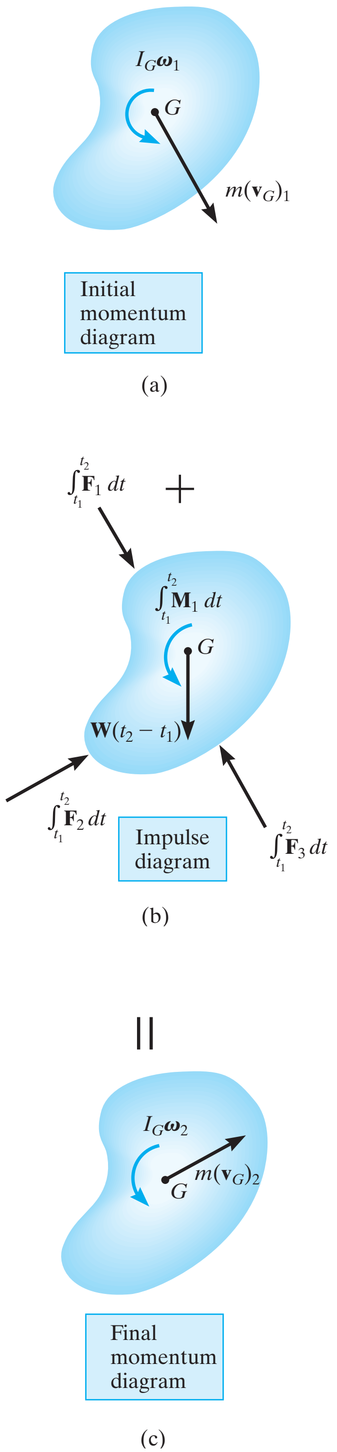


Fig. 19–4 (repeated)

To summarize these concepts, if motion occurs in the x–y plane, the following *three scalar equations* can be written to describe the *planar motion* of the body.

$$
\begin{aligned}
m(v_{Gx})_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt &= m(v_{Gx})_2 \\
m(v_{Gy})_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt &= m(v_{Gy})_2 \\
I_G\omega_1 + \Sigma\int_{t_1}^{t_2} M_G\,dt &= I_G\omega_2
\end{aligned}
\qquad (19\text{–}14)
$$

The terms in these equations can be shown graphically by drawing a set of impulse and momentum diagrams for the body, Fig. 19–4. Note that the linear momentum $m\mathbf{v}_G$ is applied at the body's mass center, Figs. 19–4*a* and 19–4*c*; whereas the angular momentum $I_G\boldsymbol{\omega}$ is a free vector, and therefore, like a couple moment, it can be applied at any point on the body. When the impulse diagram is constructed, Fig. 19–4*b*, the forces $\mathbf{F}$ and moment $\mathbf{M}$ vary with time, and are indicated by the integrals. However, if $\mathbf{F}$ and $\mathbf{M}$ are *constant* integration of the impulses yields $\mathbf{F}(t_2 - t_1)$ and $\mathbf{M}(t_2 - t_1)$, respectively. Such is the case for the body's weight $\mathbf{W}$, Fig. 19–4*b*.

Equations 19–14 can also be applied to an entire system of connected bodies rather than to each body separately. This eliminates the need to include interaction impulses which occur at the connections since they are *internal* to the system. The resultant equations may be written in symbolic form as

$$
\begin{aligned}
\left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_{x1} + \left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{impulse}\end{matrix}\right)_{x(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_{x2} \\
\left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_{y1} + \left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{impulse}\end{matrix}\right)_{y(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_{y2} \\
\left(\Sigma\begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O1} + \left(\Sigma\begin{matrix}\text{syst. angular}\\ \text{impulse}\end{matrix}\right)_{O(1-2)} &= \left(\Sigma\begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O2}
\end{aligned}
$$

(19–15)

As indicated by the third equation, the system's angular momentum and angular impulse must be computed with respect to the *same reference point O* for all the bodies of the system.

Procedure For Analysis

Impulse and momentum principles are used to solve kinetic problems that involve *velocity, force*, and *time* since these terms are involved in the formulation.

Free-Body Diagram.

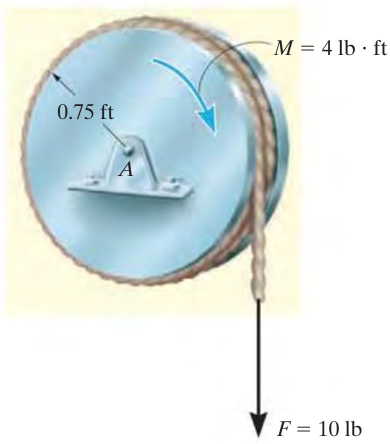
- Establish the x, y, z inertial frame of reference and draw the free-body diagram in order to account for all the forces and couple moments that produce impulses on the body.
- The direction and sense of the initial and final velocity of the body's mass center, $\mathbf{v}_G$, and the body's angular velocity $\boldsymbol{\omega}$ should be established. If any of these motions is unknown, assume that the sense of its components is in the direction of the positive inertial coordinates.
- Compute the moment of inertia I_G or I_O.
- As an alternative procedure, draw the impulse and momentum diagrams for the body or system of bodies. Each of these diagrams represents an outlined shape of the body which graphically accounts for the data required for each of the three terms in Eqs. 19–14 or 19–15, Fig. 19–4. These diagrams are particularly helpful in order to visualize the "moment" terms used in the principle of angular impulse and momentum, if application is about the *IC* or another point other than the body's mass center G or a fixed point O.

Principle of Impulse and Momentum.

- Apply the three scalar equations of impulse and momentum.
- The angular momentum of a rigid body rotating about a fixed axis is the moment of $m\mathbf{v}_G$ plus $I_G\boldsymbol{\omega}$ about the axis. This is equal to $H_O = I_O\omega$, where I_O is the moment of inertia of the body about the axis.
- All the forces acting on the body's free-body diagram will create an impulse; however, some of these forces will do no work.
- Forces that are functions of time must be integrated to obtain the impulse.
- The principle of angular impulse and momentum is often used to eliminate unknown impulsive forces that are parallel or pass through a common axis, since the moment of these forces is zero about this axis.

Kinematics.

- If more than three equations are needed for a complete solution, it may be possible to relate the velocity of the body's mass center to the body's angular velocity using *kinematics*. If the motion appears to be complicated, kinematic (velocity) diagrams may be helpful in obtaining the necessary relation.



The 20-lb disk shown in Fig. 19–5*a* is acted upon by a constant couple moment of 4 lb·ft and a force of 10 lb which is applied to a cord wrapped around its periphery. Determine the angular velocity of the disk two seconds after starting from rest. Also, what are the force components of reaction at the pin?

SOLUTION

Since angular velocity, force, and time are involved in the problems, we will apply the principles of impulse and momentum to the solution.

Free-Body Diagram. Fig. 19–5*b*. The disk's mass center does not move; however, the loading causes the disk to rotate clockwise.

The moment of inertia of the disk about its fixed axis of rotation is

$$I_A = \frac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.75\text{ ft})^2 = 0.1747\text{ slug}\cdot\text{ft}^2$$

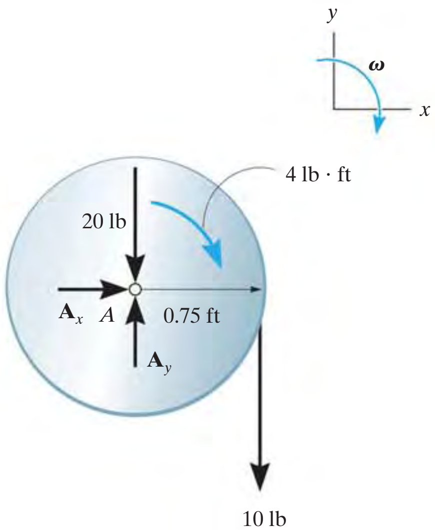


Fig. 19–5

Principle of Impulse and Momentum.

$(\overset{+}{\rightarrow})$ $$m(v_{Ax})_1 + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_{Ax})_2$$

$$0 + A_x(2\text{ s}) = 0$$

$(+\uparrow)$ $$m(v_{Ay})_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_{Ay})_2$$

$$0 + A_y(2\text{ s}) - 20\text{ lb}(2\text{ s}) - 10\text{ lb}(2\text{ s}) = 0$$

$(\circlearrowright +)$ $$I_A\omega_1 + \Sigma\int_{t_1}^{t_2} M_A\,dt = I_A\omega_2$$

$$0 + 4\text{ lb}\cdot\text{ft}(2\text{ s}) + [10\text{ lb}(2\text{ s})](0.75\text{ ft}) = 0.1747\omega_2$$

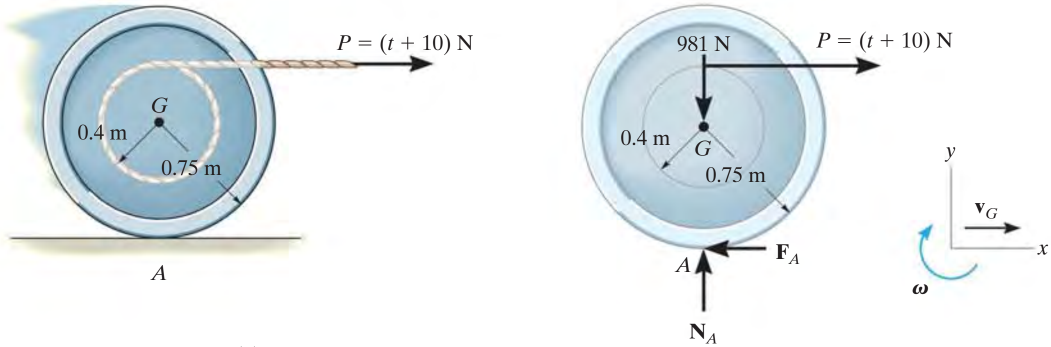
Solving these equations yields

$$A_x = 0$$ *Ans.*

$$A_y = 30\text{ lb}$$ *Ans.*

$$\omega_2 = 132\text{ rad/s}\ \circlearrowright$$ *Ans.*

The 100-kg spool shown in Fig. 19–6*a* has a radius of gyration $k_G = 0.35$ m. A cable is wrapped around the central hub of the spool, and a horizontal force having a variable magnitude of $P = (t + 10)$ N is applied, where t is in seconds. If the spool is initially at rest, determine its angular velocity in 5 s. Assume that the spool rolls without slipping at A.



(a)

(b)

Fig. 19–6

SOLUTION

Free-Body Diagram. From the free-body diagram, Fig. 19–6*b*, the *variable* force **P** will cause the friction force $\mathbf{F}_A$ to be variable, and thus the impulses created by both **P** and $\mathbf{F}_A$ must be determined by integration. Force **P** causes the mass center to have a velocity $\mathbf{v}_G$ to the right, and so the spool has a clockwise angular velocity $\boldsymbol{\omega}$.

Principle of Impulse and Momentum. A direct solution for $\boldsymbol{\omega}$ can be obtained by applying the principle of angular impulse and momentum about point A, the IC, in order to eliminate the unknown friction impulse.

$$(\circlearrowright +) \qquad I_A\omega_1 + \Sigma \int M_A\, dt = I_A\omega_2$$

$$0 + \left[\int_0^{5\text{ s}} (t + 10)\text{ N } dt\right](0.75\text{ m} + 0.4\text{ m}) = [100\text{ kg }(0.35\text{ m})^2 + (100\text{ kg})(0.75\text{ m})^2]\omega_2$$

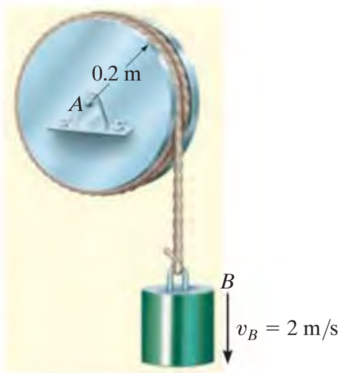
$$62.5(1.15) = 68.5\omega_2$$

$$\omega_2 = 1.05\text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

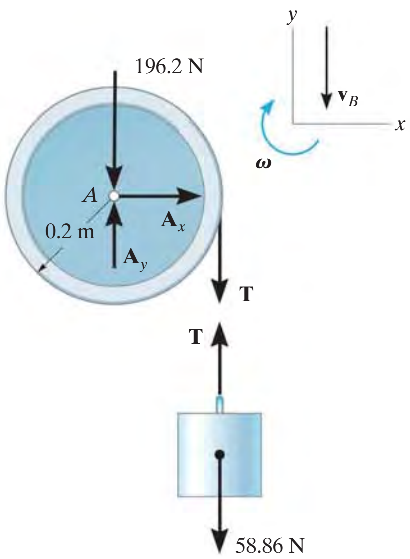
NOTE: Try solving this problem by applying the principle of impulse and momentum about G and using the principle of linear impulse and momentum in the x direction.

19

The cylinder B, shown in Fig. 19–7*a* has a mass of 6 kg. It is attached to a cord which is wrapped around the periphery of a 20-kg disk that has a moment of inertia $I_A = 0.40\ \text{kg}\cdot\text{m}^2$. If the cylinder is initially moving downward with a speed of 2 m/s, determine its speed in 3 s. Neglect the mass of the cord in the calculation.



(a)



(b)

Fig. 19–7

SOLUTION I

Free-Body Diagram. The free-body diagrams of the cylinder and disk are shown in Fig. 19–7*b*. All the forces are *constant* since the weight of the cylinder causes the motion. The downward motion of the cylinder, $\mathbf{v}_B$, causes $\boldsymbol{\omega}$ of the disk to be clockwise.

Principle of Impulse and Momentum. We can eliminate $\mathbf{A}_x$ and $\mathbf{A}_y$ from the analysis by applying the principle of angular impulse and momentum about point A. Hence

Disk

$$(\circlearrowright +) \qquad I_A\omega_1 + \Sigma \int M_A\, dt = I_A\omega_2$$

$$0.40\ \text{kg}\cdot\text{m}^2(\omega_1) + T(3\ \text{s})(0.2\ \text{m}) = (0.40\ \text{kg}\cdot\text{m}^2)\omega_2$$

Cylinder

$$(+\uparrow) \qquad m_B(v_B)_1 + \Sigma \int F_y\, dt = m_B(v_B)_2$$

$$-6\ \text{kg}(2\ \text{m/s}) + T(3\ \text{s}) - 58.86\ \text{N}(3\ \text{s}) = -6\ \text{kg}(v_B)_2$$

Kinematics. Since $\omega = v_B/r$, then $\omega_1 = (2\ \text{m/s})/(0.2\ \text{m}) = 10\ \text{rad/s}$ and $\omega_2 = (v_B)_2/0.2\ \text{m} = 5(v_B)_2$. Substituting and solving the equations simultaneously for $(v_B)_2$ yields

$$(v_B)_2 = 13.0\ \text{m/s}\downarrow \qquad \textit{Ans.}$$

SOLUTION II

Impulse and Momentum Diagrams. We can obtain $(v_B)_2$ *directly* by considering the *system* consisting of the cylinder, the cord, and the disk. The impulse and momentum diagrams have been drawn to clarify application of the principle of angular impulse and momentum about point A, Fig. 19–7*c*.

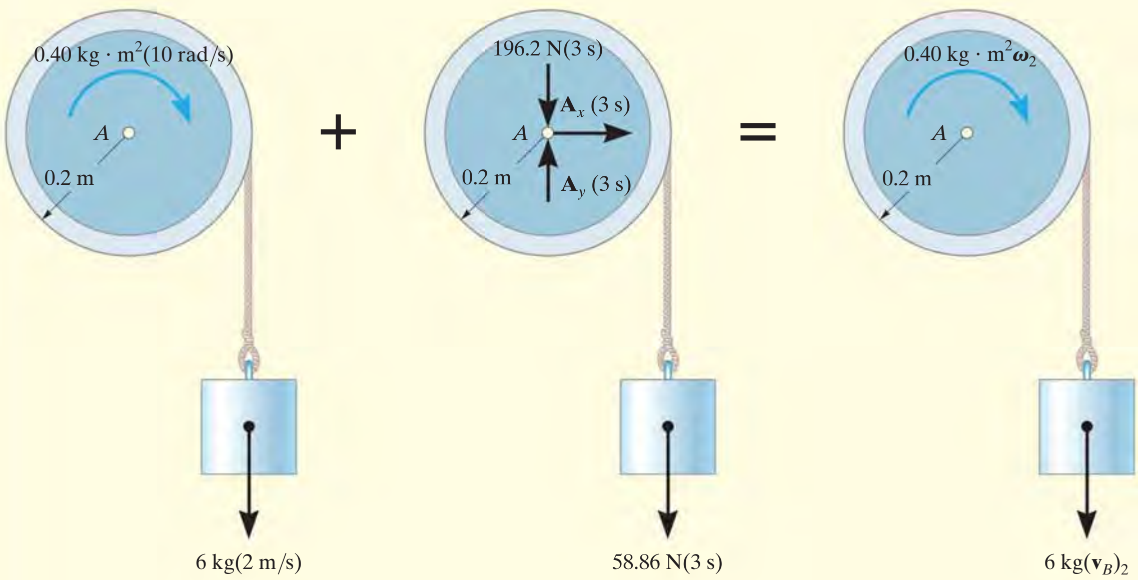
Principle of Angular Impulse and Momentum. Realizing that $\omega_1 = 10$ rad/s and $\omega_2 = 5(v_B)_2$, we have

$$(\zeta +)\left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{A1} + \left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{impulse}\end{matrix}\right)_{A(1-2)} = \left(\Sigma \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{A2}$$

$$(6\text{ kg})(2\text{ m/s})(0.2\text{ m}) + (0.40\text{ kg}\cdot\text{m}^2)(10\text{ rad/s}) + (58.86\text{ N})(3\text{ s})(0.2\text{ m})$$

$$= (6\text{ kg})(v_B)_2(0.2\text{ m}) + (0.40\text{ kg}\cdot\text{m}^2)[5(v_B)_2]$$

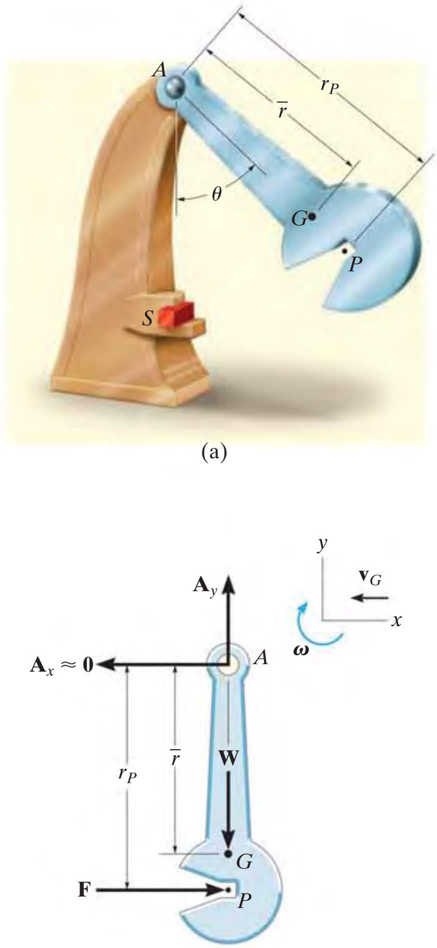
$$(v_B)_2 = 13.0\text{ m/s}\downarrow \qquad \textit{Ans.}$$



(c)

Fig. 19–7 (cont.)

The Charpy impact test is used in materials testing to determine the energy absorption characteristics of a material during impact. The test is performed using the pendulum shown in Fig. 19–8*a*, which has a mass m, mass center at G, and a radius of gyration k_G about G. Determine the distance r_P from the pin at A to the point P where the impact with the specimen S should occur so that the horizontal force at the pin A is essentially zero during the impact. For the calculation, assume the specimen absorbs all the pendulum's kinetic energy gained during the time it falls and thereby stops the pendulum from swinging when $\theta = 0°$.



(a)
(b)
Fig. 19–8

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 19–8*b*, the conditions of the problem require the horizontal force at A to be zero. Just before impact, the pendulum has a clockwise angular velocity $\boldsymbol{\omega}_1$, and the mass center of the pendulum is moving to the left at $(v_G)_1 = \bar{r}\omega_1$.

Principle of Impulse and Momentum. We will apply the principle of angular impulse and momentum about point A. Thus,

$$I_A\omega_1 + \Sigma\int M_A\,dt = I_A\omega_2$$

$$(\circlearrowright +) \qquad I_A\omega_1 - \left(\int F\,dt\right) r_P = 0$$

$$m(v_G)_1 + \Sigma\int F\,dt = m(v_G)_2$$

$$(\overset{+}{\rightarrow}) \qquad -m(\bar{r}\omega_1) + \int F\,dt = 0$$

Eliminating the impulse $\int F\,dt$ and substituting $I_A = mk_G^2 + m\bar{r}^2$ yields

$$[mk_G^2 + m\bar{r}^2]\omega_1 - m(\bar{r}\omega_1)r_P = 0$$

Factoring out $m\omega_1$ and solving for r_P, we obtain

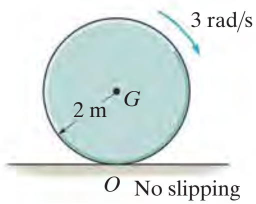
$$r_P = \bar{r} + \frac{k_G^2}{\bar{r}} \qquad \textit{Ans.}$$

NOTE: Point P, so defined, is called the *center of percussion*. By placing the striking point at P, the force developed at the pin will be minimized. Many sports rackets, clubs, etc. are designed so that collision with the object being struck occurs at the center of percussion. As a consequence, no "sting" or little sensation occurs in the hand of the player. (Also see Probs. 17–66 and 19–1.)

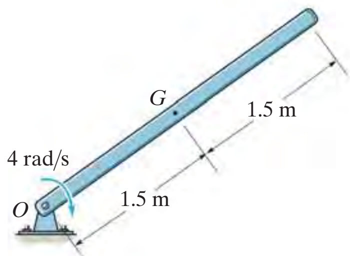
PRELIMINARY PROBLEMS

P19–1. Determine the angular momentum of the 100-kg disk or rod about point G and about point O.

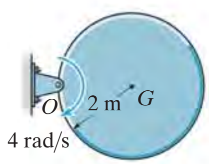
a)



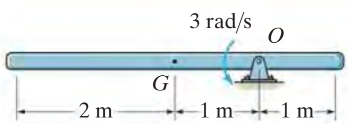
b)



c)



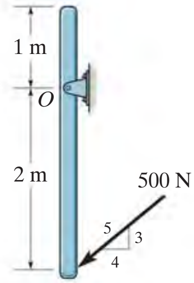
d)



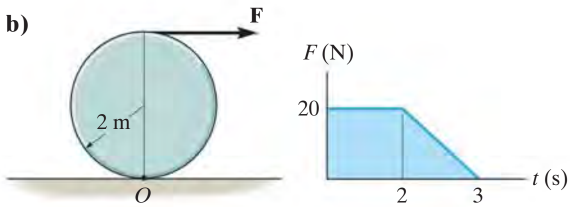
Prob. P19–1

P19–2. Determine the angular impulse about point O for $t = 3$ s.

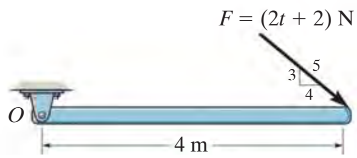
a)



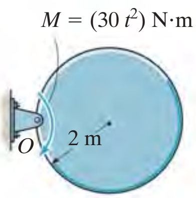
b)



c)



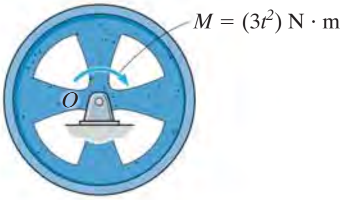
d)



Prob. P19–2

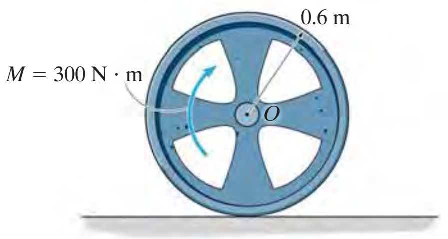
FUNDAMENTAL PROBLEMS

F19–1. The 60-kg wheel has a radius of gyration about its center O of $k_O = 300$ mm. If it is subjected to a couple moment of $M = (3t^2)$ N·m, where t is in seconds, determine the angular velocity of the wheel when $t = 4$ s, starting from rest.



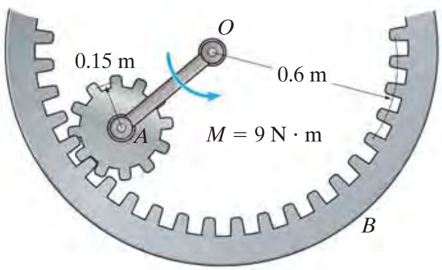
Prob. F19–1

F19–2. The 300-kg wheel has a radius of gyration about its mass center O of $k_O = 400$ mm. If the wheel is subjected to a couple moment of $M = 300$ N·m, determine its angular velocity 6 s after it starts from rest and no slipping occurs. Also, determine the friction force that the ground applies to the wheel.



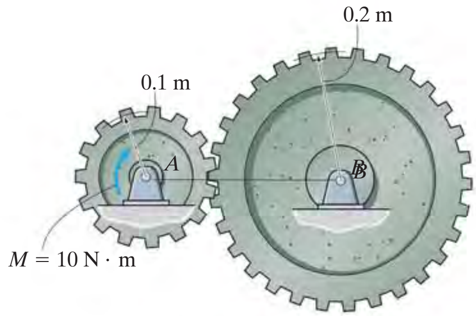
Prob. F19–2

F19–3. If rod OA of negligible mass is subjected to the couple moment $M = 9$ N·m, determine the angular velocity of the 10-kg inner gear $t = 5$ s after it starts from rest. The gear has a radius of gyration about its mass center of $k_A = 100$ mm, and it rolls on the fixed outer gear, B. Motion occurs in the horizontal plane.



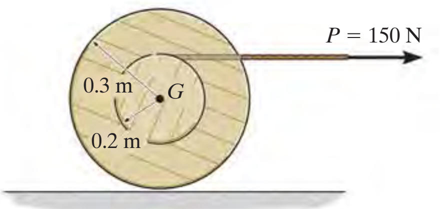
Prob. F19–3

F19–4. Gears A and B of mass 10 kg and 50 kg have radii of gyration about their respective mass centers of $k_A = 80$ mm and $k_B = 150$ mm. If gear A is subjected to the couple moment $M = 10$ N·m when it is at rest, determine the angular velocity of gear B when $t = 5$ s.



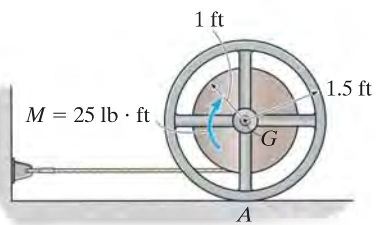
Prob. F19–4

F19–5. The 50-kg spool is subjected to a horizontal force of $P = 150$ N. If the spool rolls without slipping, determine its angular velocity 3 s after it starts from rest. The radius of gyration of the spool about its center of mass is $k_G = 175$ mm.



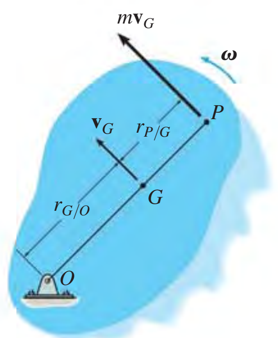
Prob. F19–5

F19–6. The reel has a weight of 150 lb and a radius of gyration about its center of gravity of $k_G = 1.25$ ft. If it is subjected to a torque of $M = 25$ lb·ft, and starts from rest when the torque is applied, determine its angular velocity in 3 seconds. The coefficient of kinetic friction between the reel and the horizontal plane is $\mu_k = 0.15$.



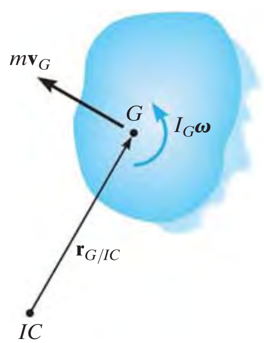
Prob. F19–6

19–1. The rigid body (slab) has a mass m and rotates with an angular velocity $\boldsymbol{\omega}$ about an axis passing through the fixed point O. Show that the momenta of all the particles composing the body can be represented by a single vector having a magnitude mv_G and acting through point P, called the *center of percussion*, which lies at a distance $r_{P/G} = k_G^2/r_{G/O}$ from the mass center G. Here k_G is the radius of gyration of the body, computed about an axis perpendicular to the plane of motion and passing through G.



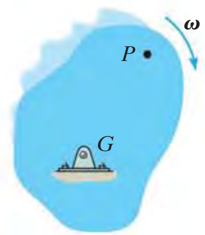
Prob. 19–1

19–2. At a given instant, the body has a linear momentum $\mathbf{L} = m\mathbf{v}_G$ and an angular momentum $\mathbf{H}_G = I_G\boldsymbol{\omega}$ computed about its mass center. Show that the angular momentum of the body computed about the instantaneous center of zero velocity IC can be expressed as $\mathbf{H}_{IC} = I_{IC}\boldsymbol{\omega}$, where I_{IC} represents the body's moment of inertia computed about the instantaneous axis of zero velocity. As shown, the IC is located at a distance $r_{G/IC}$ away from the mass center G.



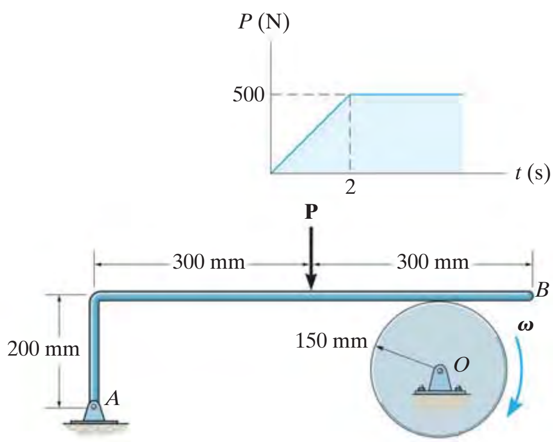
Prob. 19–2

19–3. Show that if a slab is rotating about a fixed axis perpendicular to the slab and passing through its mass center G, the angular momentum is the same when computed about any other point P.



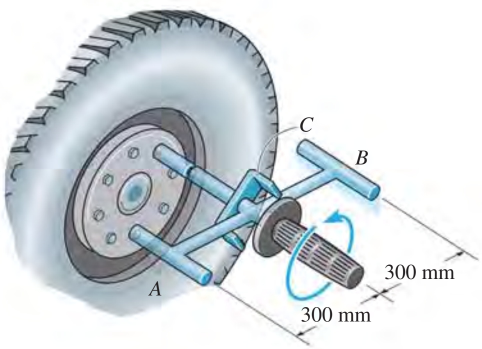
Prob. 19–3

***19–4.** The 40-kg disk is rotating at $\omega = 100$ rad/s. When the force $\mathbf{P}$ is applied to the brake as indicated by the graph. If the coefficient of kinetic friction at B is $\mu_k = 0.3$, determine the time t needed to stay the disk from rotating. Neglect the thickness of the brake.



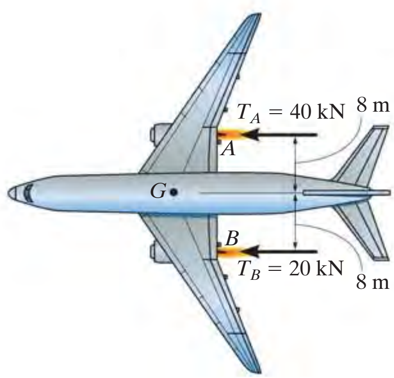
Prob. 19–4

19–5. The impact wrench consists of a slender 1-kg rod AB which is 580 mm long, and cylindrical end weights at A and B that each have a diameter of 20 mm and a mass of 1 kg. This assembly is free to turn about the handle and socket, which are attached to the lug nut on the wheel of a car. If the rod AB is given an angular velocity of 4 rad/s and it strikes the bracket C on the handle without rebounding, determine the angular impulse imparted to the lug nut.



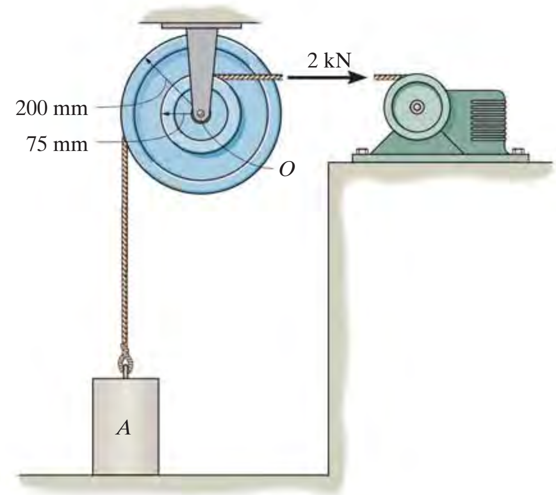
Prob. 19–5

19–6. The airplane is traveling in a straight line with a speed of 300 km/h, when the engines A and B produce a thrust of $T_A = 40$ kN and $T_B = 20$ kN, respectively. Determine the angular velocity of the airplane in $t = 5$ s. The plane has a mass of 200 Mg, its center of mass is located at G, and its radius of gyration about G is $k_G = 15$ m.



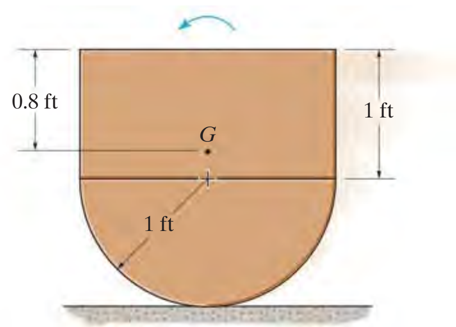
Prob. 19–6

19–7. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration of $k_O = 110$ mm. If the block at A has a mass of 40 kg, determine the speed of the block in 3 s after a constant force of 2 kN is applied to the rope wrapped around the inner hub of the pulley. The block is originally at rest.



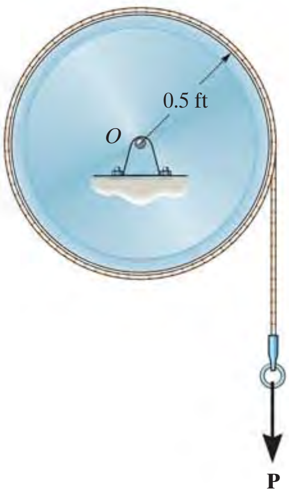
Prob. 19–7

***19–8.** The assembly weighs 10 lb and has a radius of gyration $k_G = 0.6$ ft about its center of mass G. The kinetic energy of the assembly is 31 ft · lb when it is in the position shown. If it rolls counterclockwise on the surface without slipping, determine its linear momentum at this instant.



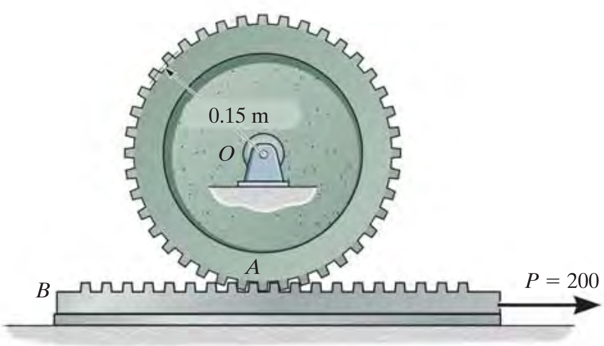
Prob. 19–8

19–9. The disk has a weight of 10 lb and is pinned at its center O. If a vertical force of $P = 2$ lb is applied to the cord wrapped around its outer rim, determine the angular velocity of the disk in four seconds starting from rest. Neglect the mass of the cord.



Prob. 19–9

19–10. The 30-kg gear A has a radius of gyration about its center of mass O of $k_O = 125$ mm. If the 20-kg gear rack B is subjected to a force of $P = 200$ N, determine the time required for the gear to obtain an angular velocity of 20 rad/s, starting from rest. The contact surface between the gear rack and the horizontal plane is smooth.

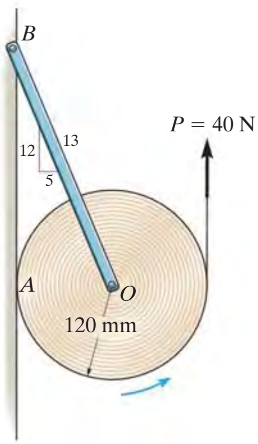


Prob. 19–10

19–11. The pulley has a weight of 8 lb and may be treated as a thin disk. A cord wrapped over its surface is subjected to forces $T_A = 4$ lb and $T_B = 5$ lb. Determine the angular velocity of the pulley when $t = 4$ s if it starts from rest when $t = 0$. Neglect the mass of the cord.

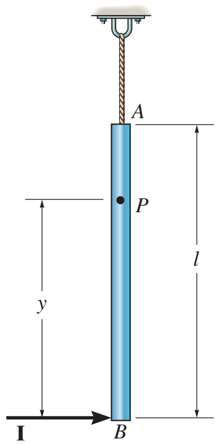
Prob. 19–11

***19–12.** The 40-kg roll of paper rests along the wall where the coefficient of kinetic friction is $\mu_k = 0.2$. If a vertical force of $P = 40$ N is applied to the paper, determine the angular velocity of the roll when $t = 6$ s starting from rest. Neglect the mass of the unraveled paper and take the radius of gyration of the spool about the axle O to be $k_O = 80$ mm.



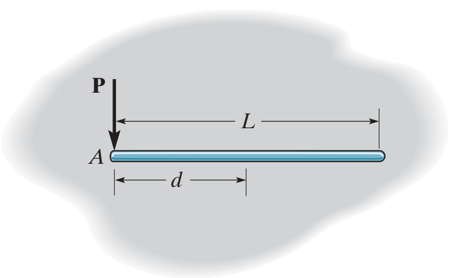
Prob. 19–12

19–13. The slender rod has a mass m and is suspended at its end A by a cord. If the rod receives a horizontal blow giving it an impulse $\mathbf{I}$ at its bottom B, determine the location y of the point P about which the rod appears to rotate during the impact.



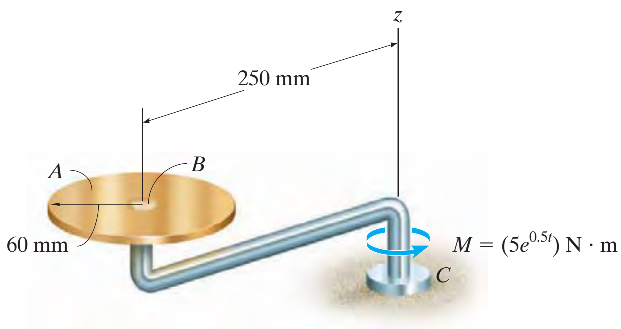
Prob. 19–13

19–14. The rod of length L and mass m lies on a smooth horizontal surface and is subjected to a force $\mathbf{P}$ at its end A as shown. Determine the location d of the point about which the rod begins to turn, i.e, the point that has zero velocity.



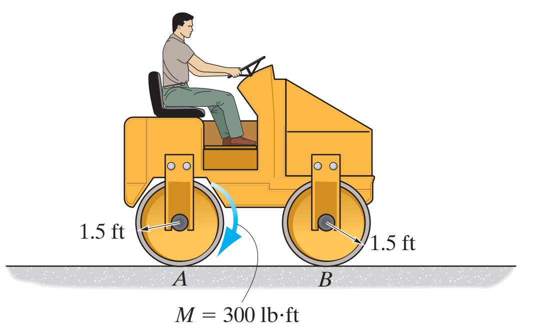
Prob. 19–14

19–15. A 4-kg disk A is mounted on arm BC, which has a negligible mass. If a torque of $M = (5e^{0.5t})$ N·m, where t is in seconds, is applied to the arm at C, determine the angular velocity of BC in 2 s starting from rest. Solve the problem assuming that (a) the disk is set in a smooth bearing at B so that it moves with curvilinear translation, (b) the disk is fixed to the shaft BC, and (c) the disk is given an initial freely spinning angular velocity of $\boldsymbol{\omega}_D = \{-80\mathbf{k}\}$ rad/s prior to application of the torque.



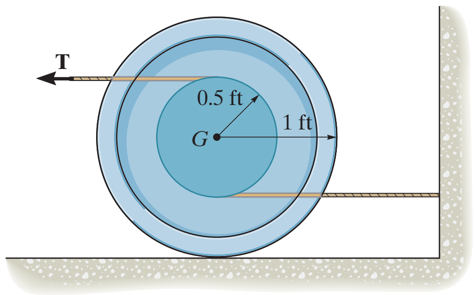
Prob. 19–15

***19–16.** The frame of a tandem drum roller has a weight of 4000 lb excluding the two rollers. Each roller has a weight of 1500 lb and a radius of gyration about its axle of 1.25 ft. If a torque of $M = 300$ lb·ft is supplied to the rear roller A, determine the speed of the drum roller 10 s later, starting from rest.



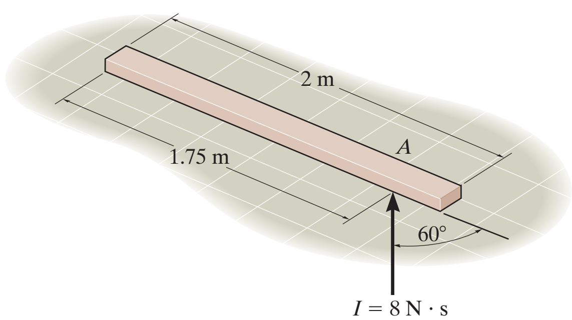
Prob. 19–16

19–17. The 100-lb wheel has a radius of gyration of $k_G = 0.75$ ft. If the upper wire is subjected to a tension of $T = 50$ lb, determine the velocity of the center of the wheel in 3 s, starting from rest. The coefficient of kinetic friction between the wheel and the surface is $\mu_k = 0.1$.



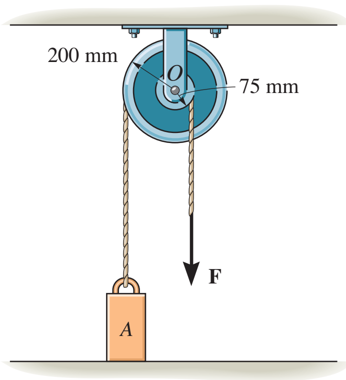
Prob. 19–17

19–18. The 4-kg slender rod rests on a smooth floor. If it is kicked so as to receive a horizontal impulse $I = 8\text{ N}\cdot\text{s}$ at point A as shown, determine its angular velocity and the speed of its mass center.



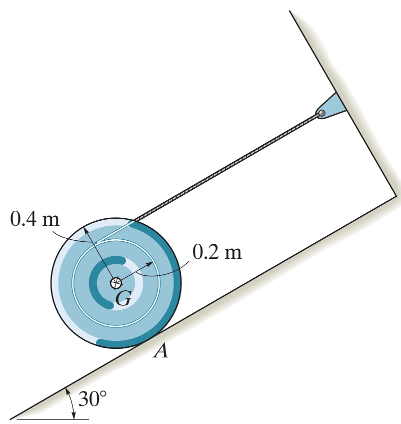
Prob. 19–18

19–19. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration $k_O = 110$ mm. If the block at A has a mass of 40 kg, determine the speed of the block in 3 s after a constant force $F = 2$ kN is applied to the rope wrapped around the inner hub of the pulley. The block is originally at rest. Neglect the mass of the rope.



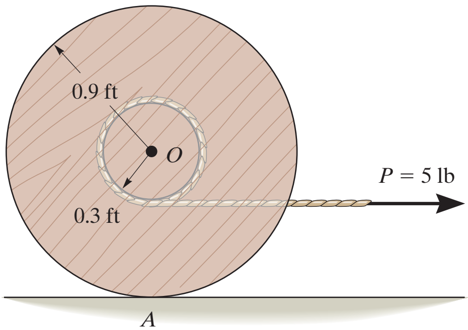
Prob. 19–19

***19–20.** The 100-kg spool is resting on the inclined surface for which the coefficient of kinetic friction is $\mu_k = 0.1$. Determine the angular velocity of the spool when $t = 4$ s after it is released from rest. The radius of gyration about the mass center is $k_G = 0.25$ m.



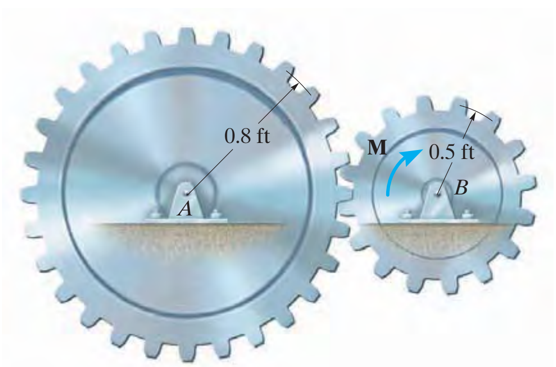
Prob. 19–20

19–21. The spool has a weight of 30 lb and a radius of gyration $k_O = 0.45$ ft. A cord is wrapped around its inner hub and the end subjected to a horizontal force $P = 5$ lb. Determine the spool's angular velocity in 4 s starting from rest. Assume the spool rolls without slipping.



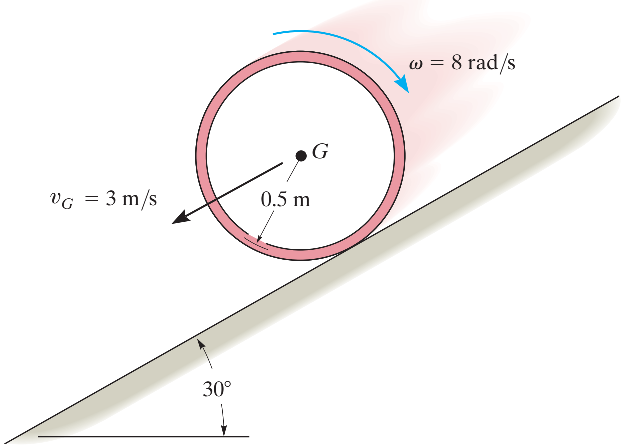
Prob. 19–21

19–22. The two gears A and B have weights and radii of gyration of $W_A = 15$ lb, $k_A = 0.5$ ft and $W_B = 10$ lb, $k_B = 0.35$ ft, respectively. If a motor transmits a couple moment to gear B of $M = 2(1 - e^{-0.5t})$ lb·ft, where t is in seconds, determine the angular velocity of gear A in $t = 5$ s, starting from rest.



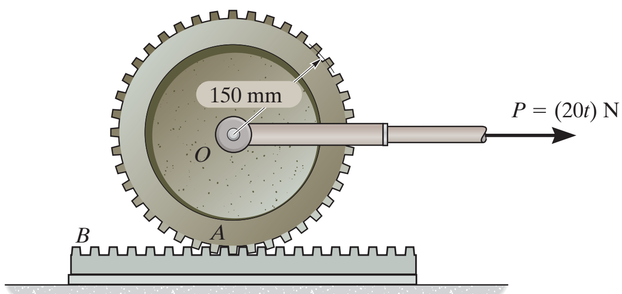
Prob. 19–22

19–23. The hoop (thin ring) has a mass of 5 kg and is released down the inclined plane such that it has a backspin $\omega = 8$ rad/s and its center has a velocity $v_G = 3$ m/s as shown. If the coefficient of kinetic friction between the hoop and the plane is $\mu_k = 0.6$, determine how long the hoop rolls before it stops slipping.



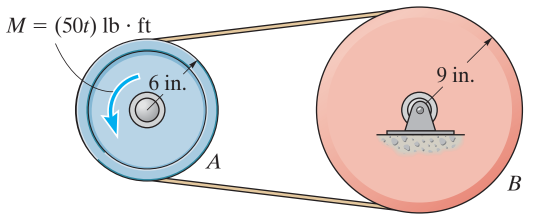
Prob. 19–23

***19–24.** The 30-kg gear is subjected to a force of $P = (20t)$ N, where t is in seconds. Determine the angular velocity of the gear at $t = 4$ s, starting from rest. Gear rack B is fixed to the horizontal plane, and the gear's radius of gyration about its mass center O is $k_O = 125$ mm.



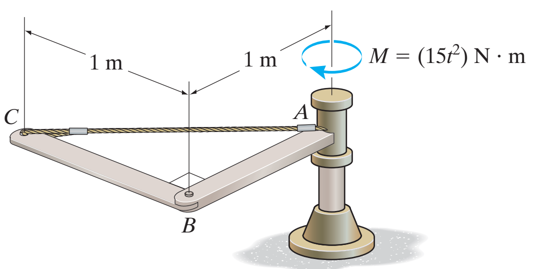
Prob. 19–24

19–25. The 30-lb flywheel A has a radius of gyration about its center of 4 in. Disk B weighs 50 lb and is coupled to the flywheel by means of a belt which does not slip at its contacting surfaces. If a motor supplies a counterclockwise torque to the flywheel of $M = (50t)\ \text{lb}\cdot\text{ft}$, where t is in seconds, determine the time required for the disk to attain an angular velocity of 60 rad/s starting from rest.



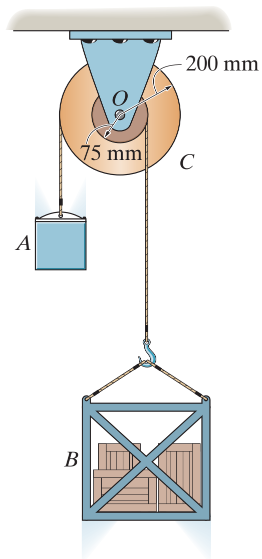
Prob. 19–25

19–26. If the shaft is subjected to a torque of $M = (15t^2)\ \text{N}\cdot\text{m}$, where t is in seconds, determine the angular velocity of the assembly when $t = 3$ s, starting from rest. Rods AB and BC each have a mass of 9 kg.



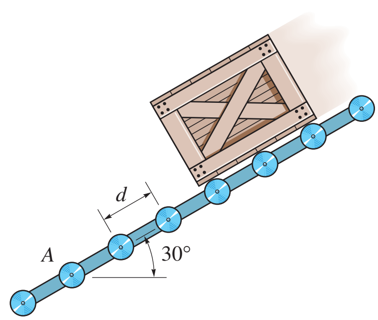
Prob. 19–26

19–27. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration of $k_O = 110$ mm. If the block at A has a mass of 40 kg and the container at B has a mass of 85 kg, including its contents, determine the speed of the container when $t = 3$ s after it is released from rest.



Prob. 19–27

***19–28.** The crate has a mass m_c. Determine the constant speed v_0 it acquires as it moves down the conveyor. The rollers each have a radius of r, mass m, and are spaced d apart. Note that friction causes each roller to rotate when the crate comes in contact with it.



Prob. 19–28

19.3 Conservation of Momentum

Conservation of Linear Momentum. If the sum of all the *linear impulses* acting on a system of connected rigid bodies is *zero* in a specific direction, then the linear momentum of the system is constant, or conserved in this direction, that is,

$$\left(\sum \begin{array}{c}\text{syst. linear}\\ \text{momentum}\end{array}\right)_1 = \left(\sum \begin{array}{c}\text{syst. linear}\\ \text{momentum}\end{array}\right)_2 \qquad (19\text{–}16)$$

This equation is referred to as the *conservation of linear momentum*.

Without introducing appreciable errors in the calculations, it may be possible to apply Eq. 19–16 in a specified direction for which the linear impulses are small or *nonimpulsive*. Specifically, nonimpulsive forces occur when small forces act over very short periods of time. Typical examples include the force of a slightly deformed spring, the initial contact force with soft ground, and in some cases the weight of the body.

Conservation of Angular Momentum. The angular momentum of a system of connected rigid bodies is conserved about the system's center of mass G, or a fixed point O, when the sum of all the angular impulses about these points is zero or appreciably small (nonimpulsive). The third of Eqs. 19–15 then becomes

$$\left(\sum \begin{array}{c}\text{syst. angular}\\ \text{momentum}\end{array}\right)_{O1} = \left(\sum \begin{array}{c}\text{syst. angular}\\ \text{momentum}\end{array}\right)_{O2} \qquad (19\text{–}17)$$

This equation is referred to as the *conservation of angular momentum*. In the case of a single rigid body, Eq. 19–17 applied to point G becomes $(I_G\omega)_1 = (I_G\omega)_2$. For example, consider a swimmer who executes a somersault after jumping off a diving board. By tucking his arms and legs in close to his chest, he *decreases* his body's moment of inertia and thus *increases* his angular velocity ($I_G\omega$ must be constant). If he straightens out just before entering the water, his body's moment of inertia is *increased*, and so his angular velocity *decreases*. Since the weight of his body creates a linear impulse during the time of motion, this example also illustrates how the angular momentum of a body can be conserved and yet the linear momentum is *not*. Such cases occur whenever the external forces creating the linear impulse pass through either the center of mass of the body or a fixed axis of rotation.

Procedure for Analysis

The conservation of linear or angular momentum should be applied using the following procedure.

Free-Body Diagram.

- Establish the *x, y* inertial frame of reference and draw the free-body diagram for the body or system of bodies during the time of impact. From this diagram classify each of the applied forces as being either "impulsive" or "nonimpulsive."

- By inspection of the free-body diagram, the *conservation of linear momentum* applies in a given direction when *no* external impulsive forces act on the body or system in that direction; whereas the *conservation of angular momentum* applies about a fixed point *O* or at the mass center *G* of a body or system of bodies when all the external impulsive forces acting on the body or system create zero moment (or zero angular impulse) about *O* or *G*.

- As an alternative procedure, draw the impulse and momentum diagrams for the body or system of bodies. These diagrams are particularly helpful in order to visualize the "moment" terms used in the conservation of angular momentum equation, when it has been decided that angular momenta are to be computed about a point other than the body's mass center *G*.

Conservation of Momentum.

- Apply the conservation of linear or angular momentum in the appropriate directions.

Kinematics.

- If the motion appears to be complicated, kinematic (velocity) diagrams may be helpful in obtaining the necessary kinematic relations.

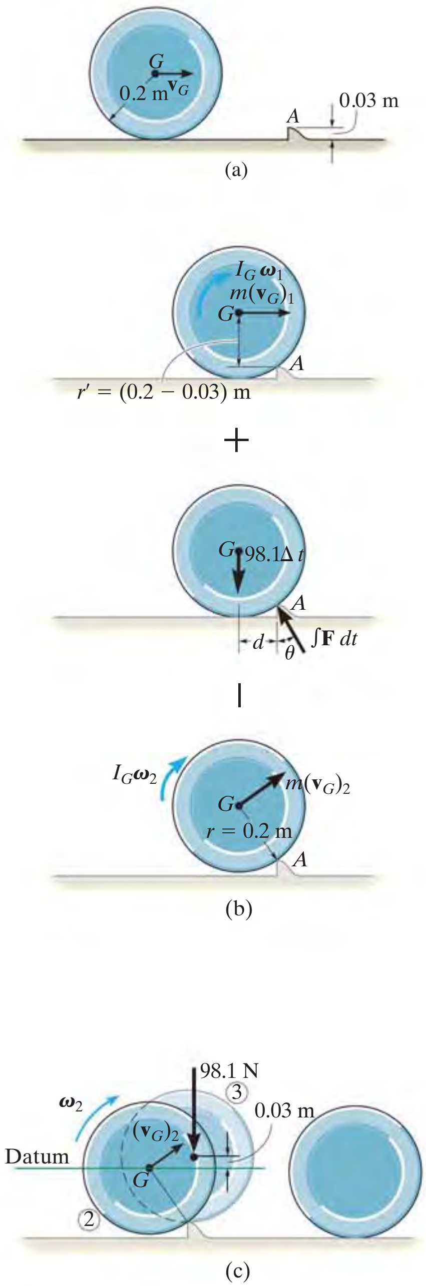


Fig. 19–9

The 10-kg wheel shown in Fig. 19–9*a* has a moment of inertia $I_G = 0.156 \text{ kg}\cdot\text{m}^2$. Assuming that the wheel does not slip or rebound, determine the minimum velocity $\mathbf{v}_G$ it must have to just roll over the obstruction at A.

SOLUTION

Impulse and Momentum Diagrams. Since no slipping or rebounding occurs, the wheel essentially *pivots* about point A during contact. This condition is shown in Fig. 19–9*b*, which indicates, respectively, the momentum of the wheel *just before impact*, the impulses given to the wheel *during impact*, and the momentum of the wheel *just after impact*. Only two impulses (forces) act on the wheel. By comparison, the force at A is much greater than that of the weight, and since the time of impact is very short, the weight can be considered nonimpulsive. The impulsive force $\mathbf{F}$ at A has both an unknown magnitude and an unknown direction θ. To eliminate this force from the analysis, note that angular momentum about A is essentially *conserved* since $(98.1\Delta t)d \approx 0$.

Conservation of Angular Momentum. With reference to Fig. 19–9*b*,

$$(\circlearrowright +) \qquad (H_A)_1 = (H_A)_2$$

$$r'm(v_G)_1 + I_G\omega_1 = rm(v_G)_2 + I_G\omega_2$$

$$(0.2 \text{ m} - 0.03 \text{ m})(10 \text{ kg})(v_G)_1 + (0.156 \text{ kg}\cdot\text{m}^2)(\omega_1) =$$
$$(0.2 \text{ m})(10 \text{ kg})(v_G)_2 + (0.156 \text{ kg}\cdot\text{m}^2)(\omega_2)$$

Kinematics. Since no slipping occurs, in general $\omega = v_G/r = v_G/0.2 \text{ m} = 5v_G$. Substituting this into the above equation and simplifying yields

$$(v_G)_2 = 0.8921(v_G)_1 \qquad (1)$$

Conservation of Energy.* In order to roll over the obstruction, the wheel must pass position 3 shown in Fig. 19–9*c*. Hence, if $(v_G)_2$ [or $(v_G)_1$] is to be a minimum, it is necessary that the kinetic energy of the wheel at position 2 be equal to the potential energy at position 3. Placing the datum through the center of gravity, as shown in the figure, and applying the conservation of energy equation, we have

$$\{T_2\} + \{V_2\} = \{T_3\} + \{V_3\}$$

$$\left\{\tfrac{1}{2}(10 \text{ kg})(v_G)_2^2 + \tfrac{1}{2}(0.156 \text{ kg}\cdot\text{m}^2)\omega_2^2\right\} + \{0\} =$$
$$\{0\} + \{(98.1 \text{ N})(0.03 \text{ m})\}$$

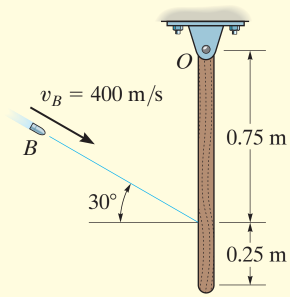
Substituting $\omega_2 = 5(v_G)_2$ and Eq. 1 into this equation, and solving,

$$(v_G)_1 = 0.729 \text{ m/s} \rightarrow \qquad \textit{Ans.}$$

*This principle *does not apply during impact*, since energy is *lost* during the collision. However, just after impact, as in Fig. 19–9*c*, it can be used.

EXAMPLE 19.7

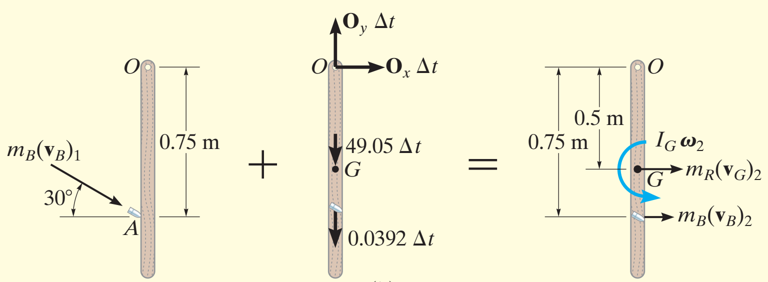
The 5-kg slender rod shown in Fig. 19–10*a* is pinned at O and is initially at rest. If a 4-g bullet is fired into the rod with a velocity of 400 m/s, as shown in the figure, determine the angular velocity of the rod just after the bullet becomes embedded in it.



(a)

SOLUTION

Impulse and Momentum Diagrams. The impulse which the bullet exerts on the rod can be eliminated from the analysis, and the angular velocity of the rod just after impact can be determined by considering the bullet and rod as a single system. To clarify the principles involved, the impulse and momentum diagrams are shown in Fig. 19–10*b*. The momentum diagrams are drawn *just before and just after impact*. During impact, the bullet and rod exert equal but *opposite internal impulses* at A. As shown on the impulse diagram, the impulses that are external to the system are due to the reactions at O and the weights of the bullet and rod. Since the time of impact, Δt, is very short, the rod moves only a slight amount, and so the "moments" of the weight impulses about point O are essentially zero. Therefore angular momentum is conserved about this point.



(b)

Conservation of Angular Momentum. From Fig. 19–10*b*, we have

$$(\circlearrowleft +) \qquad \Sigma(H_O)_1 = \Sigma(H_O)_2$$

$$m_B(v_B)_1 \cos 30°(0.75 \text{ m}) = m_B(v_B)_2(0.75 \text{ m}) + m_R(v_G)_2(0.5 \text{ m}) + I_G\omega_2$$

$$(0.004 \text{ kg})(400 \cos 30° \text{ m/s})(0.75 \text{ m}) =$$

$$(0.004 \text{ kg})(v_B)_2(0.75 \text{ m}) + (5 \text{ kg})(v_G)_2(0.5 \text{ m}) + \left[\tfrac{1}{12}(5 \text{ kg})(1 \text{ m})^2\right]\omega_2 \quad (1)$$

or

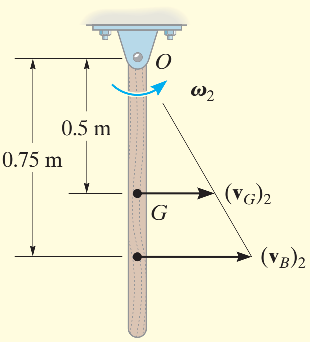
$$1.039 = 0.003(v_B)_2 + 2.50(v_G)_2 + 0.4167\omega_2$$

Kinematics. Since the rod is pinned at O, from Fig. 19–9*c* we have

$$(v_G)_2 = (0.5 \text{ m})\omega_2 \quad (v_B)_2 = (0.75 \text{ m})\omega_2$$

Substituting into Eq. 1 and solving yields

$$\omega_2 = 0.623 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

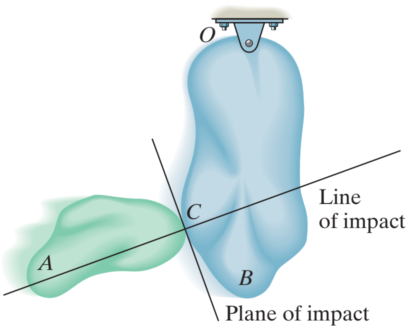


(c)

Fig. 19–10

*19.4 Eccentric Impact

The concepts involving central and oblique impact of particles were presented in Sec. 15.4. We will now expand this treatment and discuss the eccentric impact of two bodies. *Eccentric impact* occurs when the line connecting the *mass centers* of the two bodies *does not* coincide with the line of impact.* This type of impact often occurs when one or both of the bodies are constrained to rotate about a fixed axis. Consider, for example, the collision at C between the two bodies A and B, shown in Fig. 19–11a. It is assumed that just before collision B is rotating counterclockwise with an angular velocity $(\boldsymbol{\omega}_B)_1$, and the velocity of the contact point C located on A is $(\mathbf{u}_A)_1$. Kinematic diagrams for both bodies just before collision are shown in Fig. 19–11b. Provided the bodies are smooth, the *impulsive forces* they exert on each other *are directed along the line of impact*. Hence, the component of velocity of point C on body B, which is directed along the line of impact, is $(v_B)_1 = (\omega_B)_1 r$, Fig. 19–11b. Likewise, on body A the component of velocity $(\mathbf{u}_A)_1$ along the line of impact is $(\mathbf{v}_A)_1$. In order for a collision to occur, $(v_A)_1 > (v_B)_1$.



(a)

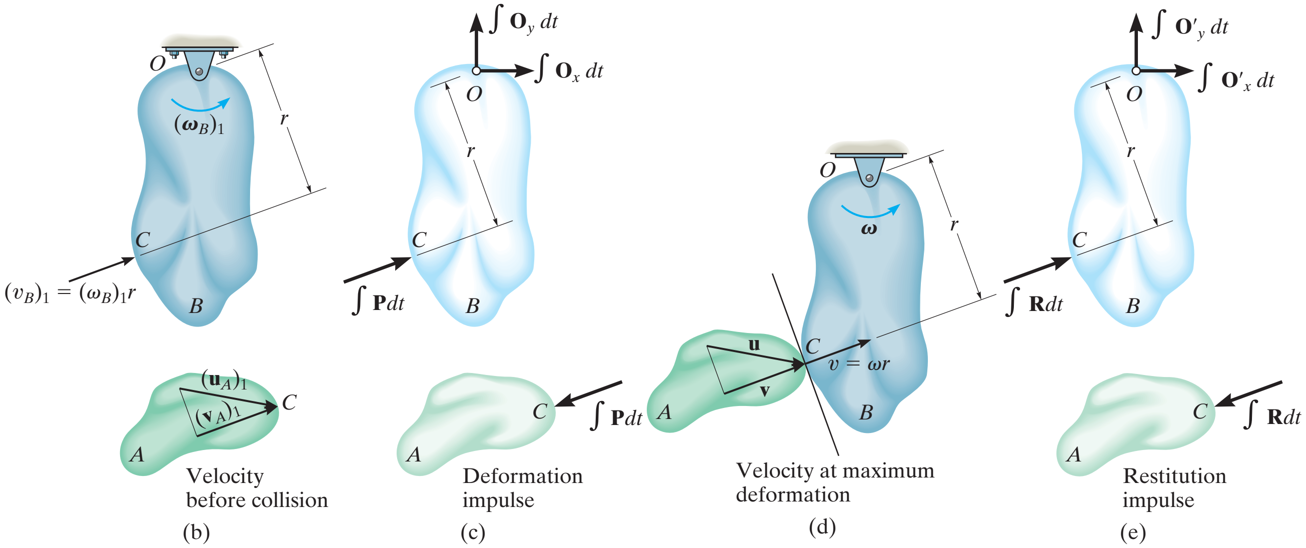
Fig. 19–11

During the impact an equal but opposite impulsive force $\mathbf{P}$ is exerted between the bodies which *deforms* their shapes at the point of contact. The resulting impulse is shown on the impulse diagrams for both bodies, Fig. 19–11c. Note that the impulsive force at point C on the rotating body creates impulsive pin reactions at O. On these diagrams it is assumed that the impact creates forces which are much larger than the nonimpulsive weights of the bodies, which are not shown. When the deformation at point C is a maximum, C on both the bodies moves with a common velocity $\mathbf{v}$ along the line of impact, Fig. 19–11d. A period of *restitution* then occurs in which the bodies tend to regain their original shapes. The restitution phase creates an equal but opposite impulsive force $\mathbf{R}$ acting between the bodies as shown on the impulse diagram, Fig. 19–11e. After restitution the bodies move apart such that point C on body B has a velocity $(\mathbf{v}_B)_2$ and point C on body A has a velocity $(\mathbf{u}_A)_2$, Fig. 19–11f, where $(v_B)_2 > (v_A)_2$.

In general, a problem involving the impact of two bodies requires determining the *two unknowns* $(v_A)_2$ and $(v_B)_2$, assuming $(v_A)_1$ and $(v_B)_1$ are known (or can be determined using kinematics, energy methods, the equations of motion, etc.). To solve such problems, two equations must be written. The *first equation* generally involves application of *the conservation of angular momentum to the two bodies*. In the case of both bodies A and B, we can state that angular momentum is conserved about point O since the impulses at C are internal to the system and the impulses at O create zero moment (or zero angular impulse) about O. The *second equation* can be obtained using the definition of the *coefficient of restitution, e*, which is a ratio of the restitution impulse to the deformation impulse.

Here is an example of eccentric impact occurring between this bowling ball and pin. (© R.C. Hibbeler)

*When these lines coincide, central impact occurs and the problem can be analyzed as discussed in Sec. 15.4.



Is is important to realize, however, that *this analysis has only a very limited application in engineering, because values of e for this case have been found to be highly sensitive to the material, geometry, and the velocity of each of the colliding bodies.* To establish a useful form of the coefficient of restitution equation we must first apply the principle of angular impulse and momentum about point O to bodies B and A separately. Combining the results, we then obtain the necessary equation. Proceeding in this manner, the principle of impulse and momentum applied to body B from the time just before the collision to the instant of maximum deformation, Figs. 19–11*b*, 19–11*c*, and 19–11*d*, becomes

$$(\circlearrowleft +) \qquad I_O(\omega_B)_1 + r\int P\,dt = I_O\omega \qquad (19\text{–}18)$$

Here I_O is the moment of inertia of body B about point O. Similarly, applying the principle of angular impulse and momentum from the instant of maximum deformation to the time just after the impact, Figs. 19–11*d*, 19–11*e*, and 19–11*f*, yields

$$(\circlearrowleft +) \qquad I_O\omega + r\int R\,dt = I_O(\omega_B)_2 \qquad (19\text{–}19)$$

Solving Eqs. 19–18 and 19–19 for $\int P\,dt$ and $\int R\,dt$, respectively, and formulating e, we have

$$e = \frac{\int R\,dt}{\int P\,dt} = \frac{r(\omega_B)_2 - r\omega}{r\omega - r(\omega_B)_1} = \frac{(v_B)_2 - v}{v - (v_B)_1}$$

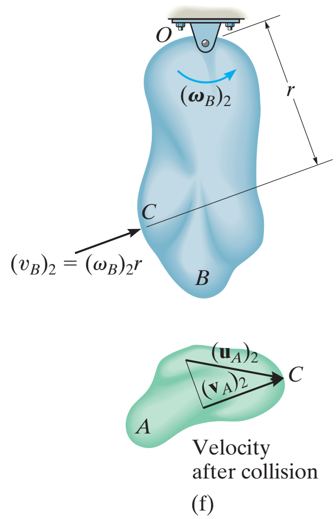


Fig. 19–11 (cont.)

In the same manner, we can write an equation which relates the magnitudes of velocity $(v_A)_1$ and $(v_A)_2$ of body A. The result is

$$e = \frac{v - (v_A)_2}{(v_A)_1 - v}$$

Combining the above two equations by eliminating the common velocity v yields the desired result, i.e.,

$$(+\nearrow) \qquad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1} \qquad (19\text{–}20)$$

This equation is identical to Eq. 15–11, which was derived for the central impact between two particles. It states that the coefficient of restitution is equal to the ratio of the relative velocity of *separation* of the points of contact (C) *just after impact* to the relative velocity at which the points *approach* one another *just* before impact. In deriving this equation, we assumed that the points of contact for both bodies move up and to the right *both* before and after impact. If motion of any one of the contacting points occurs down and to the left, the velocity of this point should be considered a negative quantity in Eq. 19–20.

During impact the columns of many highway signs are intended to break out of their supports and easily collapse at their joints. This is shown by the slotted connections at their base and the breaks at the column's midsection. (© R.C. Hibbeler)

The 10-lb slender rod is suspended from the pin at A, Fig. 19–12a. If a 2-lb ball B is thrown at the rod and strikes its center with a velocity of 30 ft/s, determine the angular velocity of the rod just after impact. The coefficient of restitution is $e = 0.4$.

SOLUTION

Conservation of Angular Momentum. Consider the ball and rod as a system, Fig. 19–12b. Angular momentum is conserved about point A since the impulsive force between the rod and ball is *internal*. Also, the *weights* of the ball and rod are *nonimpulsive*. Noting the directions of the velocities of the ball and rod just after impact as shown on the kinematic diagram, Fig. 19–12c, we require

$$(\circlearrowleft +) \qquad (H_A)_1 = (H_A)_2$$

$$m_B(v_B)_1(1.5 \text{ ft}) = m_B(v_B)_2(1.5 \text{ ft}) + m_R(v_G)_2(1.5 \text{ ft}) + I_G\omega_2$$

$$\left(\frac{2 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(30 \text{ ft/s})(1.5 \text{ ft}) = \left(\frac{2 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(v_B)_2(1.5 \text{ ft}) +$$

$$\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(v_G)_2(1.5 \text{ ft}) + \left[\frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(3 \text{ ft})^2\right]\omega_2$$

Since $(v_G)_2 = 1.5\omega_2$ then

$$2.795 = 0.09317(v_B)_2 + 0.9317\omega_2 \qquad (1)$$

Coefficient of Restitution. With reference to Fig. 19–12c, we have

$$(\overset{+}{\rightarrow}) \qquad e = \frac{(v_G)_2 - (v_B)_2}{(v_B)_1 - (v_G)_1} \qquad 0.4 = \frac{(1.5 \text{ ft})\omega_2 - (v_B)_2}{30 \text{ ft/s} - 0}$$

$$12.0 = 1.5\omega_2 - (v_B)_2 \qquad (2)$$

Solving Eqs. 1 and 2, yields

$$(v_B)_2 = -6.52 \text{ ft/s} = 6.52 \text{ ft/s} \leftarrow$$

$$\omega_2 = 3.65 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

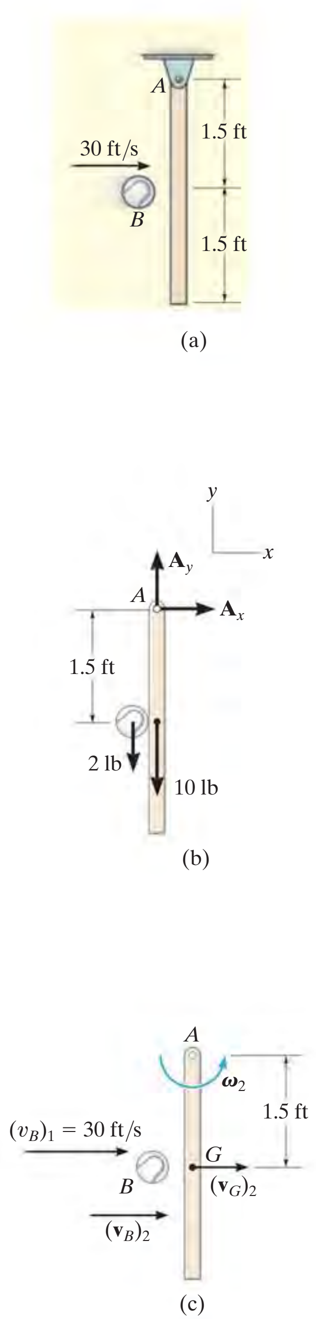
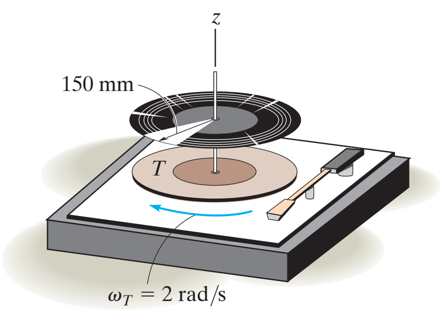


Fig. 19–12

PROBLEMS

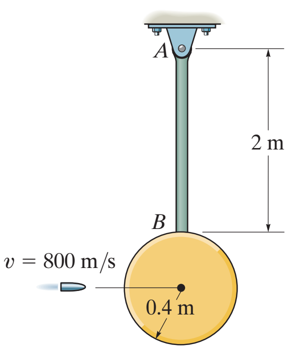
19–29. The turntable T of a record player has a mass of 0.75 kg and a radius of gyration $k_z = 125$ mm. It is *turning freely* at $\omega_T = 2$ rad/s when a 50-g record (thin disk) falls on it. Determine the final angular velocity of the turntable just after the record stops slipping on the turntable.



Prob. 19–29

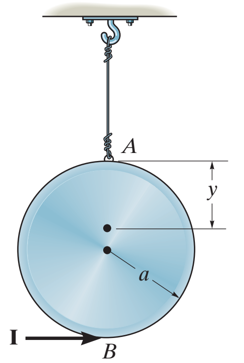
19–30. The 10-g bullet having a velocity of 800 m/s is fired into the edge of the 5-kg disk as shown. Determine the angular velocity of the disk just after the bullet becomes embedded into its edge. Also, calculate the angle θ the disk will swing when it stops. The disk is originally at rest. Neglect the mass of the rod AB.

19–31. The 10-g bullet having a velocity of 800 m/s is fired into the edge of the 5-kg disk as shown. Determine the angular velocity of the disk just after the bullet becomes embedded into its edge. Also, calculate the angle θ the disk will swing when it stops. The disk is originally at rest. The rod AB has a mass of 3 kg.



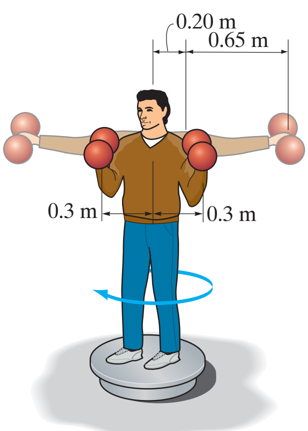
Probs. 19–30/31

***19–32.** The circular disk has a mass m and is suspended at A by the wire. If it receives a horizontal impulse **I** at its edge B, determine the location y of the point P about which the disk appears to rotate during the impact.



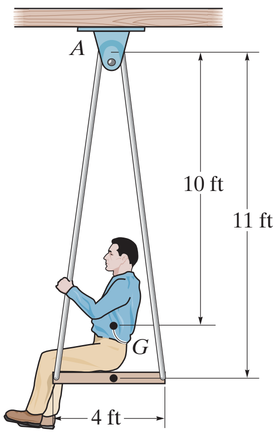
Prob. 19–32

19–33. The 80-kg man is holding two dumbbells while standing on a turntable of negligible mass, which turns freely about a vertical axis. When his arms are fully extended, the turntable is rotating with an angular velocity of 0.5 rev/s. Determine the angular velocity of the man when he retracts his arms to the position shown. When his arms are fully extended, approximate each arm as a uniform 6-kg rod having a length of 650 mm, and his body as a 68-kg solid cylinder of 400-mm diameter. With his arms in the retracted position, assume the man is an 80-kg solid cylinder of 450-mm diameter. Each dumbbell consists of two 5-kg spheres of negligible size.



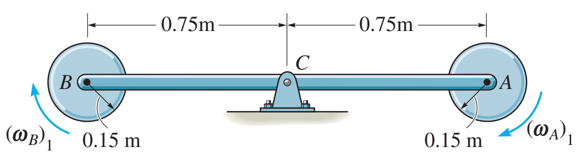
Prob. 19–33

19–34. The platform swing consists of a 200-lb flat plate suspended by four rods of negligible weight. When the swing is at rest, the 150-lb man jumps off the platform when his center of gravity G is 10 ft from the pin at A. This is done with a horizontal velocity of 5 ft/s, measured relative to the swing at the level of G. Determine the angular velocity he imparts to the swing just after jumping off.



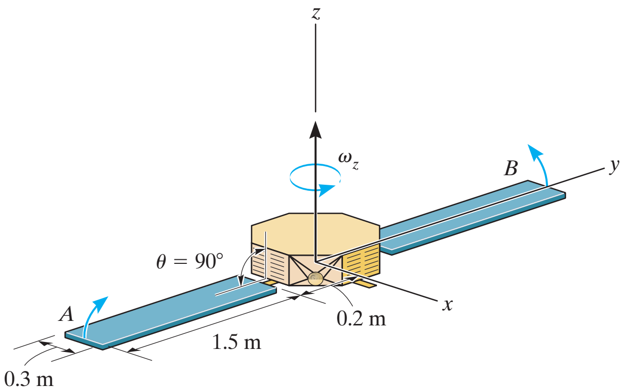
Prob. 19–34

19–35. The 2-kg rod ACB supports the two 4-kg disks at its ends. If both disks are given a clockwise angular velocity $(\omega_A)_1 = (\omega_B)_1 = 5$ rad/s while the rod is held stationary and then released, determine the angular velocity of the rod after both disks have stopped spinning relative to the rod due to frictional resistance at the pins A and B. Motion is in the *horizontal plane*. Neglect friction at pin C.



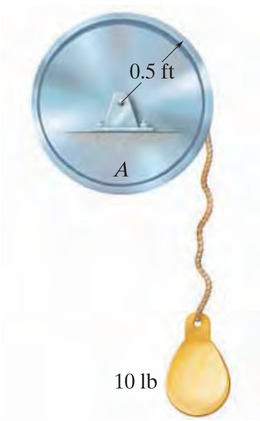
Prob. 19–35

***19–36.** The satellite has a mass of 200 kg and a radius of gyration about z axis of $k_z = 0.1$ m, excluding the two solar panels A and B. Each solar panel has a mass of 15 kg and can be approximated as a thin plate. If the satellite is originally spinning about the z axis at a constant rate $\omega_z = 0.5$ rad/s when $\theta = 90°$, determine the rate of spin if both panels are raised and reach the upward position, $\theta = 0°$, at the same instant.



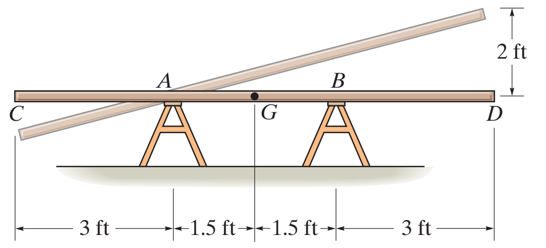
Prob. 19–36

19–37. Disk A has a weight of 20 lb. An inextensible cable is attached to the 10-lb weight and wrapped around the disk. The weight is dropped 2 ft before the slack is taken up. If the impact is perfectly elastic, i.e., $e = 1$, determine the angular velocity of the disk just after impact.



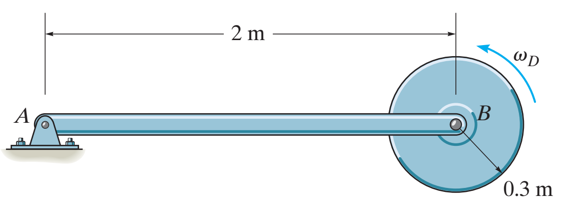
Prob. 19–37

19–38. The plank has a weight of 30 lb, center of gravity at G, and it rests on the two sawhorses at A and B. If the end D is raised 2 ft above the top of the sawhorses and is released from rest, determine how high end C will rise from the top of the sawhorses after the plank falls so that it rotates clockwise about A, strikes and pivots on the sawhorse at B, and rotates clockwise off the sawhorse at A.



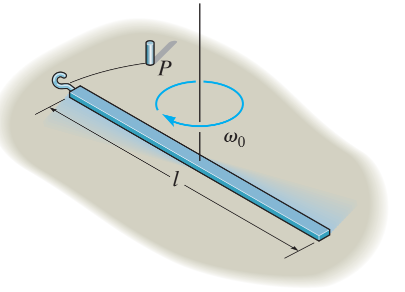
Prob. 19–38

19–39. The 12-kg rod AB is pinned to the 40-kg disk. If the disk is given an angular velocity $\omega_D = 100$ rad/s while the rod is held stationary, and the assembly is then released, determine the angular velocity of the rod after the disk has stopped spinning relative to the rod due to frictional resistance at the bearing B. Motion is in the *horizontal plane*. Neglect friction at the pin A.



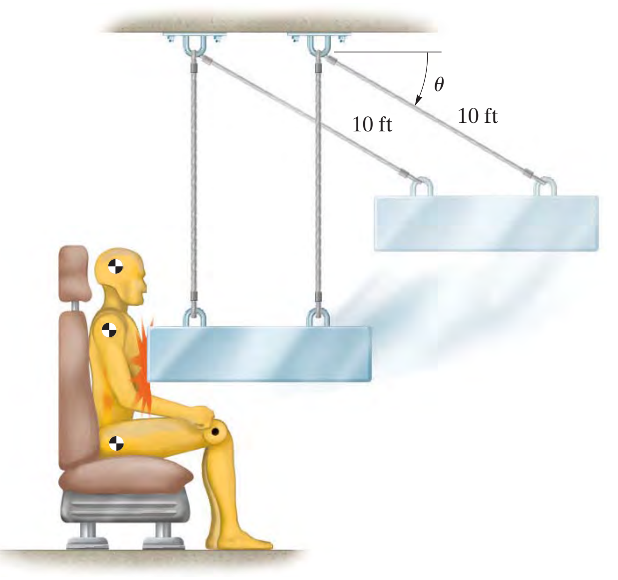
Prob. 19–39

***19–40.** A thin rod of mass m has an angular velocity $\boldsymbol{\omega}_0$ while rotating on a smooth surface. Determine its new angular velocity just after its end strikes and hooks onto the peg and the rod starts to rotate about P without rebounding. Solve the problem (a) using the parameters given, (b) setting $m = 2$ kg, $\omega_0 = 4$ rad/s, $l = 1.5$ m.



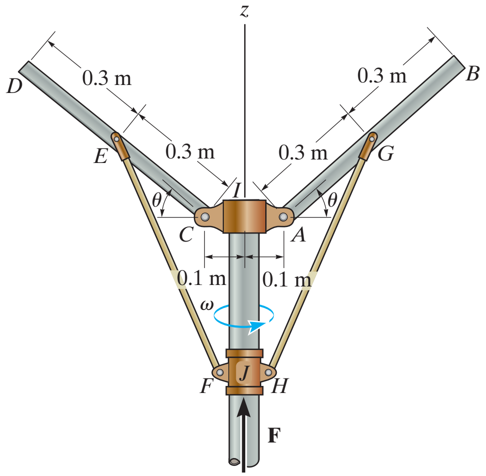
Prob. 19–40

19–41. Tests of impact on the fixed crash dummy are conducted using the 300-lb ram that is released from rest at $\theta = 30°$, and allowed to fall and strike the dummy at $\theta = 90°$. If the coefficient of restitution between the dummy and the ram is $e = 0.4$, determine the angle θ to which the ram will rebound before momentarily coming to rest.



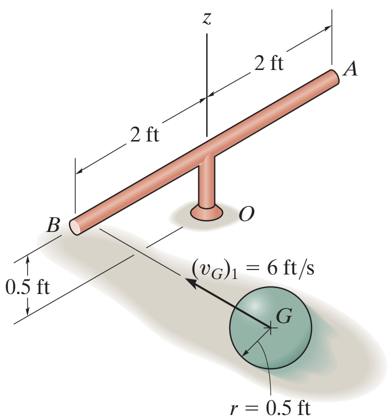
Prob. 19–41

19–42. The vertical shaft is rotating with an angular velocity of 3 rad/s when $\theta = 0°$. If a force **F** is applied to the collar so that $\theta = 90°$, determine the angular velocity of the shaft. Also, find the work done by force **F**. Neglect the mass of rods GH and EF and the collars I and J. The rods AB and CD each have a mass of 10 kg.



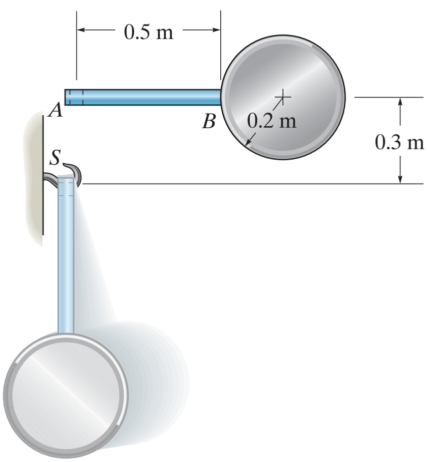
Prob. 19–42

19–43. The mass center of the 3-lb ball has a velocity of $(v_G)_1 = 6$ ft/s when it strikes the end of the smooth 5-lb slender bar which is at rest. Determine the angular velocity of the bar about the z axis just after impact if $e = 0.8$.



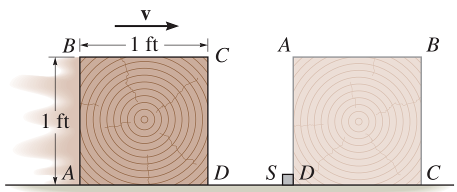
Prob. 19–43

***19–44.** The pendulum consists of a slender 2-kg rod AB and 5-kg disk. It is released from rest without rotating. When it falls 0.3 m, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity of the pendulum after it has rotated 90°. Treat the pendulum's weight during impact as a nonimpulsive force.



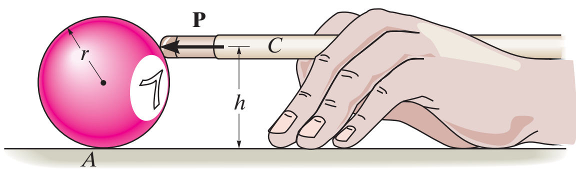
Prob. 19–44

19–45. The 10-lb block is sliding on the smooth surface when the corner D hits a stop block S. Determine the minimum velocity **v** the block should have which would allow it to tip over on its side and land in the position shown. Neglect the size of S. *Hint*: During impact consider the weight of the block to be nonimpulsive.



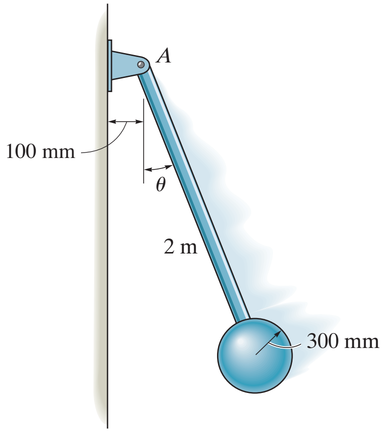
Prob. 19–45

19–46. Determine the height h at which a billiard ball of mass m must be struck so that no frictional force develops between it and the table at A. Assume that the cue C only exerts a horizontal force **P** on the ball.



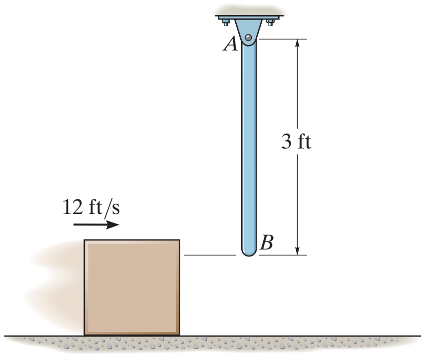
Prob. 19–46

19–47. The pendulum consists of a 15-kg solid ball and 6-kg rod. If it is released from rest when $\theta_1 = 90°$, determine the angle θ_2 after the ball strikes the wall, rebounds, and the pendulum swings up to the point of momentary rest. Take $e = 0.6$.



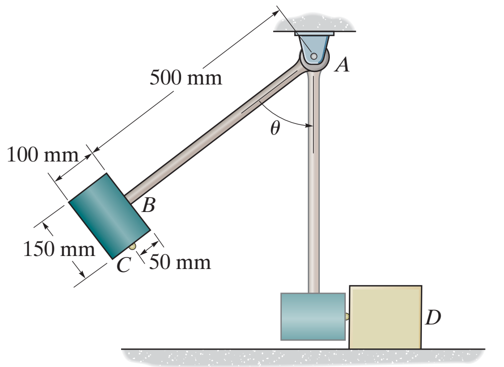
Prob. 19–47

***19–48.** The 4-lb rod AB is hanging in the vertical position. A 2-lb block, sliding on a smooth horizontal surface with a velocity of 12 ft/s, strikes the rod at its end B. Determine the velocity of the block immediately after the collision. The coefficient of restitution between the block and the rod at B is $e = 0.8$.



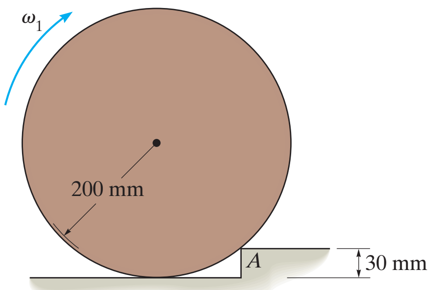
Prob. 19–48

19–49. The hammer consists of a 10-kg solid cylinder C and 6-kg uniform slender rod AB. If the hammer is released from rest when $\theta = 90°$ and strikes the 30-kg block D when $\theta = 0°$, determine the velocity of block D and the angular velocity of the hammer immediately after the impact. The coefficient of restitution between the hammer and the block is $e = 0.6$.



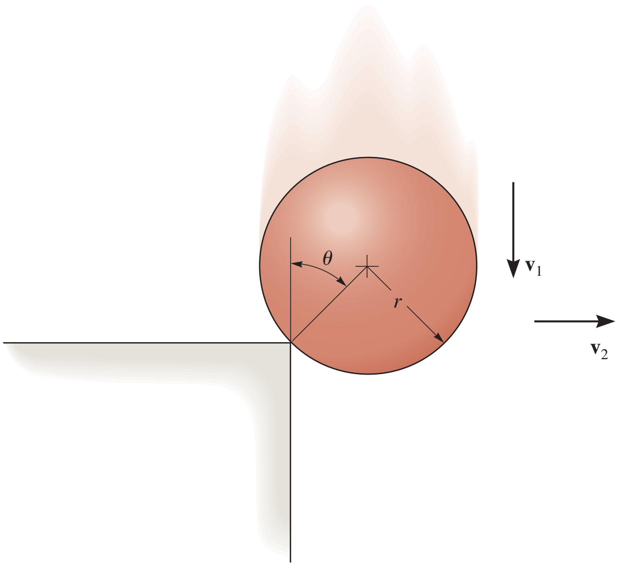
Prob. 19–49

19–50. The 20-kg disk strikes the step without rebounding. Determine the largest angular velocity ω_1 the disk can have and not lose contact with the step, A.



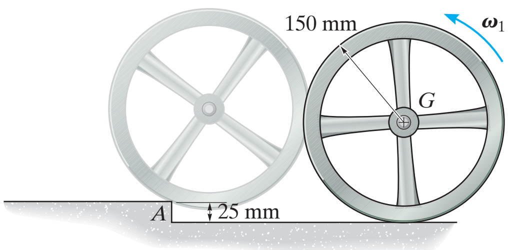
Prob. 19–50

19–51. The solid ball of mass m is dropped with a velocity $\mathbf{v}_1$ onto the edge of the rough step. If it rebounds horizontally off the step with a velocity $\mathbf{v}_2$, determine the angle θ at which contact occurs. Assume no slipping when the ball strikes the step. The coefficient of restitution is e.



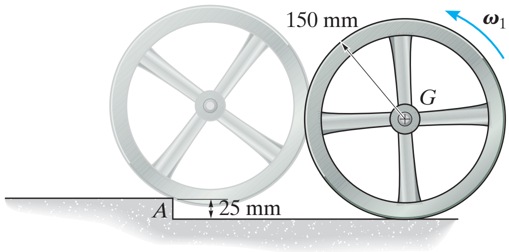
Prob. 19–51

***19–52.** The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. Determine the minimum value of the angular velocity $\boldsymbol{\omega}_1$ of the wheel, so that it strikes the step at A without rebounding and then rolls over it without slipping.



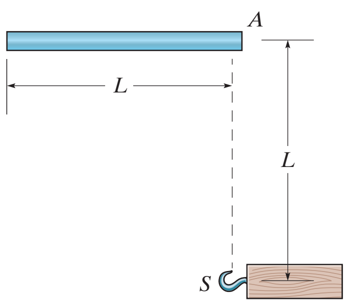
Prob. 19–52

19–53. The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. If it rolls without slipping with an angular velocity of $\boldsymbol{\omega}_1 = 5\text{ rad/s}$ before it strikes the step at A, determine its angular velocity after it rolls over the step. The wheel does not lose contact with the step when it strikes it.



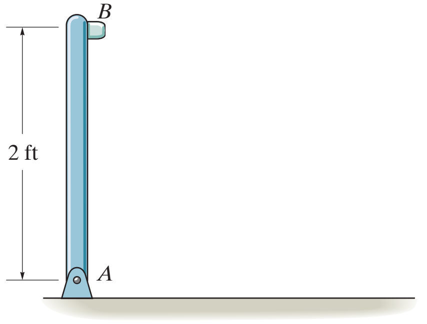
Prob. 19–53

19–54. The rod of mass m and length L is released from rest without rotating. When it falls a distance L, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity ω of the rod after it has rotated 90°. Treat the rod's weight during impact as a nonimpulsive force.



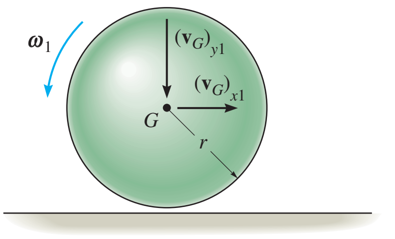
Prob. 19–54

19–55. The 15-lb rod AB is released from rest in the vertical position. If the coefficient or restitution between the floor and the cushion at B is $e = 0.7$, determine how high the end of the rod rebounds after impact with the floor.



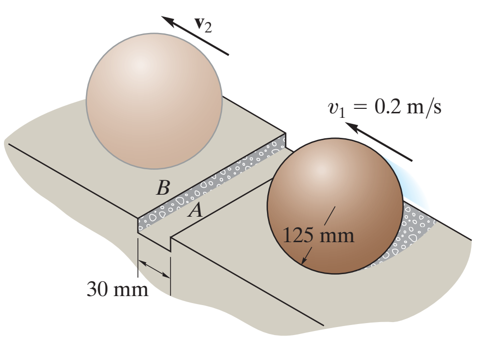
Prob. 19–55

19–57. A solid ball with a mass m is thrown on the ground such that at the instant of contact it has an angular velocity $\boldsymbol{\omega}_1$ and velocity components $(\mathbf{v}_G)_{x1}$ and $(\mathbf{v}_G)_{y1}$ as shown. If the ground is rough so no slipping occurs, determine the components of the velocity of its mass center just after impact. The coefficient of restitution is e.



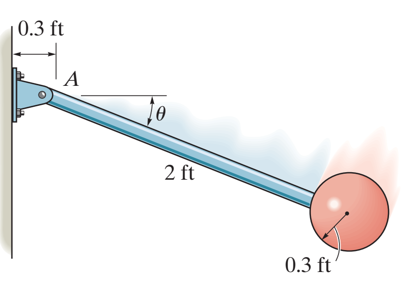
Prob. 19–57

***19–56.** A ball having a mass of 8 kg and initial speed of $v_1 = 0.2$ m/s rolls over a 30-mm-long depression. Assuming that the ball rolls off the edges of contact first A, then B, without slipping, determine its final velocity $\mathbf{v}_2$ when it reaches the other side.



Prob. 19–56

19–58. The pendulum consists of a 10-lb solid ball and 4-lb rod. If it is released from rest when $\theta_0 = 0°$, determine the angle θ_1 of rebound after the ball strikes the wall and the pendulum swings up to the point of momentary rest. Take $e = 0.6$.



Prob. 19–58

CONCEPTUAL PROBLEMS

C19–1. The soil compactor moves forward at constant velocity by supplying power to the rear wheels. Use appropriate numerical data for the wheel, roller, and body and calculate the angular momentum of this system about point A at the ground, point B on the rear axle, and point G, the center of gravity for the system.

Prob. C19–1 (© R.C. Hibbeler)

C19–2. The swing bridge opens and closes by turning 90° using a motor located under the center of the deck at A that applies a torque **M** to the bridge. If the bridge was supported at its end B, would the same torque open the bridge at the same time, or would it open slower or faster? Explain your answer using numerical values and an impulse and momentum analysis. Also, what are the benefits of making the bridge have the variable depth as shown?

Prob. C19–2 (© R.C. Hibbeler)

C19–3. Why is it necessary to have the tail blade B on the helicopter that spins perpendicular to the spin of the main blade A? Explain your answer using numerical values and an impulse and momentum analysis.

Prob. C19–3 (© R.C. Hibbeler)

C19–4. The amusement park ride consists of two gondolas A and B, and counterweights C and D that swing in opposite directions. Using realistic dimensions and mass, calculate the angular momentum of this system for any angular position of the gondolas. Explain through analysis why it is a good idea to design this system to have counterweights with each gondola.

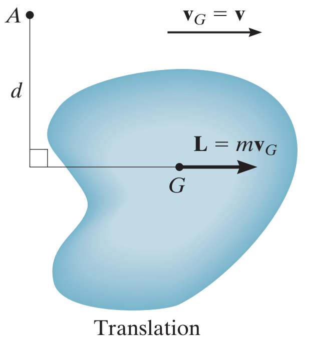
Prob. C19–4 (© R.C. Hibbeler)

CHAPTER REVIEW

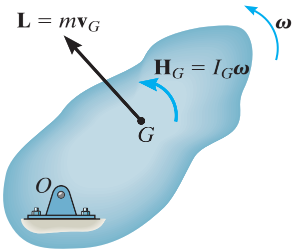
Linear and Angular Momentum

The linear and angular momentum of a rigid body can be referenced to its mass center G.

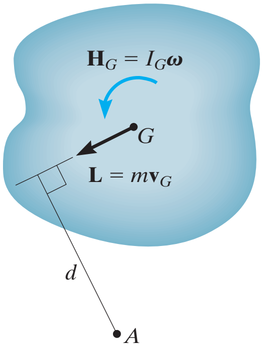
If the angular momentum is to be determined about an axis other than the one passing through the mass center, then the angular momentum is determined by summing vector $\mathbf{H}_G$ and the moment of vector $\mathbf{L}$ about this axis.



Translation



Rotation about a fixed axis



General plane motion

$L = mv_G$	$L = mv_G$	$L = mv_G$
$H_G = 0$	$H_G = I_G\omega$	$H_G = I_G\omega$
$H_A = (mv_G)d$	$H_O = I_O\omega$	$H_A = I_G\omega + (mv_G)d$

Principle of Impulse and Momentum

The principles of linear and angular impulse and momentum are used to solve problems that involve force, velocity, and time. Before applying these equations, it is important to establish the x, y, z inertial coordinate system. The free-body diagram for the body should also be drawn in order to account for all of the forces and couple moments that produce impulses on the body.

$$m(v_{Gx})_1 + \Sigma \int_{t_1}^{t_2} F_x \, dt = m(v_{Gx})_2$$

$$m(v_{Gy})_1 + \Sigma \int_{t_1}^{t_2} F_y \, dt = m(v_{Gy})_2$$

$$I_G\omega_1 + \Sigma \int_{t_1}^{t_2} M_G \, dt = I_G\omega_2$$

Conservation of Momentum

Provided the sum of the linear impulses acting on a system of connected rigid bodies is zero in a particular direction, then the linear momentum for the system is conserved in this direction. Conservation of angular momentum occurs if the impulses pass through an axis or are parallel to it. Momentum is also conserved if the external forces are small and thereby create nonimpulsive forces on the system. A free-body diagram should accompany any application in order to classify the forces as impulsive or nonimpulsive and to determine an axis about which the angular momentum may be conserved.

$$\left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_1 = \left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_2$$

$$\left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O1} = \left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O2}$$

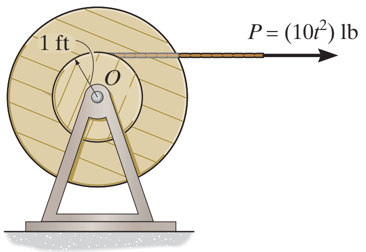
Eccentric Impact

If the line of impact does not coincide with the line connecting the mass centers of two colliding bodies, then eccentric impact will occur. If the motion of the bodies just after the impact is to be determined, then it is necessary to consider a conservation of momentum equation for the system and use the coefficient of restitution equation.

$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

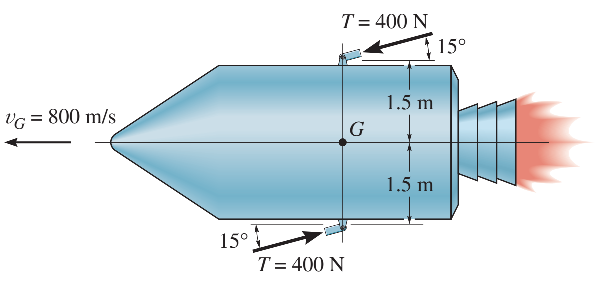
REVIEW PROBLEMS

R19–1. The cable is subjected to a force of $P = (10t^2)$ lb. where t is in seconds. Determine the angular velocity of the spool 3 s after **P** is applied, starting from rest. The spool has a weight of 150 lb and a radius of gyration of 1.25 ft about its center, O.



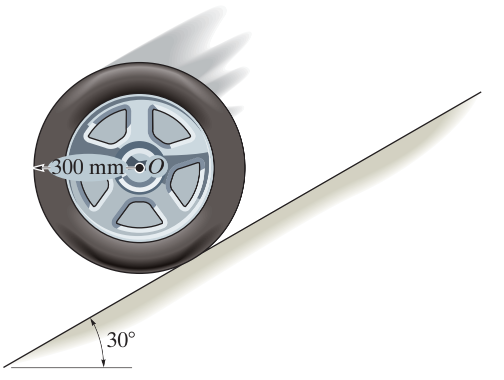
Prob. R19–1

R19–2. The space capsule has a mass of 1200 kg and a moment of inertia $I_G = 900\ \text{kg}\cdot\text{m}^2$ about an axis passing through G and directed perpendicular to the page. If it is traveling forward with a speed $v_G = 800$ m/s and executes a turn by means of two jets, which provide a constant thrust of 400 N for 0.3 s, determine the capsule's angular velocity just after the jets are turned off.



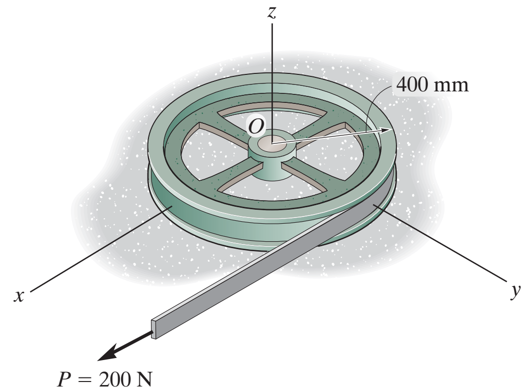
Prob. R19–2

R19–3. The tire has a mass of 9 kg and a radius of gyration $k_O = 225$ mm. If it is released from rest and rolls down the plane without slipping, determine the speed of its center O when $t = 3$ s.



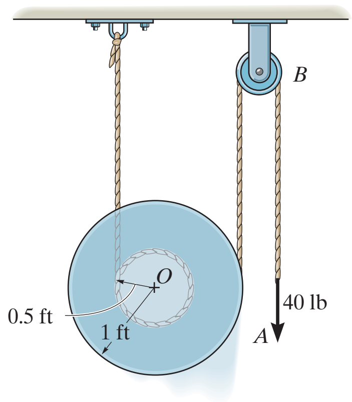
Prob. R19–3

R19–4. The wheel having a mass of 100 kg and a radius of gyration about the z axis of $k_z = 300$ mm, rests on the smooth horizontal plane. If the belt is subjected to a force of $P = 200$ N, determine the angular velocity of the wheel and the speed of its center of mass O, three seconds after the force is applied.



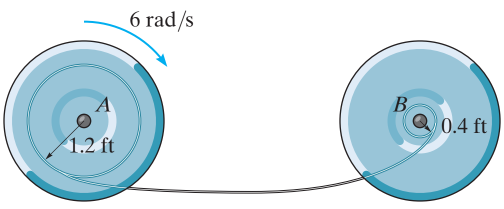
Prob. R19–4

R19–5. The spool has a weight of 30 lb and a radius of gyration $k_O = 0.65$ ft. If a force of 40 lb is applied to the cord at A, determine the angular velocity of the spool in $t = 3$ s starting from rest. Neglect the mass of the pulley and cord.



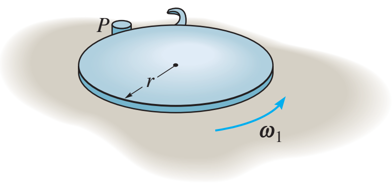
Prob. R19–5

R19–6. Spool B is at rest and spool A is rotating at 6 rad/s when the slack in the cord connecting them is taken up. If the cord does not stretch, determine the angular velocity of each spool immediately after the cord is jerked tight. The spools A and B have weights and radii of gyration $W_A = 30$ lb, $k_A = 0.8$ ft, $W_B = 15$ lb, $k_B = 0.6$ ft, respectively.



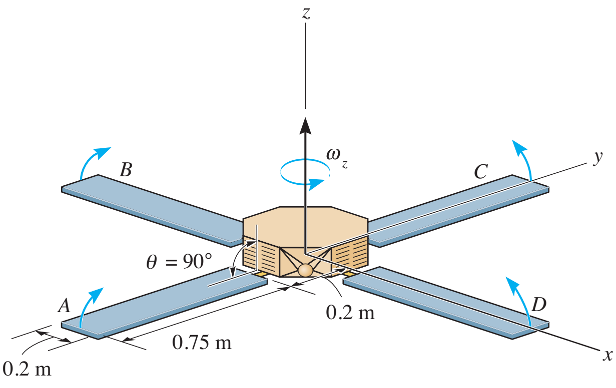
Prob. R19–6

R19–7. A thin disk of mass m has an angular velocity $\boldsymbol{\omega}_1$ while rotating on a smooth surface. Determine its new angular velocity just after the hook at its edge strikes the peg P and the disk starts to rotate about P without rebounding.



Prob. R19–7

R19–8. The space satellite has a mass of 125 kg and a moment of inertia $I_z = 0.940\ \text{kg}\cdot\text{m}^2$, excluding the four solar panels A, B, C, and D. Each solar panel has a mass of 20 kg and can be approximated as a thin plate. If the satellite is originally spinning about the z axis at a constant rate $\omega_z = 0.5$ rad/s when $\theta = 90°$, determine the rate of spin if all the panels are raised and reach the upward position, $\theta = 0°$, at the same instant.



Prob. R19–8

Chapter 20

(© Philippe Psaila/Science Source)

Design of industrial robots requires knowing the kinematics of their three-dimensional motions.

Three-Dimensional Kinematics of a Rigid Body

CHAPTER OBJECTIVES

- To analyze the kinematics of a body subjected to rotation about a fixed point and to general plane motion.
- To provide a relative-motion analysis of a rigid body using translating and rotating axes.

20.1 Rotation About a Fixed Point

When a rigid body rotates about a fixed point, the distance r from the point to a particle located on the body is the *same* for *any position* of the body. Thus, the path of motion for the particle lies on the *surface of a sphere* having a radius r and centered at the fixed point. Since motion along this path occurs only from a series of rotations made during a finite time interval, we will first develop a familiarity with some of the properties of rotational displacements.

The boom can rotate up and down, and because it is hinged at a point on the vertical axis about which it turns, it is subjected to rotation about a fixed point. (© R.C. Hibbeler)

Euler's Theorem. Euler's theorem states that two "component" rotations about different axes passing through a point are equivalent to a single resultant rotation about an axis passing through the point. If more than two rotations are applied, they can be combined into pairs, and each pair can be further reduced and combined into one rotation.

Finite Rotations. If component rotations used in Euler's theorem are *finite*, it is important that the *order* in which they are applied be maintained. To show this, consider the two finite rotations $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$ applied to the block in Fig. 20–1*a*. Each rotation has a magnitude of 90° and a direction defined by the right-hand rule, as indicated by the arrow. The final position of the block is shown at the right. When these two rotations are applied in the order $\boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$, as shown in Fig. 20–1*b*, the final position of the block is *not* the same as it is in Fig. 20–1*a*. Because *finite rotations* do not obey the commutative law of addition ($\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$), *they cannot be classified as vectors*. If smaller, yet finite, rotations had been used to illustrate this point, e.g., 10° instead of 90°, the *final position* of the block after each combination of rotations would also be different; however, in this case, the difference is only a small amount.

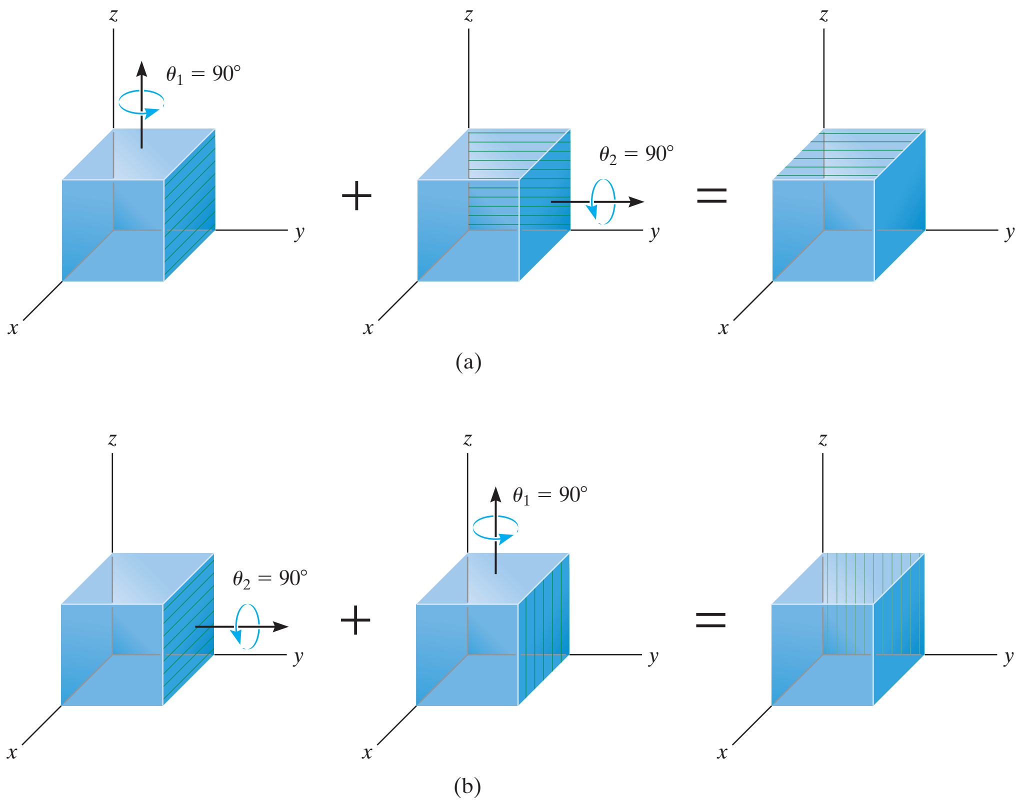


Fig. 20–1

20

Infinitesimal Rotations. When defining the angular motions of a body subjected to three-dimensional motion, only rotations which are *infinitesimally small* will be considered. *Such rotations can be classified as vectors, since they can be added vectorially in any manner.* To show this, for purposes of simplicity let us consider the rigid body itself to be a sphere which is allowed to rotate about its central fixed point O, Fig. 20–2*a*. If we impose two infinitesimal rotations $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$ on the body, it is seen that point P moves along the path $d\boldsymbol{\theta}_1 \times \mathbf{r} + d\boldsymbol{\theta}_2 \times \mathbf{r}$ and ends up at P'. Had the two successive rotations occurred in the order $d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$, then the resultant displacements of P would have been $d\boldsymbol{\theta}_2 \times \mathbf{r} + d\boldsymbol{\theta}_1 \times \mathbf{r}$. Since the vector cross product obeys the distributive law, by comparison $(d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2) \times \mathbf{r} = (d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1) \times \mathbf{r}$. Here infinitesimal rotations $d\boldsymbol{\theta}$ are vectors, since these quantities have both a magnitude and direction for which the order of (vector) addition is not important, i.e., $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2 = d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$. As a result, as shown in Fig. 20–2*a*, the two "component" rotations $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$ are equivalent to a single resultant rotation $d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$, a consequence of Euler's theorem.

Angular Velocity. If the body is subjected to an angular rotation $d\boldsymbol{\theta}$ about a fixed point, the angular velocity of the body is defined by the time derivative,

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} \tag{20–1}$$

The line specifying the direction of $\boldsymbol{\omega}$, which is collinear with $d\boldsymbol{\theta}$, is referred to as the *instantaneous axis of rotation*, Fig. 20–2*b*. In general, this axis changes direction during each instant of time. Since $d\boldsymbol{\theta}$ is a vector quantity, so too is $\boldsymbol{\omega}$, and it follows from vector addition that if the body is subjected to two component angular motions, $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$, the resultant angular velocity is $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

Angular Acceleration. The body's angular acceleration is determined from the time derivative of its angular velocity, i.e.,

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} \tag{20–2}$$

For motion about a fixed point, $\boldsymbol{\alpha}$ must account for a change in *both* the magnitude and direction of $\boldsymbol{\omega}$, so that, in general, $\boldsymbol{\alpha}$ is not directed along the instantaneous axis of rotation, Fig. 20–3.

As the direction of the instantaneous axis of rotation (or the line of action of $\boldsymbol{\omega}$) changes in space, the locus of the axis generates a fixed *space cone*, Fig. 20–4. If the change in the direction of this axis is viewed with respect to the rotating body, the locus of the axis generates a *body cone*.

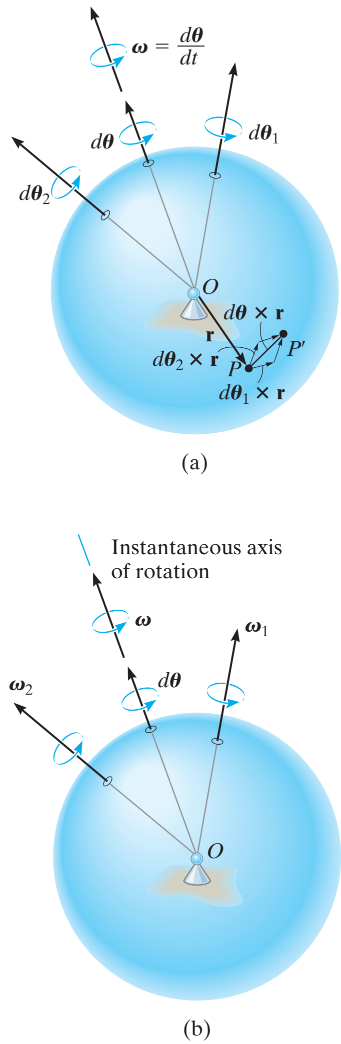


Fig. 20–2

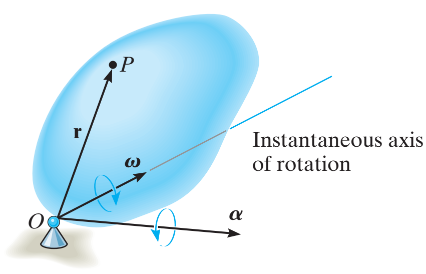


Fig. 20–3

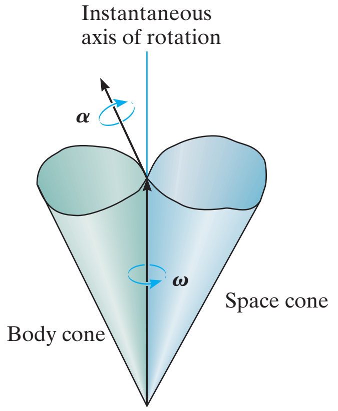
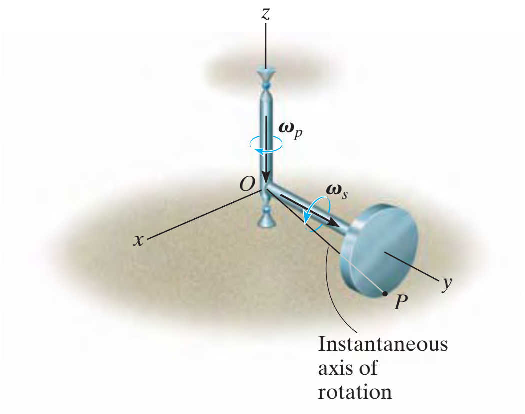
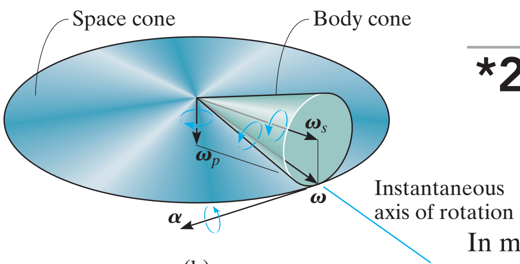


Fig. 20–4



(a)



(b)

Fig. 20–5

At any given instant, these cones meet along the instantaneous axis of rotation, and when the body is in motion, the body cone appears to roll either on the inside or the outside surface of the fixed space cone. Provided the paths defined by the open ends of the cones are described by the head of the $\boldsymbol{\omega}$ vector, then $\boldsymbol{\alpha}$ must act tangent to these paths at any given instant, since the time rate of change of $\boldsymbol{\omega}$ is equal to $\boldsymbol{\alpha}$. Fig. 20–4.

To illustrate this concept, consider the disk in Fig. 20–5*a* that spins about the rod at $\boldsymbol{\omega}_s$, while the rod and disk precess about the vertical axis at $\boldsymbol{\omega}_p$. The resultant angular velocity of the disk is therefore $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$. Since both point O and the contact point P have zero velocity, then all points on a line between these points must have zero velocity. Thus, both $\boldsymbol{\omega}$ and the instantaneous axis of rotation are along OP. Therefore, as the disk rotates, this axis appears to move along the surface of the fixed space cone shown in Fig. 20–5*b*. If the axis is observed from the rotating disk, the axis then appears to move on the surface of the body cone. At any instant, though, these two cones meet each other along the axis OP. If $\boldsymbol{\omega}$ has a constant magnitude, then $\boldsymbol{\alpha}$ indicates only the change in the direction of $\boldsymbol{\omega}$, which is tangent to the cones at the tip of $\boldsymbol{\omega}$ as shown in Fig. 20–5*b*.

Velocity. Once $\boldsymbol{\omega}$ is specified, the velocity of any point on a body rotating about a fixed point can be determined using the same methods as for a body rotating about a fixed axis. Hence, by the cross product,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{20–3}$$

Here $\mathbf{r}$ defines the position of the point measured from the fixed point O, Fig. 20–3.

Acceleration. If $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known at a given instant, the acceleration of a point can be obtained from the time derivative of Eq. 20–3, which yields

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{20–4}$$

*20.2 The Time Derivative of a Vector Measured from Either a Fixed or Translating-Rotating System

In many types of problems involving the motion of a body about a fixed point, the angular velocity $\boldsymbol{\omega}$ is specified in terms of its components. Then, if the angular acceleration $\boldsymbol{\alpha}$ of such a body is to be determined, it is often easier to compute the time derivative of $\boldsymbol{\omega}$ using a coordinate system that has a *rotation* defined by one or more of the components of $\boldsymbol{\omega}$. For example, in the case of the disk in Fig. 20–5*a*, where $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$, the x, y, z axes can be given an angular velocity of $\boldsymbol{\omega}_p$. For this reason, and for other uses later, an equation will now be derived, which relates the time derivative of any vector $\mathbf{A}$ defined from a translating-rotating reference to its time derivative defined from a fixed reference.

Consider the *x, y, z* axes of the moving frame of reference to be rotating with an angular velocity $\mathbf{\Omega}$, which is measured from the fixed *X, Y, Z* axes, Fig. 20–6*a*. In the following discussion, it will be convenient to express vector **A** in terms of its **i**, **j**, **k** components, which define the directions of the moving axes. Hence,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

In general, the time derivative of **A** must account for the change in both its magnitude and direction. However, if this derivative is taken *with respect to the moving frame of reference*, only the change in the magnitudes of the components of **A** must be accounted for, since the directions of the components do not change with respect to the moving reference. Hence,

$$(\dot{\mathbf{A}})_{xyz} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} \tag{20–5}$$

When the time derivative of **A** is taken *with respect to the fixed frame of reference*, the *directions* of **i**, **j**, and **k** change only on account of the *rotation* $\mathbf{\Omega}$ of the axes and not their translation. Hence, in general,

$$\dot{\mathbf{A}} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} + A_x\dot{\mathbf{i}} + A_y\dot{\mathbf{j}} + A_z\dot{\mathbf{k}}$$

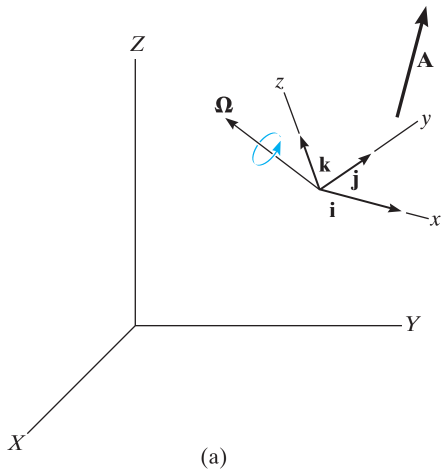
The time derivatives of the unit vectors will now be considered. For example, $\dot{\mathbf{i}} = d\mathbf{i}/dt$ represents only the change in the *direction* of **i** with respect to time, since **i** always has a magnitude of 1 unit. As shown in Fig. 20–6*b*, the change, $d\mathbf{i}$, is *tangent to the path* described by the arrowhead of **i** as **i** swings due to the rotation $\mathbf{\Omega}$. Accounting for both the magnitude and direction of $d\mathbf{i}$, we can therefore define $\dot{\mathbf{i}}$ using the cross product, $\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i}$. In general, then

$$\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \mathbf{\Omega} \times \mathbf{j} \qquad \dot{\mathbf{k}} = \mathbf{\Omega} \times \mathbf{k}$$

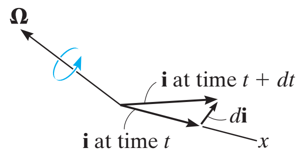
These formulations were also developed in Sec. 16.8, regarding planar motion of the axes. Substituting these results into the above equation and using Eq. 20–5 yields

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \mathbf{\Omega} \times \mathbf{A} \tag{20–6}$$

This result is important, and will be used throughout Sec. 20.4 and Chapter 21. It states that the time derivative of *any vector* **A** as observed from the fixed *X, Y, Z* frame of reference is equal to the time rate of change of **A** as observed from the *x, y, z* translating-rotating frame of reference, Eq. 20–5, plus $\mathbf{\Omega} \times \mathbf{A}$, the change of **A** caused by the rotation of the *x, y, z* frame. As a result, Eq. 20–6 should always be used whenever $\mathbf{\Omega}$ produces a change in the direction of **A** as seen from the *X, Y, Z* reference. If this change does not occur, i.e., $\mathbf{\Omega} = \mathbf{0}$, then $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz}$, and so the time rate of change of **A** as observed from both coordinate systems will be the *same*.



(a)



(b)

Fig. 20–6

EXAMPLE 20.1

The disk shown in Fig. 20–7 spins about its axle with a constant angular velocity $\omega_s = 3$ rad/s, while the horizontal platform on which the disk is mounted rotates about the vertical axis at a constant rate $\omega_p = 1$ rad/s. Determine the angular acceleration of the disk and the velocity and acceleration of point A on the disk when it is in the position shown.

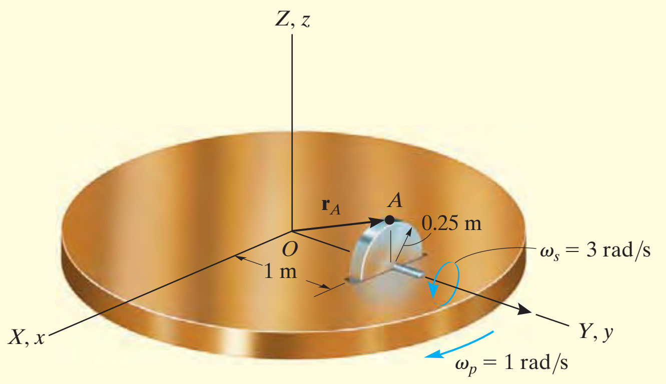


Fig. 20–7

SOLUTION

Point O represents a fixed point of rotation for the disk if one considers a hypothetical extension of the disk to this point. To determine the velocity and acceleration of point A, it is first necessary to determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the disk, since these vectors are used in Eqs. 20–3 and 20–4.

Angular Velocity. The angular velocity, which is measured from X, Y, Z, is simply the vector addition of its two component motions. Thus,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \{3\mathbf{j} - 1\mathbf{k}\} \text{ rad/s}$$

Angular Acceleration. Since the magnitude of $\boldsymbol{\omega}$ is constant, only a change in its direction, as seen from the fixed reference, creates the angular acceleration $\boldsymbol{\alpha}$ of the disk. One way to obtain $\boldsymbol{\alpha}$ is to compute the time derivative of *each of the two components* of $\boldsymbol{\omega}$ using Eq. 20–6. At the instant shown in Fig. 20–7, imagine the fixed X, Y, Z and a rotating x, y, z frame to be coincident. If the rotating x, y, z frame is chosen to have an angular velocity of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{-1\mathbf{k}\}$ rad/s, then $\boldsymbol{\omega}_s$ will *always* be directed along the y (not Y) axis, and the time rate of change of $\boldsymbol{\omega}_s$ *as seen from x, y, z is zero*; i.e., $(\dot{\boldsymbol{\omega}}_s)_{xyz} = \mathbf{0}$ (the magnitude and direction of $\boldsymbol{\omega}_s$ is constant). Thus,

$$\dot{\boldsymbol{\omega}}_s = (\dot{\boldsymbol{\omega}}_s)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s = \mathbf{0} + (-1\mathbf{k}) \times (3\mathbf{j}) = \{3\mathbf{i}\} \text{ rad/s}^2$$

By the same choice of axes rotation, $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$, or even with $\boldsymbol{\Omega} = \mathbf{0}$, the time derivative $(\dot{\boldsymbol{\omega}}_p)_{xyz} = \mathbf{0}$, since $\boldsymbol{\omega}_p$ has a constant magnitude and direction with respect to x, y, z. Hence,

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

The angular acceleration of the disk is therefore

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p = \{3\mathbf{i}\} \text{ rad/s}^2 \qquad \textit{Ans.}$$

Velocity and Acceleration. Since $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ have now been determined, the velocity and acceleration of point A can be found using Eqs. 20–3 and 20–4. Realizing that $\mathbf{r}_A = \{1\mathbf{j} + 0.25\mathbf{k}\}$ m, Fig. 20–7, we have

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A = (3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k}) = \{1.75\mathbf{i}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\begin{aligned}\mathbf{a}_A &= \boldsymbol{\alpha} \times \mathbf{r}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_A)\\ &= (3\mathbf{i}) \times (1\mathbf{j} + 0.25\mathbf{k}) + (3\mathbf{j} - 1\mathbf{k}) \times [(3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k})]\\ &= \{-2.50\mathbf{j} - 2.25\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

20

EXAMPLE 20.2

At the instant $\theta = 60°$, the gyrotop in Fig. 20–8 has three components of angular motion directed as shown and having magnitudes defined as:

Spin: $\omega_s = 10$ rad/s, increasing at the rate of 6 rad/s^2

Nutation: $\omega_n = 3$ rad/s, increasing at the rate of 2 rad/s^2

Precession: $\omega_p = 5$ rad/s, increasing at the rate of 4 rad/s^2

Determine the angular velocity and angular acceleration of the top.

SOLUTION

Angular Velocity. The top rotates about the fixed point O. If the fixed and rotating frames are coincident at the instant shown, then the angular velocity can be expressed in terms of **i**, **j**, **k** components, with reference to the x, y, z frame; i.e.,

$$\boldsymbol{\omega} = -\omega_n\mathbf{i} + \omega_s \sin\theta\mathbf{j} + (\omega_p + \omega_s\cos\theta)\mathbf{k}$$
$$= -3\mathbf{i} + 10\sin 60°\mathbf{j} + (5 + 10\cos 60°)\mathbf{k}$$
$$= \{-3\mathbf{i} + 8.66\mathbf{j} + 10\mathbf{k}\}\text{ rad/s} \qquad \textit{Ans.}$$

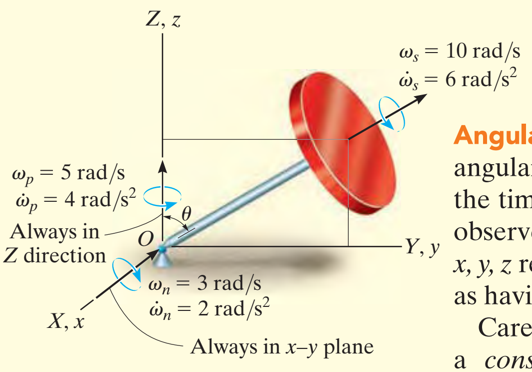


Fig. 20–8

Angular Acceleration. As in the solution of Example 20.1, the angular acceleration $\boldsymbol{\alpha}$ will be determined by investigating separately the time rate of change of *each of the angular velocity components* as observed from the fixed X, Y, Z reference. We will choose an $\boldsymbol{\Omega}$ for the x, y, z reference so that the component of $\boldsymbol{\omega}$ being considered is viewed as having a *constant direction* when observed from x, y, z.

Careful examination of the motion of the top reveals that $\boldsymbol{\omega}_s$ has a *constant direction* relative to x, y, z if these axes rotate at $\boldsymbol{\Omega} = \boldsymbol{\omega}_n + \boldsymbol{\omega}_p$. Thus,

$$\dot{\boldsymbol{\omega}}_s = (\dot{\boldsymbol{\omega}}_s)_{xyz} + (\boldsymbol{\omega}_n + \boldsymbol{\omega}_p) \times \boldsymbol{\omega}_s$$
$$= (6\sin 60°\mathbf{j} + 6\cos 60°\mathbf{k}) + (-3\mathbf{i} + 5\mathbf{k}) \times (10\sin 60°\mathbf{j} + 10\cos 60°\mathbf{k})$$
$$= \{-43.30\mathbf{i} + 20.20\mathbf{j} - 22.98\mathbf{k}\}\text{ rad/s}^2$$

Since $\boldsymbol{\omega}_n$ *always* lies in the fixed X–Y plane, this vector has a *constant direction* if the motion is viewed from axes x, y, z having a rotation of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$ (not $\boldsymbol{\Omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$). Thus,

$$\dot{\boldsymbol{\omega}}_n = (\dot{\boldsymbol{\omega}}_n)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_n = -2\mathbf{i} + (5\mathbf{k}) \times (-3\mathbf{i}) = \{-2\mathbf{i} - 15\mathbf{j}\}\text{ rad/s}^2$$

Finally, the component $\boldsymbol{\omega}_p$ is *always directed* along the Z axis so that here it is not necessary to think of x, y, z as rotating, i.e., $\boldsymbol{\Omega} = \mathbf{0}$. Expressing the data in terms of the **i**, **j**, **k** components, we therefore have

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \mathbf{0} \times \boldsymbol{\omega}_p = \{4\mathbf{k}\}\text{ rad/s}^2$$

Thus, the angular acceleration of the top is

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_n + \dot{\boldsymbol{\omega}}_p = \{-45.3\mathbf{i} + 5.20\mathbf{j} - 19.0\mathbf{k}\}\text{ rad/s}^2 \qquad \textit{Ans.}$$

20

20.3 General Motion

Shown in Fig. 20–9 is a rigid body subjected to general motion in three dimensions for which the angular velocity is $\boldsymbol{\omega}$ and the angular acceleration is $\boldsymbol{\alpha}$. If point A has a known motion of $\mathbf{v}_A$ and $\mathbf{a}_A$, the motion of any other point B can be determined by using a relative-motion analysis. In this section a *translating coordinate system* will be used to define the relative motion, and in the next section a reference that is both rotating and translating will be considered.

If the origin of the translating coordinate system x, y, z ($\boldsymbol{\Omega} = \mathbf{0}$) is located at the "base point" A, then, at the instant shown, the motion of the body can be regarded as the sum of an instantaneous translation of the body having a motion of $\mathbf{v}_A$, and $\mathbf{a}_A$, and a rotation of the body about an instantaneous axis passing through point A. Since the body is rigid, the motion of point B measured by an observer located at A is therefore the same as *the rotation of the body about a fixed point*. This relative motion occurs about the instantaneous axis of rotation and is defined by $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 20–3, and $\mathbf{a}_{B/A} = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$, Eq. 20–4. For translating axes, the relative motions are related to absolute motions by $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ and $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, Eqs. 16–15 and 16–17, so that the absolute velocity and acceleration of point B can be determined from the equations

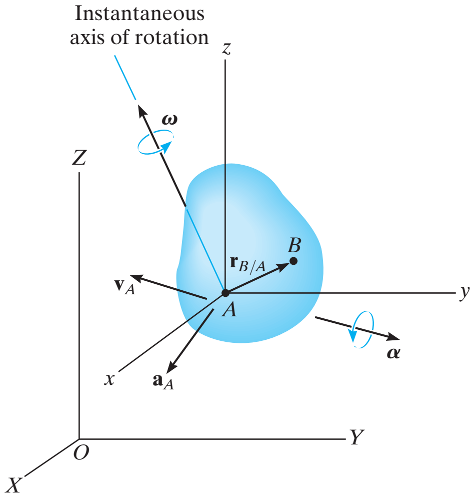


Fig. 20–9

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \qquad (20\text{–}7)$$

and

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \qquad (20\text{–}8)$$

These two equations are essentially the same as those describing the general plane motion of a rigid body, Eqs. 16–16 and 16–18. However, difficulty in application arises for three-dimensional motion, because $\boldsymbol{\alpha}$ now measures the change in *both* the magnitude and direction of $\boldsymbol{\omega}$.

Although this may be the case, a direct solution for $\mathbf{v}_B$ and $\mathbf{a}_B$ can be obtained by noting that $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$, and so Eq. 20–7 becomes $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. The cross product indicates that $\mathbf{v}_{B/A}$ is *perpendicular* to $\mathbf{r}_{B/A}$, and so, as noted by Eq. C–14 of Appendix C, we require

$$\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0 \qquad (20\text{–}9)$$

Taking the time derivative, we have

$$\mathbf{v}_{B/A} \cdot \mathbf{v}_{B/A} + \mathbf{r}_{B/A} \cdot \mathbf{a}_{B/A} = 0 \qquad (20\text{–}10)$$

Solution II of the following example illustrates application of this idea.

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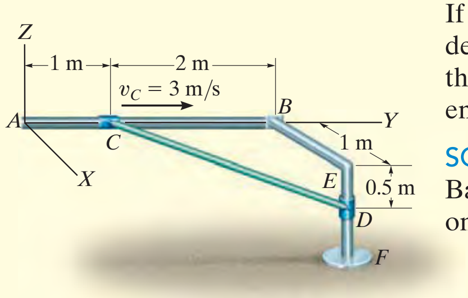
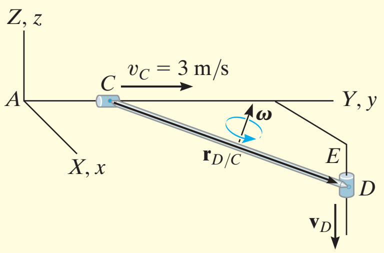
EXAMPLE 20.3



(a)



(b)

Fig. 20–10

If the collar at C in Fig. 20–10a moves toward B with a speed of 3 m/s, determine the velocity of the collar at D and the angular velocity of the bar at the instant shown. The bar is connected to the collars at its end points by ball-and-socket joints.

SOLUTION I

Bar CD is subjected to general motion. Why? The velocity of point D on the bar can be related to the velocity of point C by the equation

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{D/C}$$

The fixed and translating frames of reference are assumed to coincide at the instant considered, Fig. 20–10b. We have

$$\mathbf{v}_D = -v_D\mathbf{k} \qquad \mathbf{v}_C = \{3\mathbf{j}\} \text{ m/s}$$

$$\mathbf{r}_{D/C} = \{1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}\} \text{ m} \qquad \boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Substituting into the above equation we get

$$-v_D\mathbf{k} = 3\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 2 & -0.5 \end{vmatrix}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

$$-0.5\omega_y - 2\omega_z = 0 \qquad (1)$$

$$0.5\omega_x + 1\omega_z + 3 = 0 \qquad (2)$$

$$2\omega_x - 1\omega_y + v_D = 0 \qquad (3)$$

These equations contain four unknowns.* A fourth equation can be written if the direction of $\boldsymbol{\omega}$ is specified. In particular, any component of $\boldsymbol{\omega}$ acting along the bar's axis has no effect on moving the collars. This is because the bar is *free to rotate* about its axis. Therefore, if $\boldsymbol{\omega}$ is specified as acting *perpendicular* to the axis of the bar, then $\boldsymbol{\omega}$ must have a *unique magnitude* to satisfy the above equations. Hence,

$$\boldsymbol{\omega} \cdot \mathbf{r}_{D/C} = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \cdot (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) = 0$$

$$1\omega_x + 2\omega_y - 0.5\omega_z = 0 \qquad (4)$$

*Although this is the case, the magnitude of $\mathbf{v}_D$ can be obtained. For example, solve Eqs. 1 and 2 for ω_y and ω_x in terms of ω_z and substitute this into Eq. 3. Then ω_z will cancel out, which will allow a solution for v_D.

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Solving Eqs. 1 through 4 simultaneously yields

$$\omega_x = -4.86 \text{ rad/s} \quad \omega_y = 2.29 \text{ rad/s} \quad \omega_z = -0.571 \text{ rad/s},$$
$$v_D = 12.0 \text{ m/s, so that} \qquad \omega = 5.40 \text{ rad/s} \qquad \textit{Ans.}$$

SOLUTION II

Applying Eq. 20–9, $\mathbf{v}_{D/C} = \mathbf{v}_D - \mathbf{v}_C = -v_D\mathbf{k} - 3\mathbf{j}$, so that

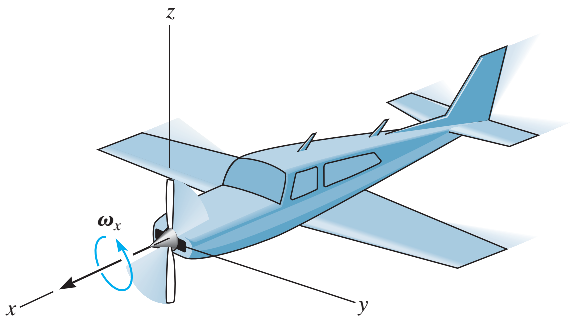
$$\mathbf{r}_{D/C} \cdot \mathbf{v}_{D/C} = (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) \cdot (-v_D\mathbf{k} - 3\mathbf{j}) = 0$$
$$(1)(0) + (2)(-3) + (-0.5)(-v_D) = 0$$
$$v_D = 12 \text{ m/s} \qquad \textit{Ans.}$$

Since $\boldsymbol{\omega}$ is *perpendicular* to $\mathbf{r}_{D/C}$ then $\mathbf{v}_{D/C} = \boldsymbol{\omega} \times \mathbf{r}_{D/C}$ or

$$v_{D/C} = \omega\, r_{D/C}$$
$$\sqrt{(-12)^2 + (-3)^2} = \omega\sqrt{(1)^2 + (2)^2 + (-0.5)^2}$$
$$\omega = 5.40 \text{ rad/s} \qquad \textit{Ans.}$$

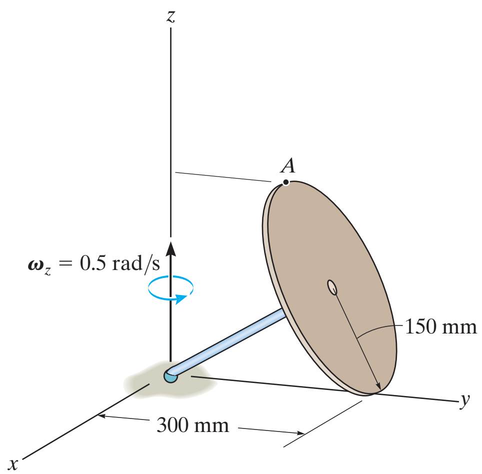
PROBLEMS

20–1. The propeller of an airplane is rotating at a constant speed $\omega_x\,\mathbf{i}$, while the plane is undergoing a turn at a constant rate ω_t. Determine the angular acceleration of the propeller if (a) the turn is horizontal, i.e., $\omega_t\mathbf{k}$, and (b) the turn is vertical, downward, i.e., $\omega_t\,\mathbf{j}$.



Prob. 20–1

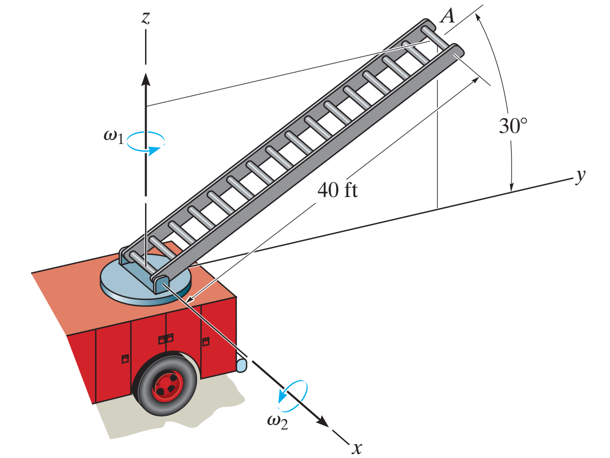
20–2. The disk rotates about the z axis at a constant rate $\omega_z = 0.5$ rad/s without slipping on the horizontal plane. Determine the velocity and the acceleration of point A on the disk.



Prob. 20–2

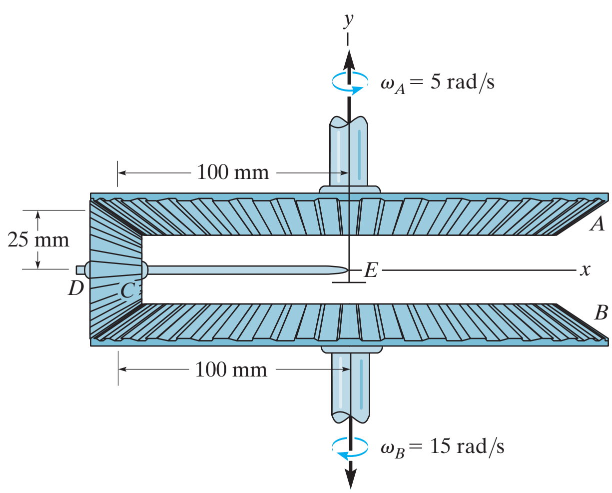
20–3. The ladder of the fire truck rotates around the z axis with an angular velocity $\omega_1 = 0.15$ rad/s, which is increasing at 0.8 rad/s^2. At the same instant it is rotating upward at a constant rate $\omega_2 = 0.6$ rad/s. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.

***20–4.** The ladder of the fire truck rotates around the z axis with an angular velocity of $\omega_1 = 0.15$ rad/s, which is increasing at 0.2 rad/s^2. At the same instant it is rotating upward at $\omega_2 = 0.6$ rad/s while increasing at 0.4 rad/s^2. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.



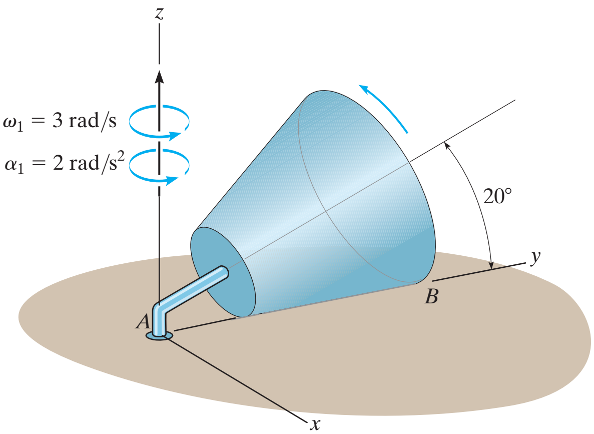
Probs. 20–3/4

20–5. If the plate gears A and B are rotating with the angular velocities shown, determine the angular velocity of gear C about the shaft DE. What is the angular velocity of DE about the y axis?



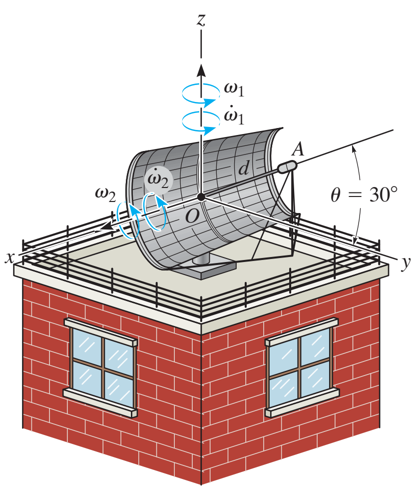
Prob. 20–5

20–6. The conical spool rolls on the plane without slipping. If the axle has an angular velocity of $\omega_1 = 3$ rad/s and an angular acceleration of $\alpha_1 = 2$ rad/s^2 at the instant shown, determine the angular velocity and angular acceleration of the spool at this instant.



Prob. 20–6

20–7. At a given instant, the antenna has an angular motion $\omega_1 = 3$ rad/s and $\dot{\omega}_1 = 2$ rad/s^2 about the z axis. At this same instant $\theta = 30°$, the angular motion about the x axis is $\omega_2 = 1.5$ rad/s, and $\dot{\omega}_2 = 4$ rad/s^2. Determine the velocity and acceleration of the signal horn A at this instant. The distance from O to A is $d = 3$ ft.



Prob. 20–7

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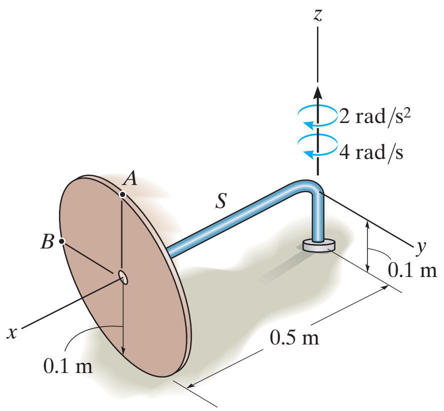
***20–8.** The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s². Determine the velocity and acceleration of point A on the disk at the instant shown. No slipping occurs.

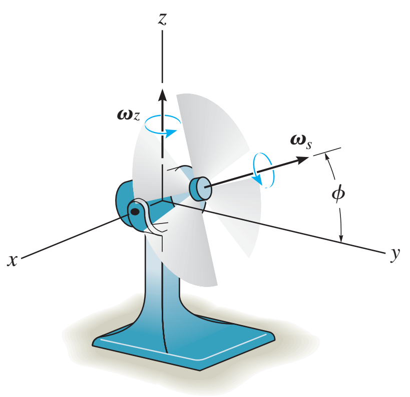
20–9. The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s². Determine the velocity and acceleration of point B on the disk at the instant shown. No slipping occurs.



Probs. 20–8/9

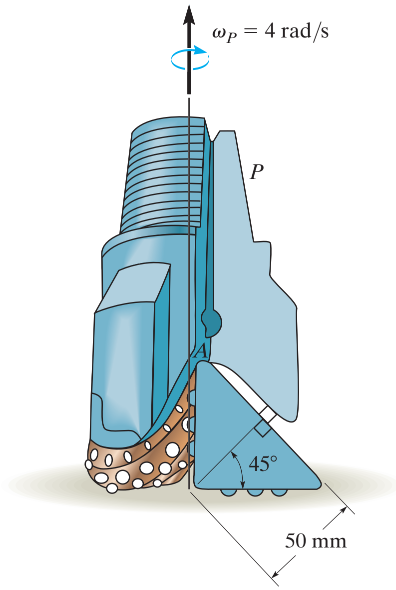
20–10. The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If $\phi = 45°$ for the motion, determine the angular velocity and the angular acceleration of the blade.

20–11. The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If at the instant $\phi = 45°$, $\dot{\phi} = 2$ rad/s for the motion, determine the angular velocity and the angular acceleration of the blade.



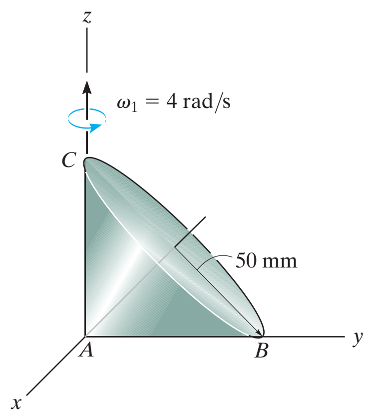
Probs. 20–10/11

***20–12.** The drill pipe P turns at a constant angular rate $\omega_P = 4$ rad/s. Determine the angular velocity and angular acceleration of the conical rock bit, which rolls without slipping. Also, what are the velocity and acceleration of point A?



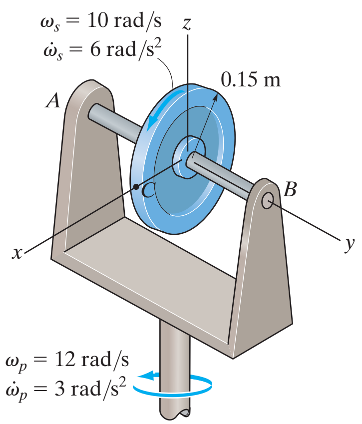
Prob. 20–12

20–13. The right circular cone rotates about the z axis at a constant rate of $\omega_1 = 4$ rad/s without slipping on the horizontal plane. Determine the magnitudes of the velocity and acceleration of points B and C.



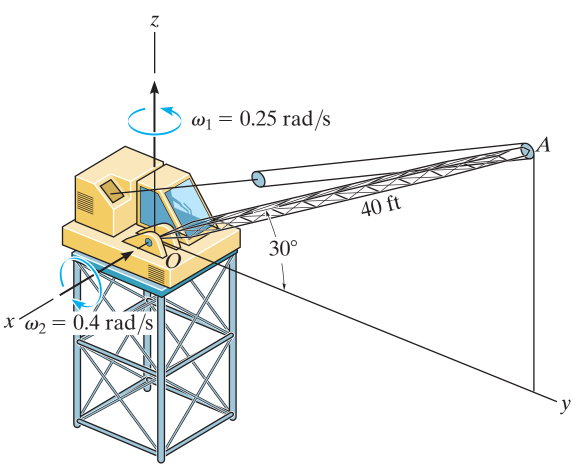
Prob. 20–13

20–14. The wheel is spinning about shaft AB with an angular velocity of $\omega_s = 10$ rad/s, which is increasing at a constant rate of $\dot{\omega}_s = 6$ rad/s^2, while the frame precesses about the z axis with an angular velocity of $\omega_p = 12$ rad/s, which is increasing at a constant rate of $\dot{\omega}_p = 3$ rad/s^2. Determine the velocity and acceleration of point C located on the rim of the wheel at this instant.



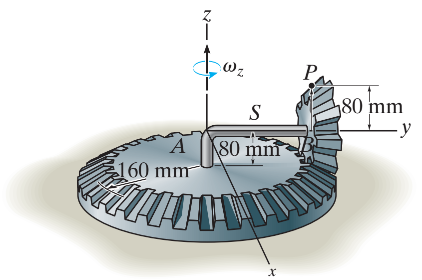
Prob. 20–14

20–15. At the instant shown, the tower crane rotates about the z axis with an angular velocity $\omega_1 = 0.25$ rad/s, which is increasing at 0.6 rad/s^2. The boom OA rotates downward with an angular velocity $\omega_2 = 0.4$ rad/s, which is increasing at 0.8 rad/s^2. Determine the velocity and acceleration of point A located at the end of the boom at this instant.



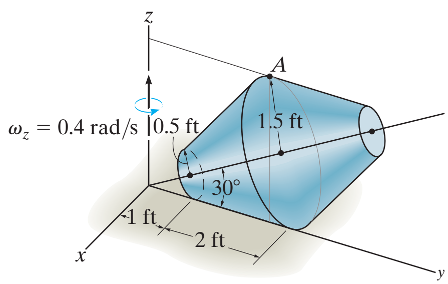
Prob. 20–15

***20–16.** Gear A is fixed while gear B is free to rotate on the shaft S. If the shaft is turning about the z axis at $\omega_z = 5$ rad/s, while increasing at 2 rad/s^2, determine the velocity and acceleration of point P at the instant shown. The face of gear B lies in a vertical plane.



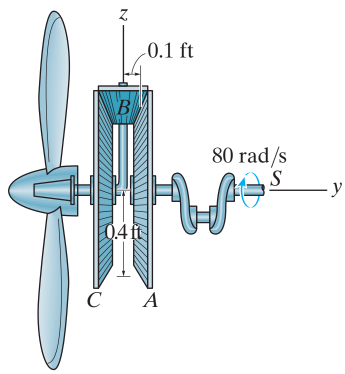
Prob. 20–16

20–17. The truncated double cone rotates about the z axis at $\omega_z = 0.4$ rad/s without slipping on the horizontal plane. If at this same instant ω_z is increasing at $\dot{\omega}_z = 0.5$ rad/s^2, determine the velocity and acceleration of point A on the cone.



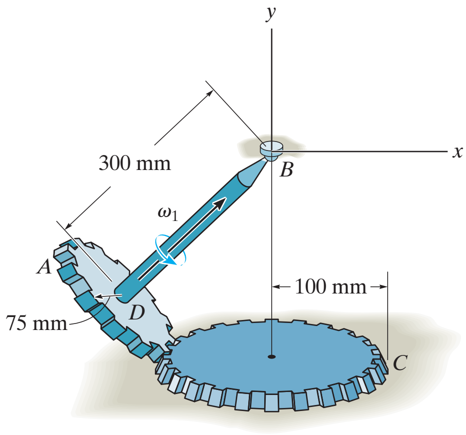
Prob. 20–17

20–18. Gear A is fixed to the crankshaft S, while gear C is fixed. Gear B and the propeller are free to rotate. The crankshaft is turning at 80 rad/s about its axis. Determine the magnitudes of the angular velocity of the propeller and the angular acceleration of gear B.



Prob. 20–18

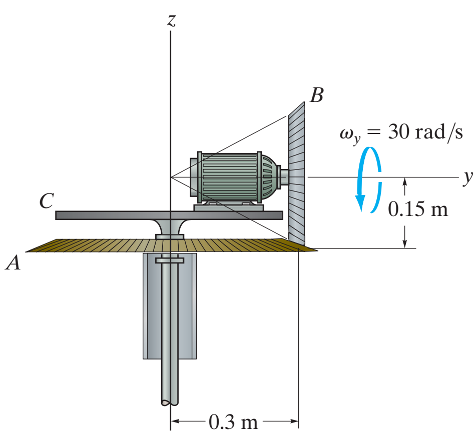
20–19. Shaft BD is connected to a ball-and-socket joint at B, and a beveled gear A is attached to its other end. The gear is in mesh with a fixed gear C. If the shaft and gear A are *spinning* with a constant angular velocity $\omega_1 = 8$ rad/s, determine the angular velocity and angular acceleration of gear A.



Prob. 20–19

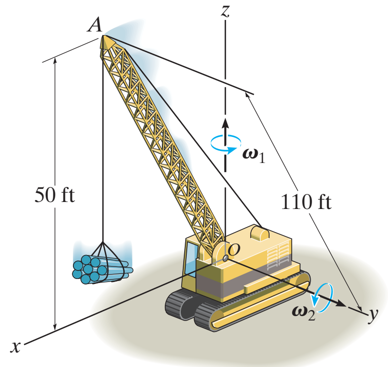
***20–20.** Gear B is driven by a motor mounted on turntable C. If gear A is held fixed, and the motor shaft rotates with a constant angular velocity of $\omega_y = 30$ rad/s, determine the angular velocity and angular acceleration of gear B.

20–21. Gear B is driven by a motor mounted on turntable C. If gear A and the motor shaft rotate with constant angular speeds of $\omega_A = \{10\mathbf{k}\}$ rad/s and $\omega_y = \{30\mathbf{j}\}$ rad/s, respectively, determine the angular velocity and angular acceleration of gear B.



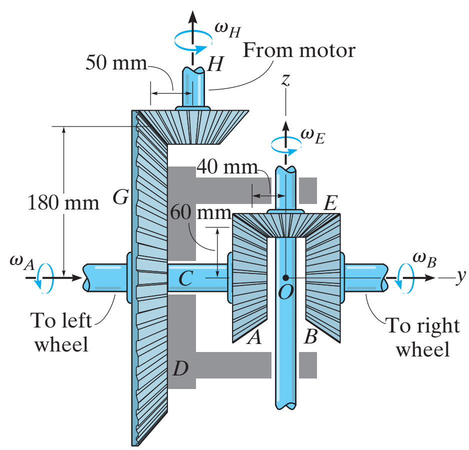
Probs. 20–20/21

20–22. The crane boom OA rotates about the z axis with a constant angular velocity of $\omega_1 = 0.15$ rad/s, while it is rotating downward with a constant angular velocity of $\omega_2 = 0.2$ rad/s. Determine the velocity and acceleration of point A located at the end of the boom at the instant shown.



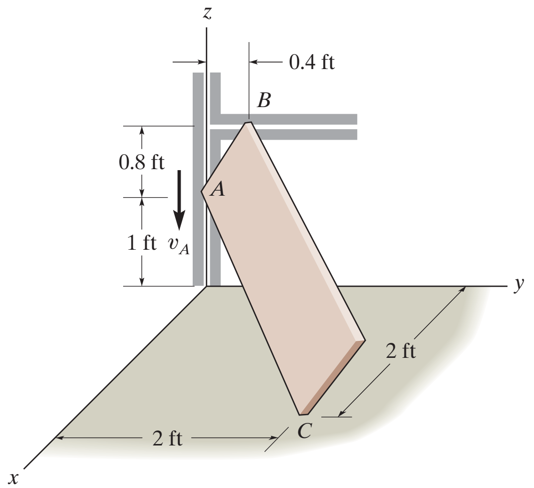
Prob. 20–22

20–23. The differential of an automobile allows the two rear wheels to rotate at different speeds when the automobile travels along a curve. For operation, the rear axles are attached to the wheels at one end and have beveled gears A and B on their other ends. The differential case D is placed over the left axle but can rotate about C independent of the axle. The case supports a pinion gear E on a shaft, which meshes with gears A and B. Finally, a ring gear G is *fixed* to the differential case so that the case rotates with the ring gear when the latter is driven by the drive pinion H. This gear, like the differential case, is free to rotate about the left wheel axle. If the drive pinion is turning at $\omega_H = 100$ rad/s and the pinion gear E is spinning about its shaft at $\omega_E = 30$ rad/s, determine the angular velocity, ω_A and ω_B, of each axle.



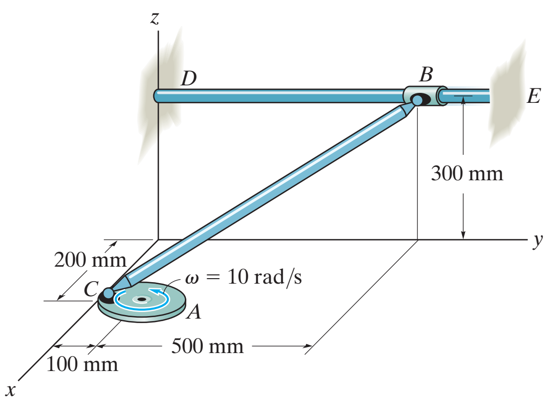
Prob. 20–23

***20–24.** The end C of the plate rests on the horizontal plane, while end points A and B are restricted to move along the grooved slots. If at the instant shown A is moving downward with a constant velocity of $v_A = 4$ ft/s, determine the angular velocity of the plate and the velocities of points B and C.



Prob. 20–24

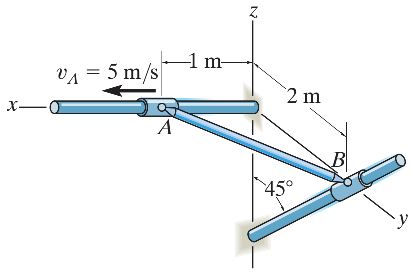
20–25. Disk A rotates at a constant angular velocity of 10 rad/s. If rod BC is joined to the disk and a collar by ball-and-socket joints, determine the velocity of collar B at the instant shown. Also, what is the rod's angular velocity $\boldsymbol{\omega}_{BC}$ if it is directed perpendicular to the axis of the rod?



Prob. 20–25

20–26. Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod at $v_A = 5$ m/s, determine the angular velocity of the rod and the velocity of collar B at the instant shown. Assume that the rod's angular velocity is directed perpendicular to the axis of the rod.

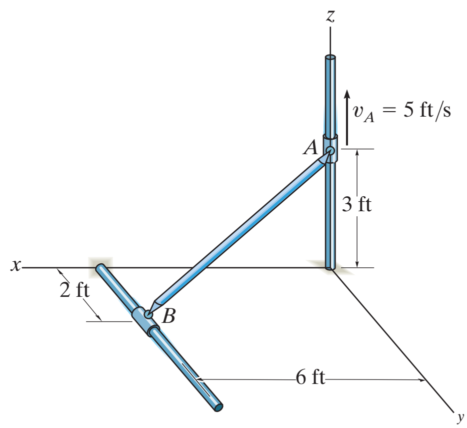
20–27. Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod with a velocity of $v_A = 5$ m/s and has an acceleration $a_A = 2$ m/s^2 at the instant shown, determine the angular acceleration of the rod and the acceleration of collar B at this instant. Assume that the rod's angular velocity and angular acceleration are directed perpendicular to the axis of the rod.



Probs. 20–26/27

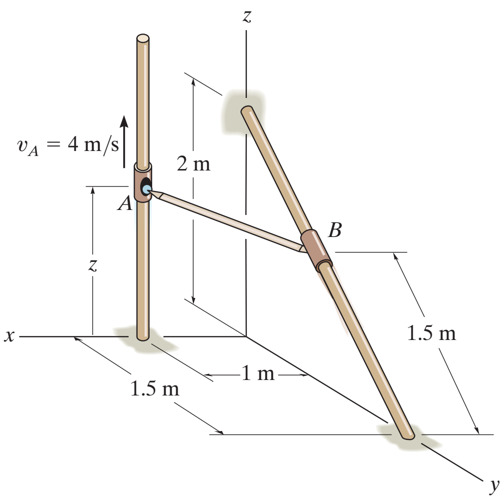
***20–28.** If the rod is attached with ball-and-socket joints to smooth collars A and B at its end points, determine the velocity of B at the instant shown if A is moving upward at a constant speed of $v_A = 5$ ft/s. Also, determine the angular velocity of the rod if it is directed perpendicular to the axis of the rod.

20–29. If the collar at A in Prob. 20–28 is moving upward with an acceleration of $\mathbf{a}_A = \{-2\mathbf{k}\}$ ft/s^2, at the instant its speed is $v_A = 5$ ft/s, determine the acceleration of the collar at B at this instant.



Probs. 20–28/29

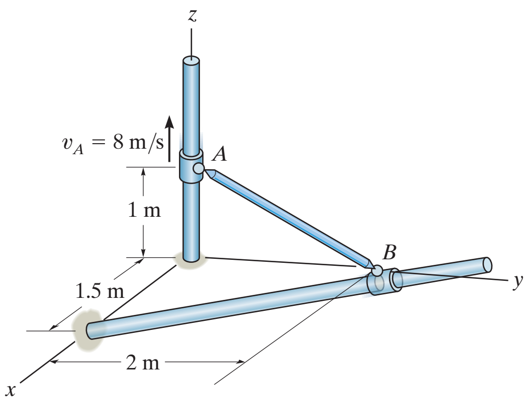
20–30. Rod AB is attached to collars at its ends by ball-and-socket joints. If collar A has a speed $v_A = 4$ m/s, determine the speed of collar B at the instant $z = 2$ m. Assume the angular velocity of the rod is directed perpendicular to the rod.



Prob. 20–30

20–31. The rod is attached to smooth collars A and B at its ends using ball-and-socket joints. Determine the speed of B at the instant shown if A is moving at $v_A = 8$ m/s. Also, determine the angular velocity of the rod if it is directed perpendicular to the axis of the rod.

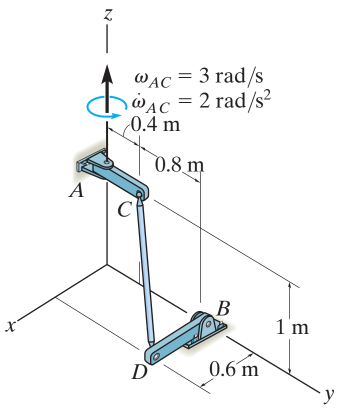
***20–32.** If the collar A in Prob. 20–31 has a deceleration of $\mathbf{a}_A = \{-5\mathbf{k}\}$ m/s^2, at the instant shown, determine the acceleration of collar B at this instant.



Probs. 20–31/32

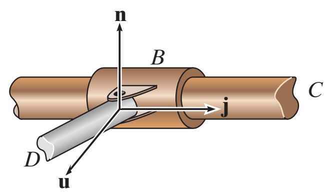
20–33. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular velocity of link BD at the instant shown.

20–34. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular acceleration of link BD at this instant.



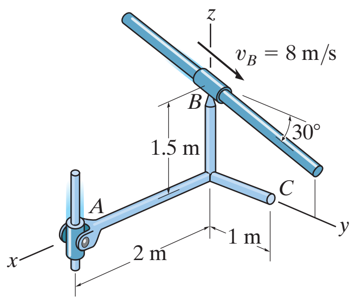
Probs. 20–33/34

20–35. Solve Prob. 20–28 if the connection at B consists of a pin as shown in the figure below, rather than a ball-and-socket joint. *Hint*: The constraint allows rotation of the rod both along the bar ($\mathbf{j}$ direction) and along the axis of the pin ($\mathbf{n}$ direction). Since there is no rotational component in the $\mathbf{u}$ direction, i.e., perpendicular to $\mathbf{n}$ and $\mathbf{j}$ where $\mathbf{u} = \mathbf{j} \times \mathbf{n}$, an additional equation for solution can be obtained from $\boldsymbol{\omega} \cdot \mathbf{u} = 0$. The vector $\mathbf{n}$ is in the same direction as $\mathbf{r}_{D/B} \times \mathbf{r}_{C/B}$.



Prob. 20–35

***20–36.** Member ABC is pin connected at A and has a ball-and-socket joint at B. If the collar at B is moving along the inclined rod at $v_B = 8$ m/s, determine the velocity of point C at the instant shown. *Hint*: See Prob. 20–35.



Prob. 20–36

*20.4 Relative-Motion Analysis Using Translating and Rotating Axes

The most general way to analyze the three-dimensional motion of a rigid body requires the use of *x, y, z* axes that both translate and rotate relative to a second frame *X, Y, Z*. This analysis also provides a means to determine the motions of two points *A* and *B* located on separate members of a mechanism, and the relative motion of one particle with respect to another when one or both particles are moving along *curved paths*.

As shown in Fig. 20–11, the locations of points *A* and *B* are specified relative to the *X, Y, Z* frame of reference by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$. The base point *A* represents the origin of the *x, y, z* coordinate system, which is translating and rotating with respect to *X, Y, Z*. At the instant considered, the velocity and acceleration of point *A* are $\mathbf{v}_A$ and $\mathbf{a}_A$, and the angular velocity and angular acceleration of the *x, y, z* axes are $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}} = d\boldsymbol{\Omega}/dt$. All these vectors are *measured* with respect to the *X, Y, Z* frame of reference, although they can be expressed in Cartesian component form along either set of axes.

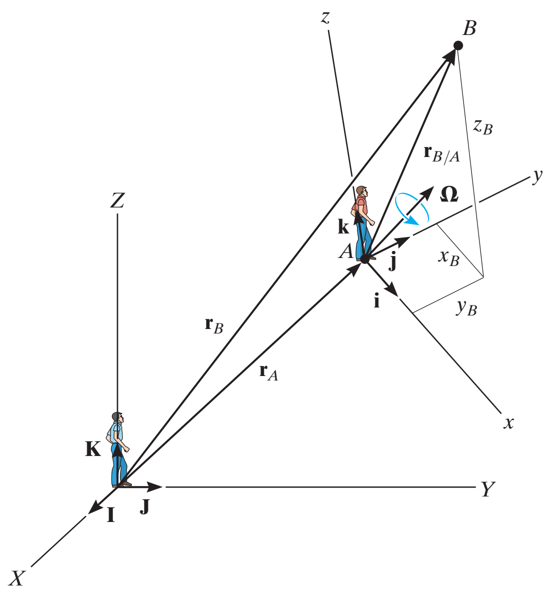


Fig. 20–11

Position. If the position of "B with respect to A" is specified by the *relative-position vector* $\mathbf{r}_{B/A}$, Fig. 20–11, then, by vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{20–11}$$

where

$\mathbf{r}_B$ = position of B
$\mathbf{r}_A$ = position of the origin A
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Velocity. The velocity of point B measured from X, Y, Z can be determined by taking the time derivative of Eq. 20–11,

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A}$$

The first two terms represent $\mathbf{v}_B$ and $\mathbf{v}_A$. The last term must be evaluated by applying Eq. 20–6, since $\mathbf{r}_{B/A}$ is measured with respect to a rotating reference. Hence,

$$\dot{\mathbf{r}}_{B/A} = (\dot{\mathbf{r}}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} \tag{20–12}$$

Therefore,

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz} \tag{20–13}$$

where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the origin A of the x, y, z frame of reference
$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference
$\boldsymbol{\Omega}$ = angular velocity of the x, y, z frame of reference
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Acceleration. The acceleration of point B measured from X, Y, Z is determined by taking the time derivative of Eq. 20–13.

$$\dot{\mathbf{v}}_B = \dot{\mathbf{v}}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times \dot{\mathbf{r}}_{B/A} + \frac{d}{dt}(\mathbf{v}_{B/A})_{xyz}$$

The time derivatives defined in the first and second terms represent $\mathbf{a}_B$ and $\mathbf{a}_A$, respectively. The fourth term can be evaluated using Eq. 20–12, and the last term is evaluated by applying Eq. 20–6, which yields

$$\frac{d}{dt}(\mathbf{v}_{B/A})_{xyz} = (\dot{\mathbf{v}}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} = (\mathbf{a}_{B/A})_{xyz} + \boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

Here $(\mathbf{a}_{B/A})_{xyz}$ is the acceleration of B with respect to A measured from x, y, z. Substituting this result and Eq. 20–12 into the above equation and simplifying, we have

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz} \tag{20–14}$$

where

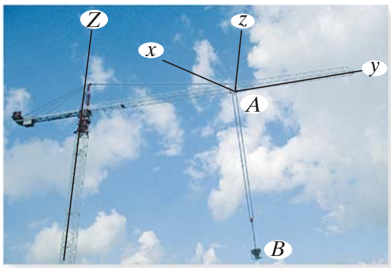
$\mathbf{a}_B$ = acceleration of B

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z frame of reference

$(\mathbf{a}_{B/A})_{xyz}, (\mathbf{v}_{B/A})_{xyz}$ = relative acceleration and relative velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference

$\dot{\boldsymbol{\Omega}}, \boldsymbol{\Omega}$ = angular acceleration and angular velocity of the x, y, z frame of reference

$\mathbf{r}_{B/A}$ = position of "B with respect to A"



Complicated spatial motion of the concrete bucket B occurs due to the rotation of the boom about the Z axis, motion of the carriage A along the boom, and extension and swinging of the cable AB. A translating-rotating x, y, z coordinate system can be established on the carriage, and a relative-motion analysis can then be applied to study this motion. (© R.C. Hibbeler)

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Equations 20–13 and 20–14 are identical to those used in Sec. 16.8 for analyzing relative plane motion.* In that case, however, application is simplified since $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$ have a *constant direction* which is always perpendicular to the plane of motion. For three-dimensional motion, $\dot{\boldsymbol{\Omega}}$ must be computed by using Eq. 20–6, since $\dot{\boldsymbol{\Omega}}$ depends on the change in *both* the magnitude and direction of $\boldsymbol{\Omega}$.

*Refer to Sec. 16.8 for an interpretation of the terms.

Procedure for Analysis

Three-dimensional motion of particles or rigid bodies can be analyzed with Eqs. 20–13 and 20–14 by using the following procedure.

Coordinate Axes.

- Select the location and orientation of the X, Y, Z and x, y, z coordinate axes. Most often solutions can be easily obtained if at the instant considered:

  (1) the origins are *coincident*

  (2) the axes are collinear

  (3) the axes are parallel

- If several components of angular velocity are involved in a problem, the calculations will be reduced if the x, y, z axes are selected such that only one component of angular velocity is observed with respect to this frame $(\boldsymbol{\Omega}_{xyz})$ and the frame rotates with $\boldsymbol{\Omega}$ defined by the other components of angular velocity.

Kinematic Equations.

- After the origin of the moving reference, A, is defined and the moving point B is specified, Eqs. 20–13 and 20–14 should then be written in symbolic form as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- If $\mathbf{r}_A$ and $\boldsymbol{\Omega}$ appear to *change direction* when observed from the fixed X, Y, Z reference then use a set of primed reference axes, x', y', z' having a rotation $\boldsymbol{\Omega}' = \boldsymbol{\Omega}$. Equation 20–6 is then used to determine $\dot{\boldsymbol{\Omega}}$ and the motion $\mathbf{v}_A$ and $\mathbf{a}_A$ of the origin of the moving x, y, z axes.

- If $\mathbf{r}_{B/A}$ and $\boldsymbol{\Omega}_{xyz}$ appear to change direction as observed from x, y, z, then use a set of double-primed reference axes x'', y'', z'' having $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}_{xyz}$ and apply Eq. 20–6 to determine $\dot{\boldsymbol{\Omega}}_{xyz}$ and the relative motion $(\mathbf{v}_{B/A})_{xyz}$ and $(\mathbf{a}_{B/A})_{xyz}$.

- After the final forms of $\dot{\boldsymbol{\Omega}}$, $\mathbf{v}_A$, $\mathbf{a}_A$, $\dot{\boldsymbol{\Omega}}_{xyz}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$ are obtained, numerical problem data can be substituted and the kinematic terms evaluated. The components of all these vectors can be selected either along the X, Y, Z or along the x, y, z axes. The choice is arbitrary, provided a consistent set of unit vectors is used.

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EXAMPLE 20.4

A motor and attached rod AB have the angular motions shown in Fig. 20–12. A collar C on the rod is located 0.25 m from A and is moving downward along the rod with a velocity of 3 m/s and an acceleration of 2 m/s^2. Determine the velocity and acceleration of C at this instant.

SOLUTION

Coordinate Axes.

The origin of the fixed X, Y, Z reference is chosen at the center of the platform, and the origin of the moving x, y, z frame at point A, Fig. 20–12. Since the collar is subjected to two components of angular motion, $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_M$, it will be viewed as having an angular velocity of $\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M$ in x, y, z. Therefore, the x, y, z axes will be attached to the platform so that $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$.

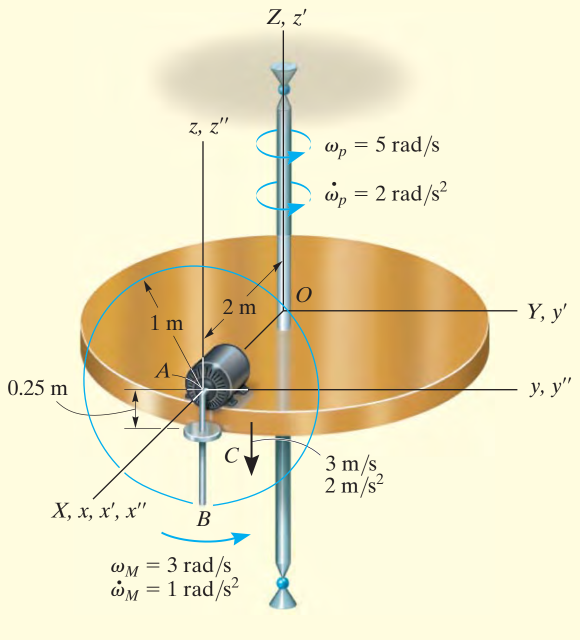

Fig. 20–12

Kinematic Equations. Equations 20–13 and 20–14, applied to points C and A, become

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz}$$

$$\mathbf{a}_C = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz}$$

Motion of A. Here $\mathbf{r}_A$ changes direction relative to X, Y, Z. To find the time derivatives of $\mathbf{r}_A$ we will use a set of x', y', z' axes coincident with the X, Y, Z axes that rotate at $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$. Thus,

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{5\mathbf{k}\} \text{ rad/s } (\boldsymbol{\Omega} \text{ does not change direction relative to } X, Y, Z.)$$

$$\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}_p = \{2\mathbf{k}\} \text{ rad/s}^2$$

$$\mathbf{r}_A = \{2\mathbf{i}\} \text{ m}$$

$$\mathbf{v}_A = \dot{\mathbf{r}}_A = (\dot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times \mathbf{r}_A = \mathbf{0} + 5\mathbf{k} \times 2\mathbf{i} = \{10\mathbf{j}\} \text{ m/s}$$

$$\begin{aligned}\mathbf{a}_A = \ddot{\mathbf{r}}_A &= [(\ddot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times (\dot{\mathbf{r}}_A)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_A + \boldsymbol{\omega}_p \times \dot{\mathbf{r}}_A \\ &= [\mathbf{0} + \mathbf{0}] + 2\mathbf{k} \times 2\mathbf{i} + 5\mathbf{k} \times 10\mathbf{j} = \{-50\mathbf{i} + 4\mathbf{j}\} \text{ m/s}^2\end{aligned}$$

Motion of C with Respect to A. Here $\mathbf{r}_{C/A}$ changes direction relative to x, y, z, and so to find its time derivatives use a set of x'', y'', z'' axes that rotate at $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M$. Thus,

$$\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M = \{3\mathbf{i}\} \text{ rad/s } (\boldsymbol{\Omega}_{xyz} \text{ does not change direction relative to } x, y, z.)$$

$$\dot{\boldsymbol{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_M = \{1\mathbf{i}\} \text{ rad/s}^2$$

$$\mathbf{r}_{C/A} = \{-0.25\mathbf{k}\} \text{ m}$$

$$\begin{aligned}(\mathbf{v}_{C/A})_{xyz} &= (\dot{\mathbf{r}}_{C/A})_{xyz} = (\dot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times \mathbf{r}_{C/A} \\ &= -3\mathbf{k} + [3\mathbf{i} \times (-0.25\mathbf{k})] = \{0.75\mathbf{j} - 3\mathbf{k}\} \text{ m/s}\end{aligned}$$

$$\begin{aligned}(\mathbf{a}_{C/A})_{xyz} &= (\ddot{\mathbf{r}}_{C/A})_{xyz} = [(\ddot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_M \times \mathbf{r}_{C/A} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{xyz} \\ &= [-2\mathbf{k} + 3\mathbf{i} \times (-3\mathbf{k})] + (1\mathbf{i}) \times (-0.25\mathbf{k}) + (3\mathbf{i}) \times (0.75\mathbf{j} - 3\mathbf{k}) \\ &= \{18.25\mathbf{j} + 0.25\mathbf{k}\} \text{ m/s}^2\end{aligned}$$

Motion of C.

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz} \\ &= 10\mathbf{j} + [5\mathbf{k} \times (-0.25\mathbf{k})] + (0.75\mathbf{j} - 3\mathbf{k}) \\ &= \{10.75\mathbf{j} - 3\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz} \\ &= (-50\mathbf{i} + 4\mathbf{j}) + [2\mathbf{k} \times (-0.25\mathbf{k})] + 5\mathbf{k} \times [5\mathbf{k} \times (-0.25\mathbf{k})] \\ &\quad + 2[5\mathbf{k} \times (0.75\mathbf{j} - 3\mathbf{k})] + (18.25\mathbf{j} + 0.25\mathbf{k}) \\ &= \{-57.5\mathbf{i} + 22.25\mathbf{j} + 0.25\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

EXAMPLE 20.5

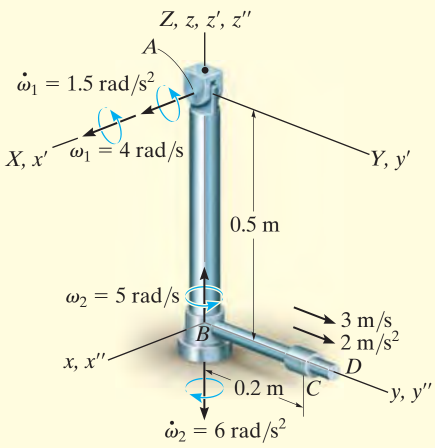

Fig. 20–13

The pendulum shown in Fig. 20–13 consists of two rods; AB is pin supported at A and swings only in the Y–Z plane, whereas a bearing at B allows the attached rod BD to spin about rod AB. At a given instant, the rods have the angular motions shown. Also, a collar C, located 0.2 m from B, has a velocity of 3 m/s and an acceleration of 2 m/s² along the rod. Determine the velocity and acceleration of the collar at this instant.

SOLUTION I

Coordinate Axes. The origin of the fixed X, Y, Z frame will be placed at A. Motion of the collar is conveniently observed from B, so the origin of the x, y, z frame is located at this point. We will choose $\mathbf{\Omega} = \boldsymbol{\omega}_1$ and $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2$.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/B} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/B}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$

Motion of B. To find the time derivatives of $\mathbf{r}_B$ let the x', y', z' axes rotate with $\mathbf{\Omega}' = \boldsymbol{\omega}_1$. Then

$$\mathbf{\Omega}' = \boldsymbol{\omega}_1 = \{4\mathbf{i}\} \text{ rad/s} \quad \dot{\mathbf{\Omega}}' = \dot{\boldsymbol{\omega}}_1 = \{1.5\mathbf{i}\} \text{ rad/s}^2$$

$$\mathbf{r}_B = \{-0.5\mathbf{k}\} \text{ m}$$

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B = \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\} \text{ m/s}$$

$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B$$
$$= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\} \text{ m/s}^2$$

Motion of C with Respect to B. To find the time derivatives of $\mathbf{r}_{C/B}$ relative to x, y, z, let the x'', y'', z'' axes rotate with $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2$. Then

$$\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_2 = \{5\mathbf{k}\} \text{ rad/s} \quad \dot{\mathbf{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_2 = \{-6\mathbf{k}\} \text{ rad/s}^2$$

$$\mathbf{r}_{C/B} = \{0.2\mathbf{j}\} \text{ m}$$

$$(\mathbf{v}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times \mathbf{r}_{C/B} = 3\mathbf{j} + 5\mathbf{k} \times 0.2\mathbf{j} = \{-1\mathbf{i} + 3\mathbf{j}\} \text{ m/s}$$

$$(\mathbf{a}_{C/B})_{xyz} = (\ddot{\mathbf{r}}_{C/B})_{xyz} = [(\ddot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{C/B} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{xyz}$$
$$= (2\mathbf{j} + 5\mathbf{k} \times 3\mathbf{j}) + (-6\mathbf{k} \times 0.2\mathbf{j}) + [5\mathbf{k} \times (-1\mathbf{i} + 3\mathbf{j})]$$
$$= \{-28.8\mathbf{i} - 3\mathbf{j}\} \text{ m/s}^2$$

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Motion of C.

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz} = 2\mathbf{j} + 4\mathbf{i} \times 0.2\mathbf{j} + (-1\mathbf{i} + 3\mathbf{j})$$
$$= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/B} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{C/B}) + 2\mathbf{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$
$$= (0.75\mathbf{j} + 8\mathbf{k}) + (1.5\mathbf{i} \times 0.2\mathbf{j}) + [4\mathbf{i} \times (4\mathbf{i} \times 0.2\mathbf{j})]$$
$$+ 2[4\mathbf{i} \times (-1\mathbf{i} + 3\mathbf{j})] + (-28.8\mathbf{i} - 3\mathbf{j})$$
$$= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

SOLUTION II

Coordinate Axes. Here we will let the *x, y, z* axes rotate at

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \{4\mathbf{i} + 5\mathbf{k}\} \text{ rad/s}$$

Then $\boldsymbol{\Omega}_{xyz} = \mathbf{0}$.

Motion of *B*. From the constraints of the problem $\boldsymbol{\omega}_1$ does not change direction relative to *X, Y, Z*; however, the direction of $\boldsymbol{\omega}_2$ is changed by $\boldsymbol{\omega}_1$. Thus, to obtain $\dot{\boldsymbol{\Omega}}$ consider x', y', z' axes coincident with the *X, Y, Z* axes at *A*, so that $\boldsymbol{\Omega}' = \boldsymbol{\omega}_1$. Then taking the derivative of the components of $\boldsymbol{\Omega}$,

$$\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = [(\dot{\boldsymbol{\omega}}_1)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_1] + [(\dot{\boldsymbol{\omega}}_2)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]$$
$$= [1.5\mathbf{i} + \mathbf{0}] + [-6\mathbf{k} + 4\mathbf{i} \times 5\mathbf{k}] = \{1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}\} \text{ rad/s}^2$$

Also, $\boldsymbol{\omega}_1$ changes the direction of $\mathbf{r}_B$ so that the time derivatives of $\mathbf{r}_B$ can be found using the primed axes defined above. Hence,

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B$$
$$= \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\} \text{ m/s}$$
$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B$$
$$= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\} \text{ m/s}^2$$

Motion of *C* with Respect to *B*.

$$\boldsymbol{\Omega}_{xyz} = \mathbf{0}$$
$$\dot{\boldsymbol{\Omega}}_{xyz} = \mathbf{0}$$
$$\mathbf{r}_{C/B} = \{0.2\mathbf{j}\} \text{ m}$$
$$(\mathbf{v}_{C/B})_{xyz} = \{3\mathbf{j}\} \text{ m/s}$$
$$(\mathbf{a}_{C/B})_{xyz} = \{2\mathbf{j}\} \text{ m/s}^2$$

Motion of *C*.

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$
$$= 2\mathbf{j} + [(4\mathbf{i} + 5\mathbf{k}) \times (0.2\mathbf{j})] + 3\mathbf{j}$$
$$= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$
$$= (0.75\mathbf{j} + 8\mathbf{k}) + [(1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}) \times (0.2\mathbf{j})]$$
$$+ (4\mathbf{i} + 5\mathbf{k}) \times [(4\mathbf{i} + 5\mathbf{k}) \times 0.2\mathbf{j}] + 2[(4\mathbf{i} + 5\mathbf{k}) \times 3\mathbf{j}] + 2\mathbf{j}$$
$$= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

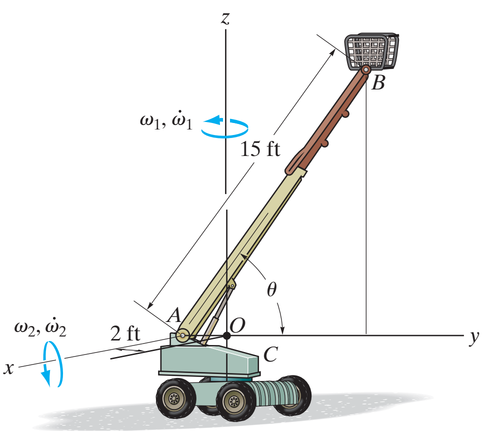
PROBLEMS

20–37. Solve Example 20.5 such that the x, y, z axes move with curvilinear translation, $\mathbf{\Omega} = \mathbf{0}$ in which case the collar appears to have both an angular velocity $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ and radial motion.

20–38. Solve Example 20.5 by fixing x, y, z axes to rod BD so that $\mathbf{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. In this case the collar appears only to move radially outward along BD; hence $\mathbf{\Omega}_{xyz} = \mathbf{0}$.

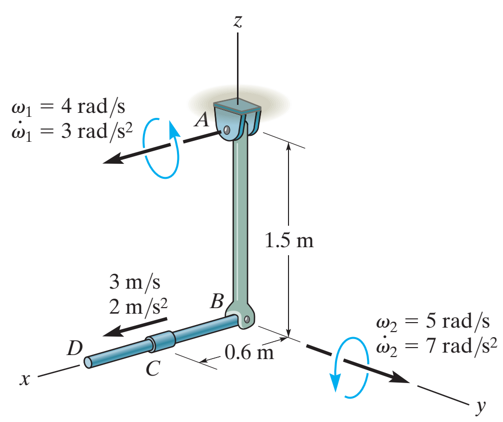
20–39. At the instant $\theta = 60°$, the telescopic boom AB of the construction lift is rotating with a constant angular velocity about the z axis of $\omega_1 = 0.5$ rad/s and about the pin at A with a constant angular speed of $\omega_2 = 0.25$ rad/s. Simultaneously, the boom is extending with a velocity of 1.5 ft/s, and it has an acceleration of 0.5 ft/s^2, both measured relative to the construction lift. Determine the velocity and acceleration of point B located at the end of the boom at this instant.

***20–40.** At the instant $\theta = 60°$, the construction lift is rotating about the z axis with an angular velocity of $\omega_1 = 0.5$ rad/s and an angular acceleration of $\dot{\omega}_1 = 0.25$ rad/s^2 while the telescopic boom AB rotates about the pin at A with an angular velocity of $\omega_2 = 0.25$ rad/s and angular acceleration of $\dot{\omega}_2 = 0.1$ rad/s^2. Simultaneously, the boom is extending with a velocity of 1.5 ft/s, and it has an acceleration of 0.5 ft/s^2, both measured relative to the frame. Determine the velocity and acceleration of point B located at the end of the boom at this instant.



Probs. 20–39/40

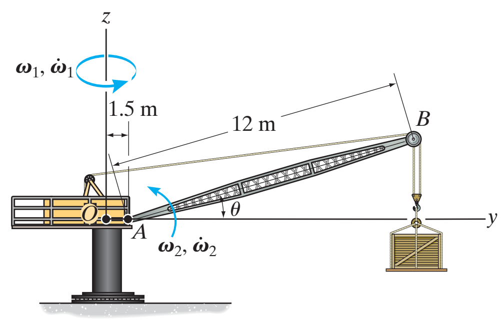
20–41. At the instant shown, the arm AB is rotating about the fixed pin A with an angular velocity $\omega_1 = 4$ rad/s and angular acceleration $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, rod BD is rotating relative to rod AB with an angular velocity $\omega_2 = 5$ rad/s, which is increasing at $\dot{\omega}_2 = 7$ rad/s^2. Also, the collar C is moving along rod BD with a velocity of 3 m/s and an acceleration of 2 m/s^2, both measured relative to the rod. Determine the velocity and acceleration of the collar at this instant.



Prob. 20–41

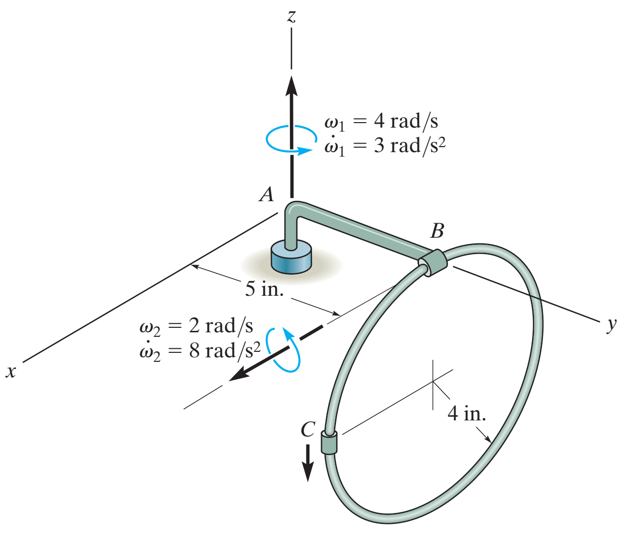
20–42. At the instant $\theta = 30°$, the frame of the crane and the boom AB rotate with a constant angular velocity of $\omega_1 = 1.5$ rad/s and $\omega_2 = 0.5$ rad/s, respectively. Determine the velocity and acceleration of point B at this instant.

20–43. At the instant $\theta = 30°$, the frame of the crane is rotating with an angular velocity of $\omega_1 = 1.5$ rad/s and angular acceleration of $\dot{\omega}_1 = 0.5$ rad/s^2, while the boom AB rotates with an angular velocity of $\omega_2 = 0.5$ rad/s and angular acceleration of $\dot{\omega}_2 = 0.25$ rad/s^2. Determine the velocity and acceleration of point B at this instant.



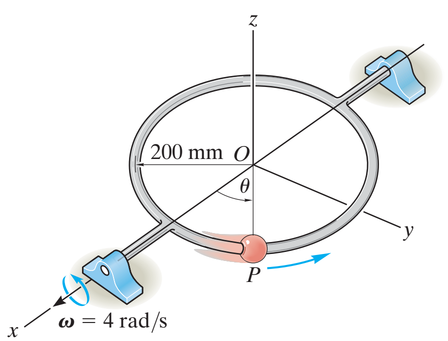
Probs. 20–42/43

***20–44.** At the instant shown, the rod AB is rotating about the z axis with an angular velocity $\omega_1 = 4$ rad/s and an angular acceleration $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, the circular rod has an angular motion relative to the rod as shown. If the collar C is moving down around the circular rod with a speed of 3 in./s, which is increasing at 8 in./s^2, both measured relative to the rod, determine the collar's velocity and acceleration at this instant.



Prob. 20–44

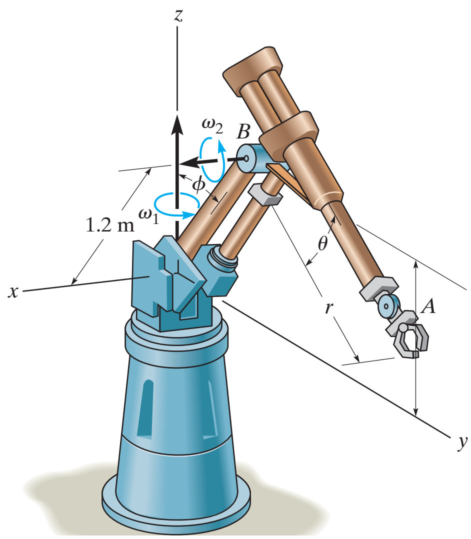
20–45. The particle P slides around the circular hoop with a constant angular velocity of $\dot{\theta} = 6$ rad/s, while the hoop rotates about the x axis at a constant rate of $\omega = 4$ rad/s. If at the instant shown the hoop is in the x–y plane and the angle $\theta = 45°$, determine the velocity and acceleration of the particle at this instant.



Prob. 20–45

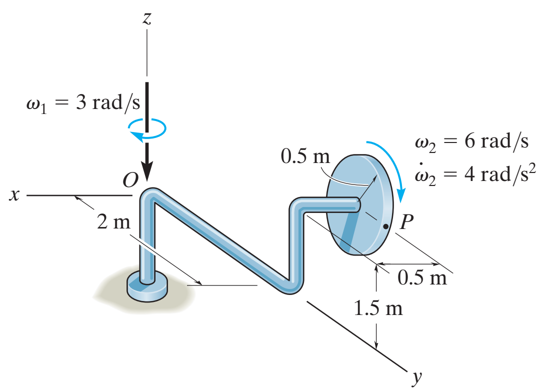
20–46. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and about joint B at $\omega_2 = 2$ rad/s. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.

20–47. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and $\dot{\omega}_1 = 2$ rad/s^2; and about joint B at $\omega_2 = 2$ rad/s and $\dot{\omega}_2 = 3$ rad/s^2. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.



Probs. 20–46/47

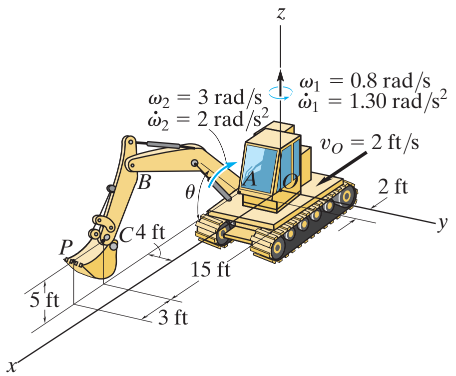
***20–48.** At the given instant, the rod is turning about the z axis with a constant angular velocity $\omega_1 = 3$ rad/s. At this same instant, the disk is spinning at $\omega_2 = 6$ rad/s when $\dot{\omega}_2 = 4$ rad/s^2, both measured *relative* to the rod. Determine the velocity and acceleration of point P on the disk at this instant.



Prob. 20–48

20

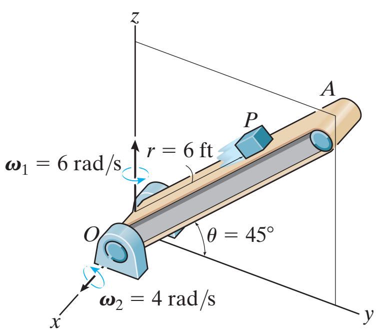
20–49. At the instant shown, the backhoe is traveling forward at a constant speed $v_O = 2$ ft/s, and the boom ABC is rotating about the z axis with an angular velocity $\omega_1 = 0.8$ rad/s and an angular acceleration $\dot{\omega}_1 = 1.30$ rad/s^2. At this same instant the boom is rotating with $\omega_2 = 3$ rad/s when $\dot{\omega}_2 = 2$ rad/s^2, both measured relative to the frame. Determine the velocity and acceleration of point P on the bucket at this instant.



Prob. 20–49

20–50. At the instant shown, the arm OA of the conveyor belt is rotating about the z axis with a constant angular velocity $\omega_1 = 6$ rad/s, while at the same instant the arm is rotating upward at a constant rate $\omega_2 = 4$ rad/s. If the conveyor is running at a constant rate $\dot{r} = 5$ ft/s, determine the velocity and acceleration of the package P at the instant shown. Neglect the size of the package.

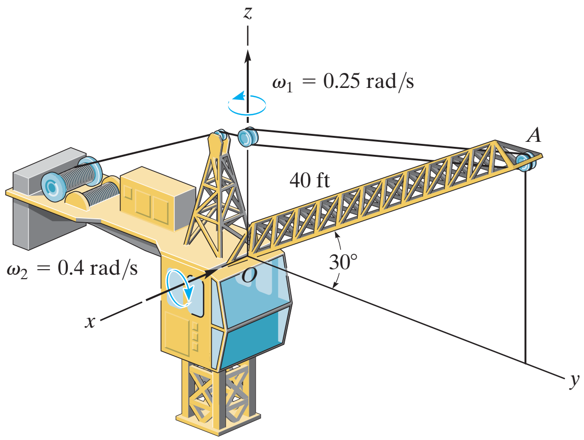
20–51. At the instant shown, the arm OA of the conveyor belt is rotating about the z axis with a constant angular velocity $\omega_1 = 6$ rad/s, while at the same instant the arm is rotating upward at a constant rate $\omega_2 = 4$ rad/s. If the conveyor is running at a rate $\dot{r} = 5$ ft/s, which is increasing at $\ddot{r} = 8$ ft/s^2, determine the velocity and acceleration of the package P at the instant shown. Neglect the size of the package.



Probs. 20–50/51

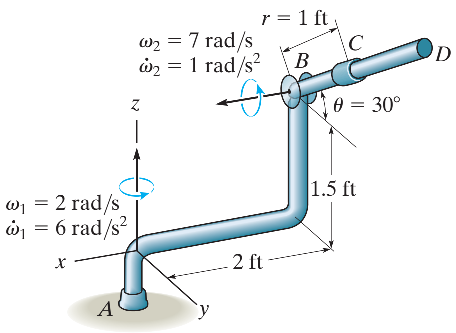
***20–52.** The crane is rotating about the z axis with a constant rate $\omega_1 = 0.25$ rad/s, while the boom OA is rotating downward with a constant rate $\omega_2 = 0.4$ rad/s. Compute the velocity and acceleration of point A located at the top of the boom at the instant shown.

20–53. Solve Prob. 20–52 if the angular motions are increasing at $\dot{\omega}_1 = 0.4$ rad/s^2 and $\dot{\omega}_2 = 0.8$ rad/s^2 at the instant shown.



Probs. 20–52/53

20–54. At the instant shown, the arm AB is rotating about the fixed bearing with an angular velocity $\omega_1 = 2$ rad/s and angular acceleration $\dot{\omega}_1 = 6$ rad/s^2. At the same instant, rod BD is rotating relative to rod AB at $\omega_2 = 7$ rad/s, which is increasing at $\dot{\omega}_2 = 1$ rad/s^2. Also, the collar C is moving along rod BD with a velocity $\dot{r} = 2$ ft/s and a *deceleration* $\ddot{r} = -0.5$ ft/s^2, both measured relative to the rod. Determine the velocity and acceleration of the collar at this instant.



Prob. 20–54

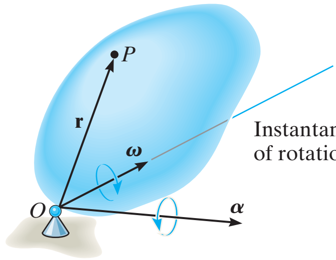
CHAPTER REVIEW

Rotation About a Fixed Point

When a body rotates about a fixed point O, then points on the body follow a path that lies on the surface of a sphere centered at O.

Since the angular acceleration is a time rate of change in the angular velocity, then it is necessary to account for both the magnitude and directional changes of $\boldsymbol{\omega}$ when finding its time derivative. To do this, the angular velocity is often specified in terms of its component motions, such that the direction of some of these components will remain constant relative to rotating x, y, z axes. If this is the case, then the time derivative relative to the fixed axis can be determined using $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{A}$.

Once $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known, the velocity and acceleration of any point P in the body can then be determined.



$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_P = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

General Motion

If the body undergoes general motion, then the motion of a point B on the body can be related to the motion of another point A using a relative motion analysis, with translating axes attached to A.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

Relative Motion Analysis Using Translating and Rotating Axes

The motion of two points A and B on a body, a series of connected bodies, or each point located on two different paths, can be related using a relative motion analysis with rotating and translating axes at A.

When applying the equations, to find $\mathbf{v}_B$ and $\mathbf{a}_B$, it is important to account for both the magnitude and directional changes of $\mathbf{r}_A$, $\mathbf{r}_{B/A}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Omega}_{xyz}$ when taking their time derivatives to find $\mathbf{v}_A$, $\mathbf{a}_A$, $(\mathbf{v}_{B/A})_{xyz}$, $(\mathbf{a}_{B/A})_{xyz}$, $\dot{\boldsymbol{\Omega}}$, and $\dot{\boldsymbol{\Omega}}_{xyz}$. To do this properly, one must use Eq. 20–6.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

Chapter 21

(© Derek Watt/Alamy)

The forces acting on each of these motorcycles can be determined using the equations of motion as discussed in this chapter.

Three-Dimensional Kinetics of a Rigid Body

CHAPTER OBJECTIVES

- To introduce the methods for finding the moments of inertia and products of inertia of a body about various axes.
- To show how to apply the principles of work and energy and linear and angular impulse and momentum to a rigid body having three-dimensional motion.
- To develop and apply the equations of motion in three dimensions.
- To study gyroscopic and torque-free motion.

*21.1 Moments and Products of Inertia

When studying the planar kinetics of a body, it was necessary to introduce the moment of inertia I_G, which was computed about an axis perpendicular to the plane of motion and passing through the body's mass center G. For the kinetic analysis of three-dimensional motion it will sometimes be necessary to calculate six inertial quantities. These terms, called the moments and products of inertia, describe in a particular way the distribution of mass for a body relative to a given coordinate system that has a specified orientation and point of origin.

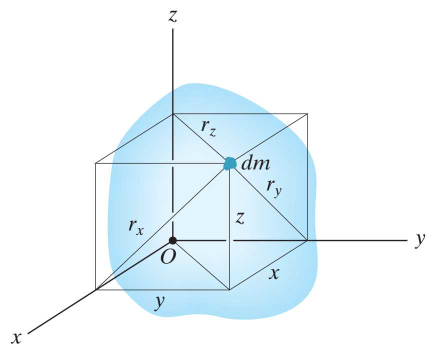


Fig. 21–1

Moment of Inertia. Consider the rigid body shown in Fig. 21–1. The *moment of inertia* for a differential element dm of the body about any one of the three coordinate axes is defined as the product of the mass of the element and the square of the shortest distance from the axis to the element. For example, as noted in the figure, $r_x = \sqrt{y^2 + z^2}$, so that the mass moment of inertia of the element about the x axis is

$$dI_{xx} = r_x^2\, dm = (y^2 + z^2)\, dm$$

The moment of inertia I_{xx} for the body can be determined by integrating this expression over the entire mass of the body. Hence, for each of the axes, we can write

$$\begin{aligned} I_{xx} &= \int_m r_x^2 dm = \int_m (y^2 + z^2)\, dm \\ I_{yy} &= \int_m r_y^2 dm = \int_m (x^2 + z^2)\, dm \\ I_{zz} &= \int_m r_z^2 dm = \int_m (x^2 + y^2)\, dm \end{aligned} \tag{21–1}$$

Here it is seen that the moment of inertia is *always a positive quantity*, since it is the summation of the product of the mass dm, which is always positive, and the distances squared.

Product of Inertia. The *product of inertia* for a differential element dm with respect to a set of *two orthogonal planes* is defined as the product of the mass of the element and the perpendicular (or shortest) distances from the planes to the element. For example, this distance is x to the y–z plane and it is y to the x–z plane, Fig. 21–1. The product of inertia dI_{xy} for the element is therefore

$$dI_{xy} = xy\, dm$$

Note also that $dI_{yx} = dI_{xy}$. By integrating over the entire mass, the products of inertia of the body with respect to each combination of planes can be expressed as

$$\begin{aligned} I_{xy} &= I_{yx} = \int_m xy\, dm \\ I_{yz} &= I_{zy} = \int_m yz\, dm \\ I_{xz} &= I_{zx} = \int_m xz\, dm \end{aligned} \tag{21–2}$$

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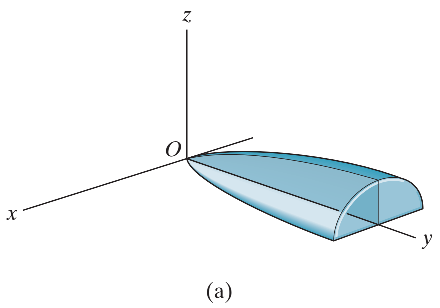

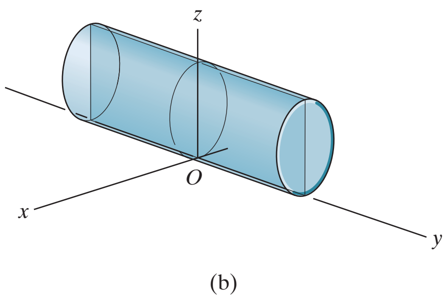

Fig. 21–2

Unlike the moment of inertia, which is always positive, the product of inertia may be positive, negative, or zero. The result depends on the algebraic signs of the two defining coordinates, which vary independently from one another. In particular, if either one or both of the orthogonal planes are *planes of symmetry* for the mass, the *product of inertia* with respect to these planes will be *zero*. In such cases, elements of mass will occur in *pairs* located on each side of the plane of symmetry. On one side of the plane the product of inertia for the element will be positive, while on the other side the product of inertia of the corresponding element will be negative, the sum therefore yielding zero. Examples of this are shown in Fig. 21–2. In the first case, Fig. 21–2*a*, the y–z plane is a plane of symmetry, and hence $I_{xy} = I_{xz} = 0$. Calculation of I_{yz} will yield a *positive* result, since all elements of mass are located using only positive y and z coordinates. For the cylinder, with the coordinate axes located as shown in Fig. 21–2*b*, the x–z and y–z planes are both planes of symmetry. Thus, $I_{xy} = I_{yz} = I_{zx} = 0$.

Parallel-Axis and Parallel-Plane Theorems. The techniques of integration used to determine the moment of inertia of a body were described in Sec. 17.1. Also discussed were methods to determine the moment of inertia of a composite body, i.e., a body that is composed of simpler segments, as tabulated on the inside back cover. In both of these cases the *parallel-axis theorem* is often used for the calculations. This theorem, which was developed in Sec. 17.1, allows us to transfer the moment of inertia of a body from an axis passing through its mass center G to a parallel axis passing through some other point. If G has coordinates x_G, y_G, z_G defined with respect to the x, y, z axes, Fig. 21–3, then the parallel-axis equations used to calculate the moments of inertia about the x, y, z axes are

$$\begin{aligned} I_{xx} &= (I_{x'x'})_G + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{y'y'})_G + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{z'z'})_G + m(x_G^2 + y_G^2) \end{aligned} \qquad (21\text{–}3)$$

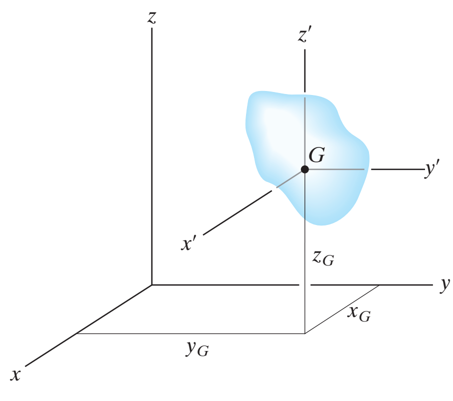

Fig. 21–3

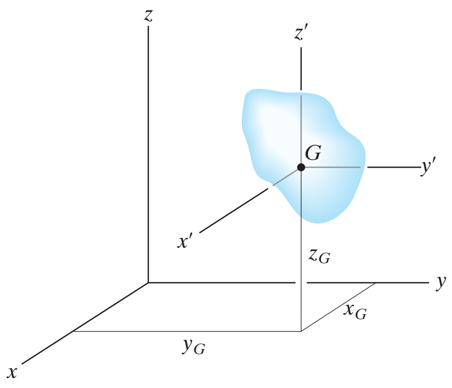

Fig. 21–3 (repeated)

The products of inertia of a composite body are computed in the same manner as the body's moments of inertia. Here, however, the *parallel-plane theorem* is important. This theorem is used to transfer the products of inertia of the body with respect to a set of three orthogonal planes passing through the body's mass center to a corresponding set of three parallel planes passing through some other point O. Defining the perpendicular distances between the planes as x_G, y_G, and z_G, Fig. 21–3, the parallel-plane equations can be written as

$$\begin{aligned} I_{xy} &= (I_{x'y'})_G + m x_G y_G \\ I_{yz} &= (I_{y'z'})_G + m y_G z_G \\ I_{zx} &= (I_{z'x'})_G + m z_G x_G \end{aligned} \tag{21–4}$$

The derivation of these formulas is similar to that given for the parallel-axis equation, Sec. 17.1.

Inertia Tensor. The inertial properties of a body are therefore completely characterized by nine terms, six of which are independent of one another. This set of terms is defined using Eqs. 21–1 and 21–2 and can be written as

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

This array is called an *inertia tensor*.* It has a unique set of values for a body when it is determined for each location of the origin O and orientation of the coordinate axes.

In general, for point O we can specify a unique axes inclination for which the products of inertia for the body are zero when computed with respect to these axes. When this is done, the inertia tensor is said to be "diagonalized" and may be written in the simplified form

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Here $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$ are termed the *principal moments of inertia* for the body, which are computed with respect to the *principal axes of inertia*. Of these three principal moments of inertia, one will be a maximum and another a minimum of the body's moment of inertia.

The dynamics of the space shuttle while it orbits the earth can be predicted only if its moments and products of inertia are known relative to its mass center. (©Ablestock/Getty Images)

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*The negative signs are here as a consequence of the development of angular momentum, Eqs. 21–10.

The mathematical determination of the directions of principal axes of inertia will not be discussed here (see Prob. 21–22). However, there are many cases in which the principal axes can be determined by inspection. From the previous discussion it was noted that if the coordinate axes are oriented such that *two* of the three orthogonal planes containing the axes are planes of *symmetry* for the body, then all the products of inertia for the body are zero with respect to these coordinate planes, and hence these coordinate axes are principal axes of inertia. For example, the x, y, z axes shown in Fig. 21–2*b* represent the principal axes of inertia for the cylinder at point O.

Moment of Inertia About an Arbitrary Axis. Consider the body shown in Fig. 21–4, where the nine elements of the inertia tensor have been determined with respect to the x, y, z axes having an origin at O. Here we wish to determine the moment of inertia of the body about the Oa axis, which has a direction defined by the unit vector $\mathbf{u}_a$. By definition $I_{Oa} = \int b^2\,dm$, where b is the *perpendicular distance* from dm to Oa. If the position of dm is located using $\mathbf{r}$, then $b = r\sin\theta$, which represents the *magnitude* of the cross product $\mathbf{u}_a \times \mathbf{r}$. Hence, the moment of inertia can be expressed as

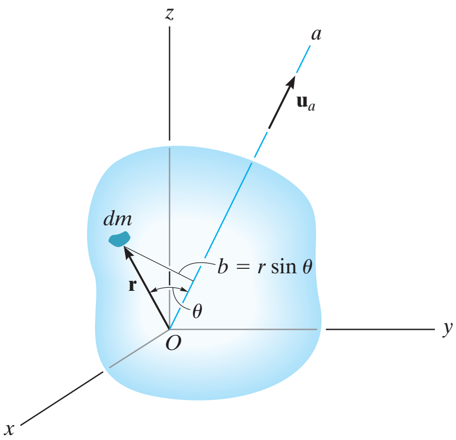


Fig. 21–4

$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r}) \cdot (\mathbf{u}_a \times \mathbf{r}) dm$$

Provided $\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$. After substituting and performing the dot-product operation, the moment of inertia is

$$\begin{aligned}
I_{Oa} &= \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2] dm \\
&= u_x^2 \int_m (y^2 + z^2) dm + u_y^2 \int_m (z^2 + x^2) dm + u_z^2 \int_m (x^2 + y^2)\, dm \\
&\quad - 2u_x u_y \int_m xy\, dm - 2u_y u_z \int_m yz\, dm - 2u_z u_x \int_m zx\, dm
\end{aligned}$$

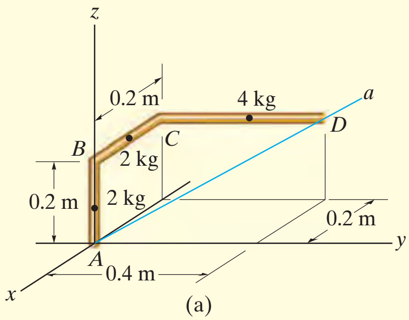
Recognizing the integrals to be the moments and products of inertia of the body, Eqs. 21–1 and 21–2, we have

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x \tag{21–5}$$

Thus, if the inertia tensor is specified for the x, y, z axes, the moment of inertia of the body about the inclined Oa axis can be found. For the calculation, the direction cosines u_x, u_y, u_z of the axes must be determined. These terms specify the cosines of the coordinate direction angles α, β, γ made between the positive Oa axis and the positive x, y, z axes, respectively (see Appendix B).

EXAMPLE 21.1

Determine the moment of inertia of the bent rod shown in Fig. 21–5*a* about the *Aa* axis. The mass of each of the three segments is given in the figure.



(a)

SOLUTION

Before applying Eq. 21–5, it is first necessary to determine the moments and products of inertia of the rod with respect to the *x, y, z* axes. This is done using the formula for the moment of inertia of a slender rod, $I = \frac{1}{12}ml^2$, and the parallel-axis and parallel-plane theorems, Eqs. 21–3 and 21–4. Dividing the rod into three parts and locating the mass center of each segment, Fig. 21–5*b*, we have

$$I_{xx} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + [0 + 2(0.2)^2] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((0.2)^2 + (0.2)^2)\right] = 0.480\text{ kg}\cdot\text{m}^2$$

$$I_{yy} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2((-0.1)^2 + (0.2)^2)\right] + [0 + 4((-0.2)^2 + (0.2)^2)] = 0.453\text{ kg}\cdot\text{m}^2$$

$$I_{zz} = [0 + 0] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2(-0.1)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((-0.2)^2 + (0.2)^2)\right] = 0.400\text{ kg}\cdot\text{m}^2$$

$$I_{xy} = [0 + 0] + [0 + 0] + [0 + 4(-0.2)(0.2)] = -0.160\text{ kg}\cdot\text{m}^2$$

$$I_{yz} = [0 + 0] + [0 + 0] + [0 + 4(0.2)(0.2)] = 0.160\text{ kg}\cdot\text{m}^2$$

$$I_{zx} = [0 + 0] + [0 + 2(0.2)(-0.1)] + [0 + 4(0.2)(-0.2)] = -0.200\text{ kg}\cdot\text{m}^2$$

The *Aa* axis is defined by the unit vector

$$\mathbf{u}_{Aa} = \frac{\mathbf{r}_D}{r_D} = \frac{-0.2\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{\sqrt{(-0.2)^2 + (0.4)^2 + (0.2)^2}} = -0.408\mathbf{i} + 0.816\mathbf{j} + 0.408\mathbf{k}$$

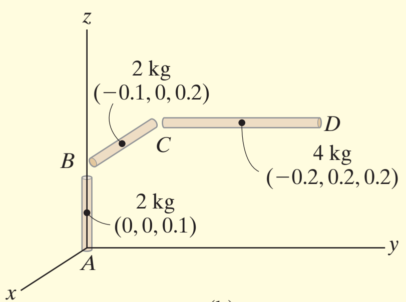
Thus,

$$u_x = -0.408 \quad u_y = 0.816 \quad u_z = 0.408$$

Substituting these results into Eq. 21–5 yields

$$\begin{aligned} I_{Aa} &= I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x \\ &= 0.480(-0.408)^2 + (0.453)(0.816)^2 + 0.400(0.408)^2 \\ &\quad - 2(-0.160)(-0.408)(0.816) - 2(0.160)(0.816)(0.408) \\ &\quad - 2(-0.200)(0.408)(-0.408) \\ &= 0.169\text{ kg}\cdot\text{m}^2 \end{aligned}$$

Ans.



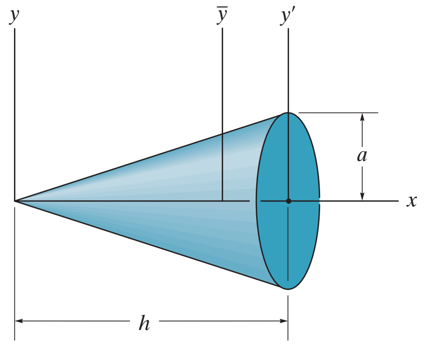
(b)

Fig. 21–5

21

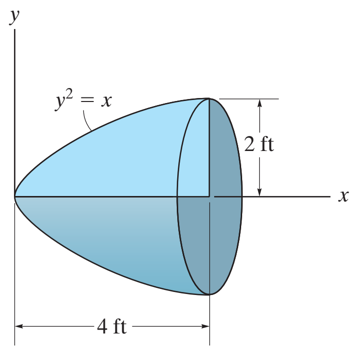
21–1. Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the x, y, z axes and thus depends only on the location of the origin.

21–2. Determine the moment of inertia of the cone with respect to a vertical $\overline{y}$ axis passing through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



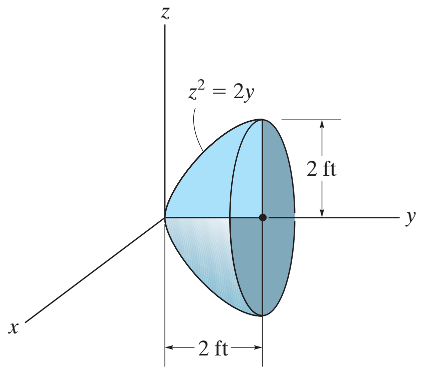
Prob. 21–2

21–3. Determine moment of inertia I_y of the solid formed by revolving the shaded area around the x axis. The density of the material is $\rho = 12 \text{ slug/ft}^3$.



Prob. 21–3

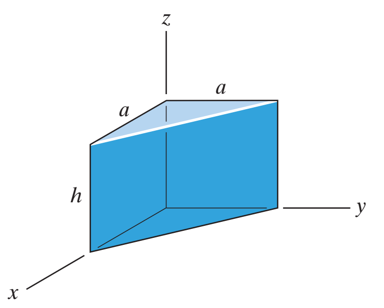
***21–4.** Determine the moments of inertia I_x and I_y of the paraboloid of revolution. The mass of the paraboloid is 20 slug.



Prob. 21–4

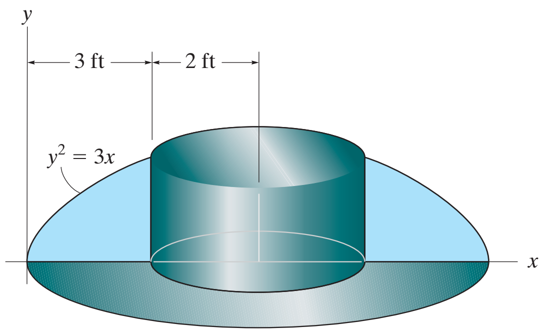
21–5. Determine by direct integration the product of inertia I_{yz} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.

21–6. Determine by direct integration the product of inertia I_{xy} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



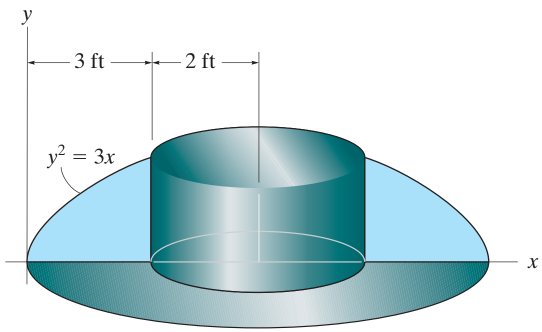
Probs. 21–5/6

21–7. Determine the product of inertia I_{xy} of the object formed by revolving the shaded area about the line $x = 5$ ft. Express the result in terms of the density of the material, ρ.



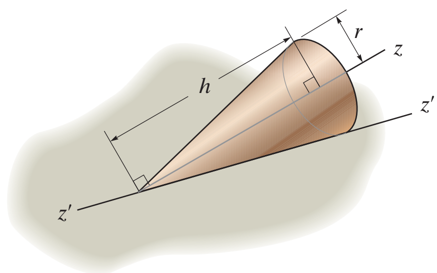
Prob. 21–7

***21–8.** Determine the moment of inertia I_y of the object formed by revolving the shaded area about the line $x = 5$ ft. Express the result in terms of the density of the material, ρ.



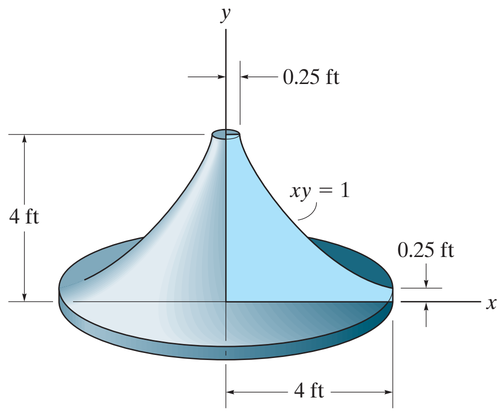
Prob. 21–8

21–9. Determine the moment of inertia of the cone about the z' axis. The weight of the cone is 15 lb, the *height* is $h = 1.5$ ft and the radius is $r = 0.5$ ft.



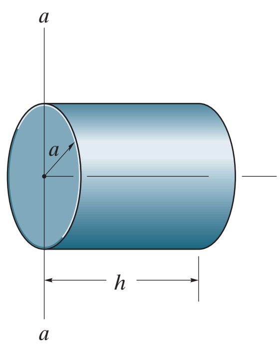
Prob. 21–9

21–10. Determine the radii of gyration k_x and k_y for the solid formed by revolving the shaded area about the y axis. The density of the material is ρ.



Prob. 21–10

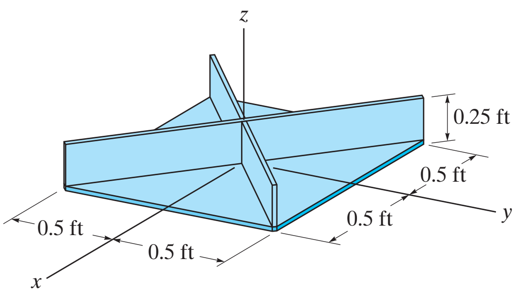
21–11. Determine the moment of inertia of the cylinder with respect to the a–a axis of the cylinder. The cylinder has a mass m.



Prob. 21–11

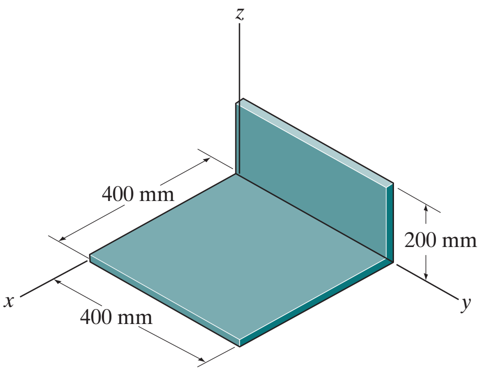
***21–12.** Determine the moment of inertia I_{xx} of the composite plate assembly. The plates have a specific weight of 6 lb/ft^2.

21–13. Determine the product of inertia I_{yz} of the composite plate assembly. The plates have a weight of 6 lb/ft^2.



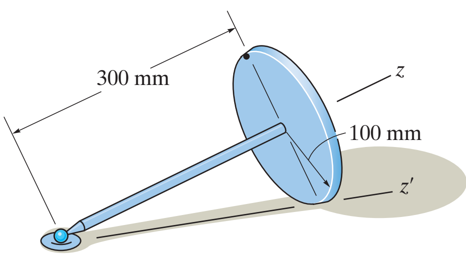
Probs. 21–12/13

21–14. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz}, of the thin plate. The material has a density per unit area of 50 kg/m^2.



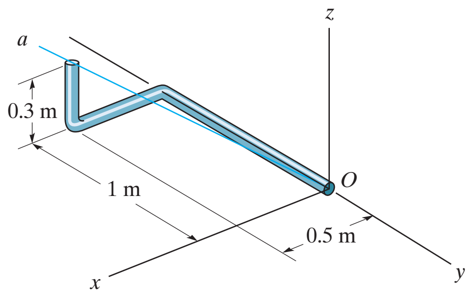
Prob. 21–14

21–15. Determine the moment of inertia of both the 1.5-kg rod and 4-kg disk about the z' axis.



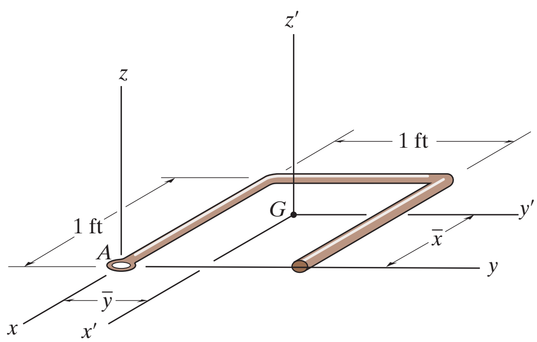
Prob. 21–15

***21–16.** The bent rod has a mass of 3 kg/m. Determine the moment of inertia of the rod about the O–a axis.



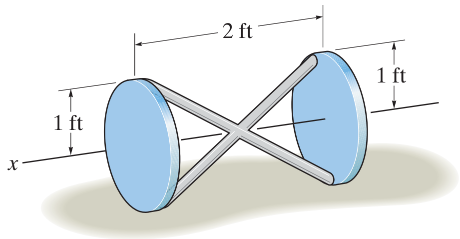
Prob. 21–16

21–17. The bent rod has a weight of 1.5 lb/ft. Locate the center of gravity $G(\bar{x}, \bar{y})$ and determine the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$ of the rod with respect to the x', y', z' axes.



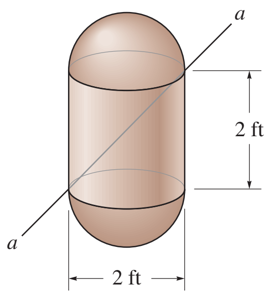
Prob. 21–17

21–18. Determine the moment of inertia of the rod-and-disk assembly about the x axis. The disks each have a weight of 12 lb. The two rods each have a weight of 4 lb, and their ends extend to the rims of the disks.



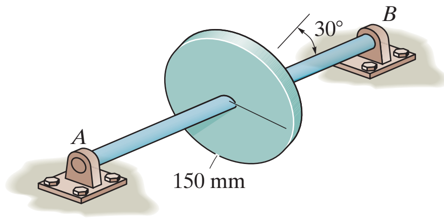
Prob. 21–18

21–19. Determine the moment of inertia of the composite body about the aa axis. The cylinder weighs 20 lb, and each hemisphere weighs 10 lb.



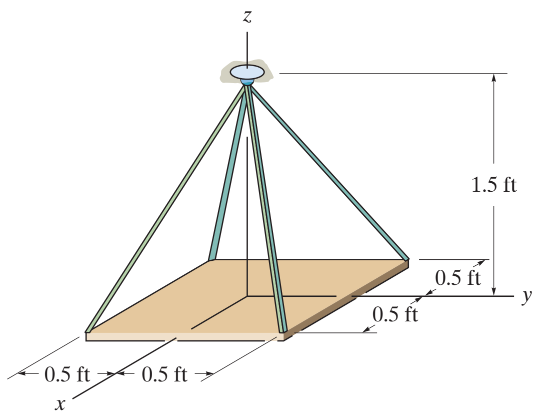
Prob. 21–19

***21–20.** Determine the moment of inertia of the disk about the axis of shaft AB. The disk has a mass of 15 kg.



Prob. 21–20

21–21. The thin plate has a weight of 5 lb and each of the four rods weighs 3 lb. Determine the moment of inertia of the assembly about the z axis.



Prob. 21–21

21.2 Angular Momentum

In this section we will develop the necessary equations used to determine the angular momentum of a rigid body about an arbitrary point. These equations will provide a means for developing both the principle of impulse and momentum and the equations of rotational motion for a rigid body.

Consider the rigid body in Fig. 21–6, which has a mass m and center of mass at G. The X, Y, Z coordinate system represents an inertial frame of reference, and hence, its axes are fixed or translate with a constant velocity. The angular momentum as measured from this reference will be determined relative to the arbitrary point A. The position vectors $\mathbf{r}_A$ and $\boldsymbol{\rho}_A$ are drawn from the origin of coordinates to point A and from A to the ith particle of the body. If the particle's mass is m_i, the angular momentum about point A is

$$(\mathbf{H}_A)_i = \boldsymbol{\rho}_A \times m_i \mathbf{v}_i$$

where $\mathbf{v}_i$ represents the particle's velocity measured from the X, Y, Z coordinate system. If the body has an angular velocity $\boldsymbol{\omega}$ at the instant considered, $\mathbf{v}_i$ may be related to the velocity of A by applying Eq. 20–7, i.e.,

$$\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$$

Thus,

$$\begin{aligned}(\mathbf{H}_A)_i &= \boldsymbol{\rho}_A \times m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= (\boldsymbol{\rho}_A m_i) \times \mathbf{v}_A + \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i\end{aligned}$$

Summing the moments of all the particles of the body requires an integration. Since $m_i \rightarrow dm$, we have

$$\mathbf{H}_A = \left(\int_m \boldsymbol{\rho}_A\, dm\right) \times \mathbf{v}_A + \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)\, dm \qquad (21\text{–}6)$$

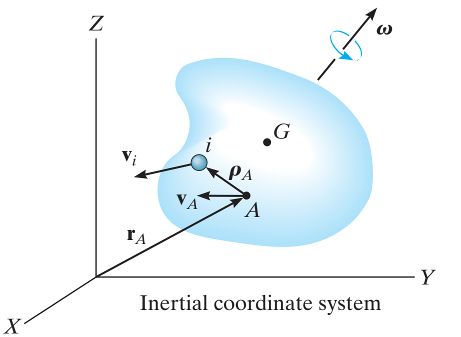
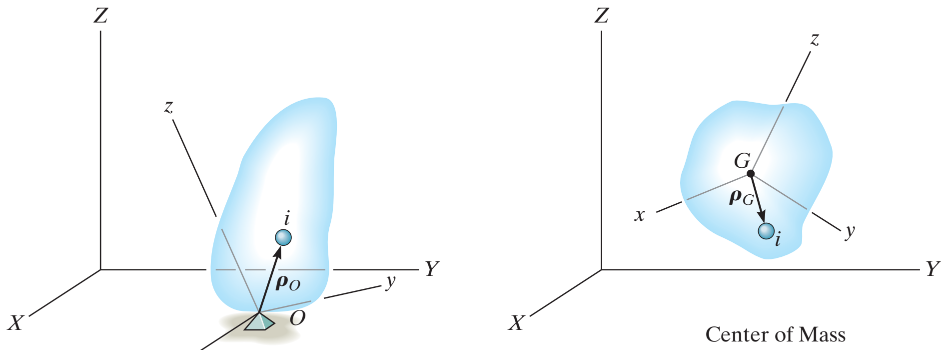


Fig. 21–6



Fixed Point

(a)

Center of Mass

(b)

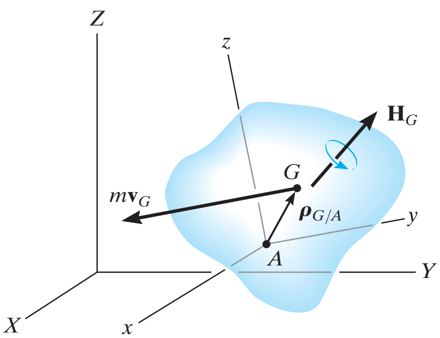
Fig. 21–7

Fixed Point *O*. If A becomes a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$ and Eq. 21–6 reduces to

$$\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\, dm \tag{21–7}$$

Center of Mass *G*. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int_m \boldsymbol{\rho}_A\, dm = \mathbf{0}$ and

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\, dm \tag{21–8}$$



Arbitrary Point

(c)

Arbitrary Point *A*. In general, A can be a point other than O or G, Fig. 21–7*c*, in which case Eq. 21–6 may nevertheless be simplified to the following form (see Prob. 21–23).

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G \tag{21–9}$$

Here the angular momentum consists of two parts—the moment of the linear momentum $m\mathbf{v}_G$ of the body about point A added (vectorially) to the angular momentum $\mathbf{H}_G$. Equation 21–9 can also be used to determine the angular momentum of the body about a fixed point O. The results, of course, will be the same as those found using the more convenient Eq. 21–7.

Rectangular Components of *H*. To make practical use of Eqs. 21–7 through 21–9, the angular momentum must be expressed in terms of its scalar components. For this purpose, it is convenient to

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choose a second set of x, y, z axes having an arbitrary orientation relative to the X, Y, Z axes, Fig. 21–7, and for a general formulation, note that Eqs. 21–7 and 21–8 are both of the form

$$\mathbf{H} = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm$$

Expressing $\mathbf{H}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ in terms of x, y, z components, we have

$$H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = \int_m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times [(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})]dm$$

Expanding the cross products and combining terms yields

$$\begin{aligned} H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = & \left[\omega_x \int_m (y^2 + z^2)dm - \omega_y \int_m xy\, dm - \omega_z \int_m xz\, dm\right]\mathbf{i} \\ & + \left[-\omega_x \int_m xy\, dm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yz\, dm\right]\mathbf{j} \\ & + \left[-\omega_x \int_m zx\, dm - \omega_y \int_m yz\, dm + \omega_z \int_m (x^2 + y^2)dm\right]\mathbf{k} \end{aligned}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and recognizing that the integrals represent the moments and products of inertia, we obtain

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \qquad (21\text{–}10)$$

These equations can be simplified further if the x, y, z coordinate axes are oriented such that they become *principal axes of inertia* for the body at the point. When these axes are used, the products of inertia $I_{xy} = I_{yz} = I_{zx} = 0$, and if the principal moments of inertia about the x, y, z axes are represented as $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$, the three components of angular momentum become

$$H_x = I_x\omega_x \quad H_y = I_y\omega_y \quad H_z = I_z\omega_z \qquad (21\text{–}11)$$

The motion of the astronaut is controlled by use of small directional jets attached to his or her space suit. The impulses these jets provide must be carefully specified in order to prevent tumbling and loss of orientation. (© NASA)

Principle of Impulse and Momentum. Now that the formulation of the angular momentum for a body has been developed, the *principle of impulse and momentum*, as discussed in Sec. 19.2, can be used to solve kinetic problems which involve *force, velocity, and time*. For this case, the following two vector equations are available:

$$m(\mathbf{v}_G)_1 + \Sigma\int_{t_1}^{t_2}\mathbf{F}\,dt = m(\mathbf{v}_G)_2 \qquad (21\text{–}12)$$

$$(\mathbf{H}_O)_1 + \Sigma\int_{t_1}^{t_2}\mathbf{M}_O\,dt = (\mathbf{H}_O)_2 \qquad (21\text{–}13)$$

In three dimensions each vector term can be represented by three scalar components, and therefore a total of *six scalar equations* can be written. Three equations relate the linear impulse and momentum in the *x, y, z* directions, and the other three equations relate the body's angular impulse and momentum about the *x, y, z* axes. Before applying Eqs. 21–12 and 21–13 to the solution of problems, the material in Secs. 19.2 and 19.3 should be reviewed.

21.3 Kinetic Energy

In order to apply the principle of work and energy to solve problems involving general rigid body motion, it is first necessary to formulate expressions for the kinetic energy of the body. To do this, consider the rigid body shown in Fig. 21–8, which has a mass m and center of mass at G. The kinetic energy of the ith particle of the body having a mass m_i and velocity $\mathbf{v}_i$, measured relative to the inertial X, Y, Z frame of reference, is

$$T_i = \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i(\mathbf{v}_i\cdot\mathbf{v}_i)$$

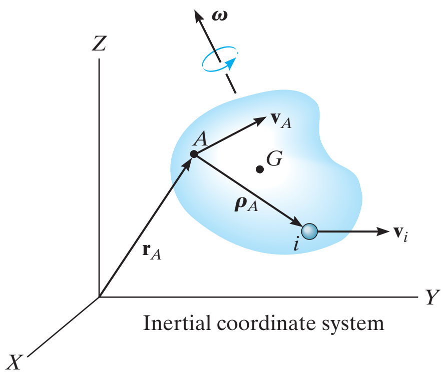


Fig. 21–8

Provided the velocity of an arbitrary point A in the body is known, $\mathbf{v}_i$ can be related to $\mathbf{v}_A$ by the equation $\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega}\times\boldsymbol{\rho}_A$, where $\boldsymbol{\omega}$ is the angular velocity of the body, measured from the X, Y, Z coordinate system, and $\boldsymbol{\rho}_A$ is a position vector extending from A to i. Using this expression, the kinetic energy for the particle can be written as

$$\begin{aligned} T_i &= \tfrac{1}{2}m_i(\mathbf{v}_A + \boldsymbol{\omega}\times\boldsymbol{\rho}_A)\cdot(\mathbf{v}_A + \boldsymbol{\omega}\times\boldsymbol{\rho}_A) \\ &= \tfrac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_A)m_i + \mathbf{v}_A\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_A)m_i + \tfrac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{\rho}_A)\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_A)m_i \end{aligned}$$

The kinetic energy for the entire body is obtained by summing the kinetic energies of all the particles of the body. This requires an integration. Since $m_i \rightarrow dm$, we get

$$T = \tfrac{1}{2}m(\mathbf{v}_A\cdot\mathbf{v}_A) + \mathbf{v}_A\cdot\left(\boldsymbol{\omega}\times\int_m \boldsymbol{\rho}_A\,dm\right) + \tfrac{1}{2}\int_m(\boldsymbol{\omega}\times\boldsymbol{\rho}_A)\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_A)\,dm$$

The last term on the right can be rewritten using the vector identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, where $\mathbf{a} = \boldsymbol{\omega}$, $\mathbf{b} = \boldsymbol{\rho}_A$, and $\mathbf{c} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. The final result is

$$T = \tfrac{1}{2}m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \tfrac{1}{2}\boldsymbol{\omega} \cdot \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \qquad (21\text{–}14)$$

This equation is rarely used because of the computations involving the integrals. Simplification occurs, however, if the reference point A is either a fixed point or the center of mass.

Fixed Point O. If A is a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$, and using Eq. 21–7, we can express Eq. 21–14 as

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_O$$

If the x, y, z axes represent the principal axes of inertia for the body, then $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$ and $\mathbf{H}_O = I_x\omega_x \mathbf{i} + I_y\omega_y \mathbf{j} + I_z\omega_z \mathbf{k}$. Substituting into the above equation and performing the dot-product operations yields

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (21\text{–}15)$$

Center of Mass G. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int \boldsymbol{\rho}_A \, dm = \mathbf{0}$ and, using Eq. 21–8, we can write Eq. 21–14 as

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G$$

In a manner similar to that for a fixed point, the last term on the right side may be represented in scalar form, in which case

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (21\text{–}16)$$

Here it is seen that the kinetic energy consists of two parts; namely, the translational kinetic energy of the mass center, $\tfrac{1}{2}mv_G^2$, and the body's rotational kinetic energy.

Principle of Work and Energy. Having formulated the kinetic energy for a body, the *principle of work and energy* can be applied to solve kinetics problems which involve *force, velocity, and displacement*. For this case only one scalar equation can be written for each body, namely,

$$T_1 + \Sigma U_{1\text{–}2} = T_2 \qquad (21\text{–}17)$$

Before applying this equation, the material in Chapter 18 should be reviewed.

EXAMPLE 21.2

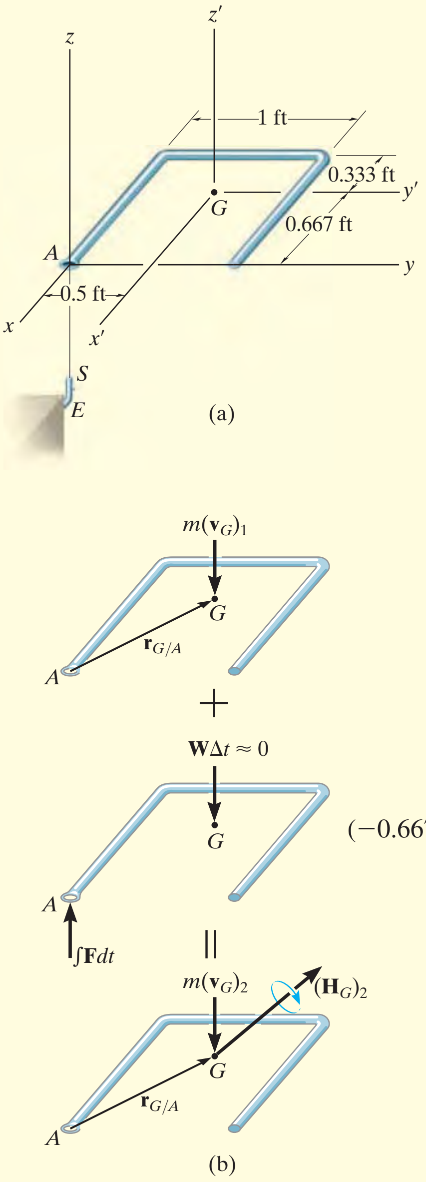


Fig. 21–9

The rod in Fig. 21–9*a* has a weight per unit length of 1.5 lb/ft. Determine its angular velocity just after the end A falls onto the hook at E. The hook provides a permanent connection for the rod due to the spring-lock mechanism S. Just before striking the hook the rod is falling downward with a speed $(v_G)_1 = 10$ ft/s.

SOLUTION

The principle of impulse and momentum will be used since impact occurs.

Impulse and Momentum Diagrams. Fig. 21–9*b*. During the short time Δt, the impulsive force $\mathbf{F}$ acting at A changes the momentum of the rod. (The impulse created by the rod's weight $\mathbf{W}$ during this time is small compared to $\int \mathbf{F}\, dt$, so that it can be neglected, i.e., the weight is a nonimpulsive force.) Hence, the angular momentum of the rod is *conserved* about point A since the moment of $\int \mathbf{F}\, dt$ about A is zero.

Conservation of Angular Momentum. Equation 21–9 must be used to find the angular momentum of the rod, since A does not become a *fixed point* until *after* the impulsive interaction with the hook. Thus, with reference to Fig. 21–9*b*, $(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$, or

$$\mathbf{r}_{G/A} \times m(\mathbf{v}_G)_1 = \mathbf{r}_{G/A} \times m(\mathbf{v}_G)_2 + (\mathbf{H}_G)_2 \qquad (1)$$

From Fig. 21–9*a*, $\mathbf{r}_{G/A} = \{-0.667\mathbf{i} + 0.5\mathbf{j}\}$ ft. Furthermore, the primed axes are principal axes of inertia for the rod because $I_{x'y'} = I_{x'z'} = I_{z'y'} = 0$. Hence, from Eqs. 21–11, $(\mathbf{H}_G)_2 = I_{x'}\omega_x\mathbf{i} + I_{y'}\omega_y\mathbf{j} + I_{z'}\omega_z\mathbf{k}$. The principal moments of inertia are $I_{x'} = 0.0272\ \text{slug}\cdot\text{ft}^2$, $I_{y'} = 0.0155\ \text{slug}\cdot\text{ft}^2$, $I_{z'} = 0.0427\ \text{slug}\cdot\text{ft}^2$ (see Prob. 21–17). Substituting into Eq. 1, we have

$$(-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-10\mathbf{k})\right] = (-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-v_G)_2\mathbf{k}\right] + 0.0272\omega_x\mathbf{i} + 0.0155\omega_y\mathbf{j} + 0.0427\omega_z\mathbf{k}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

$$-0.699 = -0.0699(v_G)_2 + 0.0272\omega_x \qquad (2)$$

$$-0.932 = -0.0932(v_G)_2 + 0.0155\omega_y \qquad (3)$$

$$0 = 0.0427\omega_z \qquad (4)$$

Kinematics. There are four unknowns in the above equations; however, another equation may be obtained by relating $\boldsymbol{\omega}$ to $(\mathbf{v}_G)_2$ using *kinematics*. Since $\omega_z = 0$ (Eq. 4) and after impact the rod rotates about the fixed point A, Eq. 20–3 can be applied, in which case $(\mathbf{v}_G)_2 = \boldsymbol{\omega} \times \mathbf{r}_{G/A}$, or

$$-(v_G)_2\mathbf{k} = (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-0.667\mathbf{i} + 0.5\mathbf{j})$$

$$-(v_G)_2 = 0.5\omega_x + 0.667\omega_y \qquad (5)$$

Solving Eqs. 2, 3 and 5 simultaneously yields

$$(\mathbf{v}_G)_2 = \{-8.41\mathbf{k}\}\ \text{ft/s} \qquad \boldsymbol{\omega} = \{-4.09\mathbf{i} - 9.55\mathbf{j}\}\ \text{rad/s} \qquad \textit{Ans.}$$

21

A 5-N · m torque is applied to the vertical shaft CD shown in Fig. 21–10a, which allows the 10-kg gear A to turn freely about CE. Assuming that gear A starts from rest, determine the angular velocity of CD after it has turned two revolutions. Neglect the mass of shaft CD and axle CE and assume that gear A can be approximated by a thin disk. Gear B is fixed.

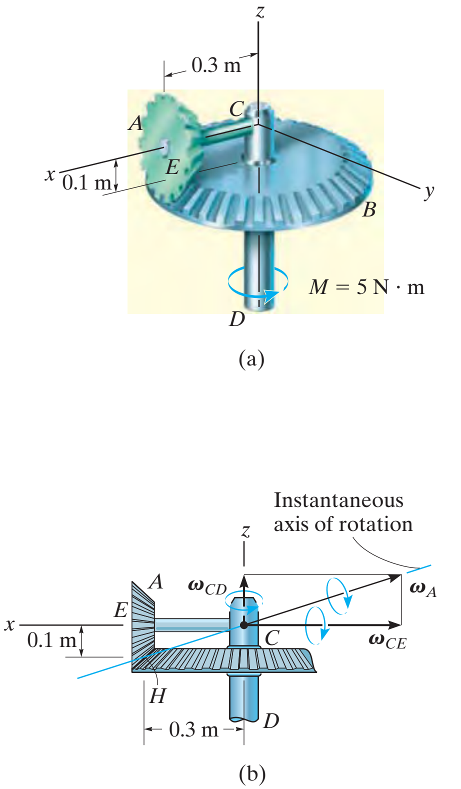


Fig. 21–10

SOLUTION

The principle of work and energy may be used for the solution. Why?

Work. If shaft CD, the axle CE, and gear A are considered as a system of connected bodies, only the applied torque $\mathbf{M}$ does work. For two revolutions of CD, this work is $\Sigma U_{1-2} = (5\ \text{N}\cdot\text{m})(4\pi\ \text{rad}) = 62.83\ \text{J}$.

Kinetic Energy. Since the gear is initially at rest, its initial kinetic energy is zero. A kinematic diagram for the gear is shown in Fig. 21–10b. If the angular velocity of CD is taken as $\boldsymbol{\omega}_{CD}$, then the angular velocity of gear A is $\boldsymbol{\omega}_A = \boldsymbol{\omega}_{CD} + \boldsymbol{\omega}_{CE}$. The gear may be imagined as a portion of a massless extended body which is rotating about the *fixed point C*. The instantaneous axis of rotation for this body is along line CH, because both points C and H on the body (gear) have zero velocity and must therefore lie on this axis. This requires that the components $\boldsymbol{\omega}_{CD}$ and $\boldsymbol{\omega}_{CE}$ be related by the equation $\omega_{CD}/0.1\ \text{m} = \omega_{CE}/0.3\ \text{m}$ or $\omega_{CE} = 3\omega_{CD}$. Thus,

$$\boldsymbol{\omega}_A = -\omega_{CE}\mathbf{i} + \omega_{CD}\mathbf{k} = -3\omega_{CD}\mathbf{i} + \omega_{CD}\mathbf{k} \qquad (1)$$

The x, y, z axes in Fig. 21–10a represent *principal axes of inertia* at C for the gear. Since point C is a fixed point of rotation, Eq. 21–15 may be applied to determine the kinetic energy, i.e.,

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (2)$$

Using the parallel-axis theorem, the moments of inertia of the gear about point C are as follows:

$$I_x = \tfrac{1}{2}(10\ \text{kg})(0.1\ \text{m})^2 = 0.05\ \text{kg}\cdot\text{m}^2$$

$$I_y = I_z = \tfrac{1}{4}(10\ \text{kg})(0.1\ \text{m})^2 + 10\ \text{kg}(0.3\ \text{m})^2 = 0.925\ \text{kg}\cdot\text{m}^2$$

Since $\omega_x = -3\omega_{CD}$, $\omega_y = 0$, $\omega_z = \omega_{CD}$, Eq. 2 becomes

$$T_A = \tfrac{1}{2}(0.05)(-3\omega_{CD})^2 + 0 + \tfrac{1}{2}(0.925)(\omega_{CD})^2 = 0.6875\omega_{CD}^2$$

Principle of Work and Energy. Applying the principle of work and energy, we obtain

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 62.83 = 0.6875\omega_{CD}^2$$

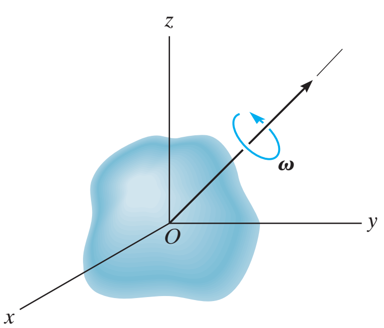
$$\omega_{CD} = 9.56\ \text{rad/s} \qquad \textit{Ans.}$$

PROBLEMS

21–22. If a body contains *no planes of symmetry*, the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$, directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is I, the angular momentum can be expressed as $\mathbf{H} = I\boldsymbol{\omega} = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I\omega_z\mathbf{k}$. The components of $\mathbf{H}$ may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of both expressions for $\mathbf{H}$ and consider ω_x, ω_y, and ω_z to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

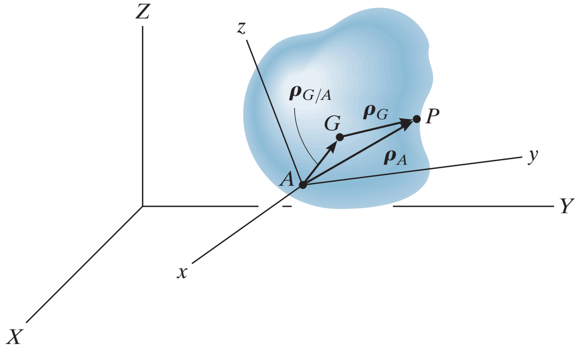
$$I^3 - (I_{xx} + I_{yy} + I_{zz})I^2$$
$$+ (I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)I$$
$$- (I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2) = 0$$

The three positive roots of I, obtained from the solution of this equation, represent the principal moments of inertia I_x, I_y, and I_z.



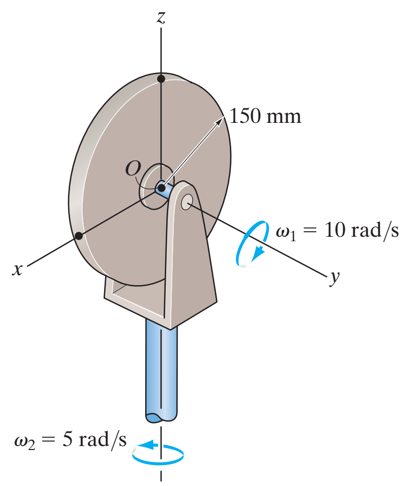
Prob. 21–22

21–23. Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\boldsymbol{\rho}_A = \boldsymbol{\rho}_G + \boldsymbol{\rho}_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \boldsymbol{\rho}_G\, dm = \mathbf{0}$ by definition of the mass center and $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{G/A}$.



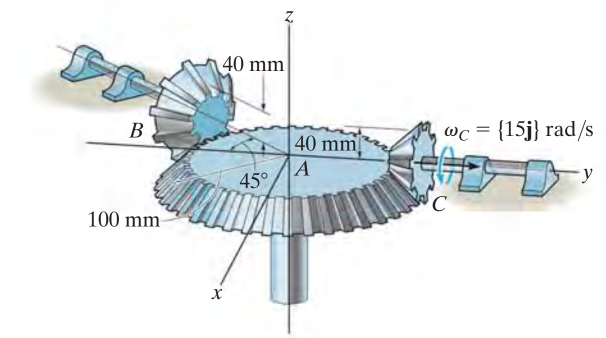
Prob. 21–23

***21–24.** The 15-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 10$ rad/s. Simultaneously, the yoke is rotating with a constant angular velocity of $\omega_2 = 5$ rad/s. Determine the angular momentum of the disk about its center of mass O, and its kinetic energy.



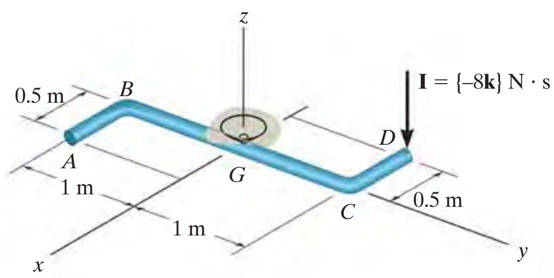
Prob. 21–24

21–25. The large gear has a mass of 5 kg and a radius of gyration of $k_z = 75$ mm. Gears B and C each have a mass of 200 g and a radius of gyration about the axis of their connecting shaft of 15 mm. If the gears are in mesh and C has an angular velocity of $\boldsymbol{\omega}_c = \{15\mathbf{j}\}$ rad/s, determine the total angular momentum for the system of three gears about point A.



Prob. 21–25

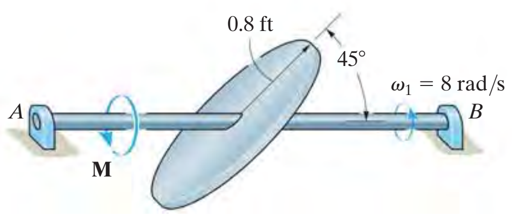
***21–28.** The rod assembly is supported at G by a ball-and-socket joint. Each segment has a mass of 0.5 kg/m. If the assembly is originally at rest and an impulse of $\mathbf{I} = \{-8\mathbf{k}\}$ N·s is applied at D, determine the angular velocity of the assembly just after the impact.



Prob. 21–28

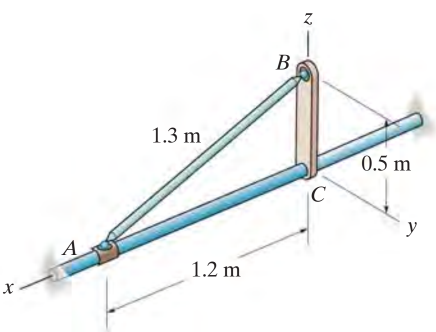
21–26. The circular disk has a weight of 15 lb and is mounted on the shaft AB at an angle of 45° with the horizontal. Determine the angular velocity of the shaft when $t = 3$ s if a constant torque $M = 2$ lb·ft is applied to the shaft. The shaft is originally spinning at $\omega_1 = 8$ rad/s when the torque is applied.

21–27. The circular disk has a weight of 15 lb and is mounted on the shaft AB at an angle of 45° with the horizontal. Determine the angular velocity of the shaft when $t = 2$ s if a torque $M = (4e^{0.1t})$ lb·ft, where t is in seconds, is applied to the shaft. The shaft is originally spinning at $\omega_1 = 8$ rad/s when the torque is applied.



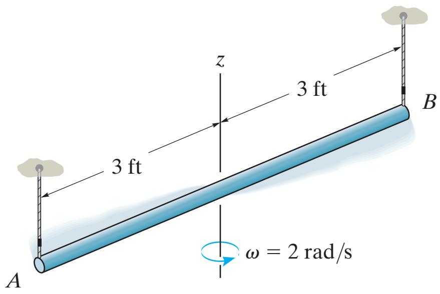
Probs. 21–26/27

21–29. The 4-lb rod AB is attached to the 1-lb collar at A and a 2-lb link BC using ball-and-socket joints. If the rod is released from rest in the position shown, determine the angular velocity of the link after it has rotated 180°.



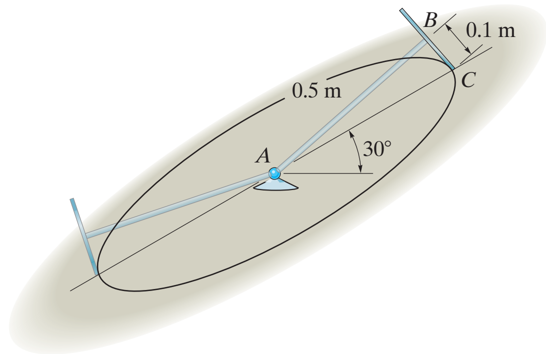
Prob. 21–29

21–30. The rod weighs 3 lb/ft and is suspended from parallel cords at A and B. If the rod has an angular velocity of 2 rad/s about the z axis at the instant shown, determine how high the center of the rod rises at the instant the rod momentarily stops swinging.



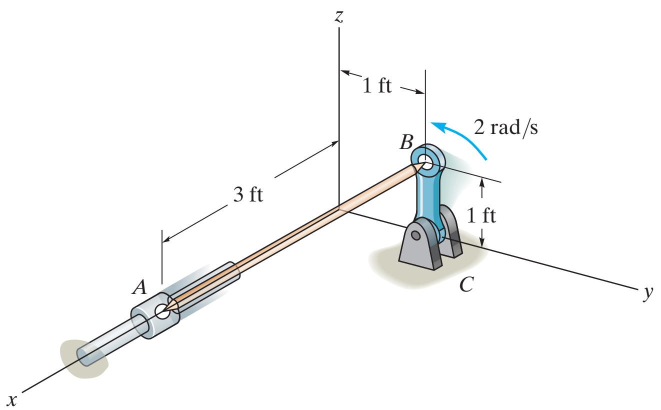
Prob. 21–30

***21–32.** The 2-kg thin disk is connected to the slender rod which is fixed to the ball-and-socket joint at A. If it is released from rest in the position shown, determine the spin of the disk about the rod when the disk reaches its lowest position. Neglect the mass of the rod. The disk rolls without slipping.



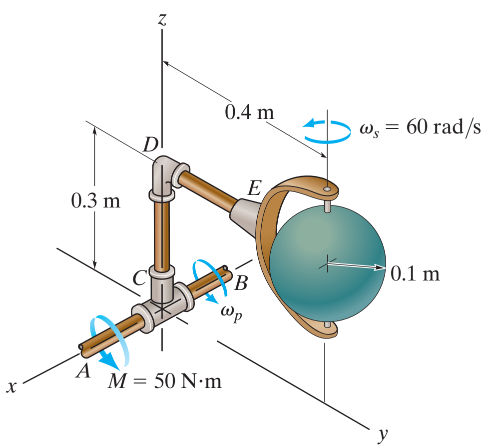
Prob. 21–32

21–31. The 4-lb rod AB is attached to the rod BC and collar A using ball-and-socket joints. If BC has a constant angular velocity of 2 rad/s, determine the kinetic energy of AB when it is in the position shown. Assume the angular velocity of AB is directed perpendicular to the axis of AB.



Prob. 21–31

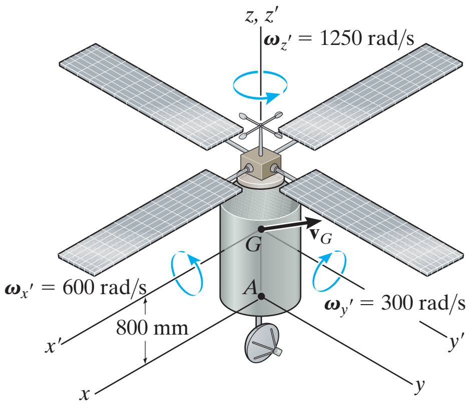
21–33. The 20-kg sphere rotates about the axle with a constant angular velocity of $\omega_s = 60$ rad/s. If shaft AB is subjected to a torque of $M = 50$ N·m, causing it to rotate, determine the value of ω_p after the shaft has turned 90° from the position shown. Initially, $\omega_p = 0$. Neglect the mass of arm CDE.



Prob. 21–33

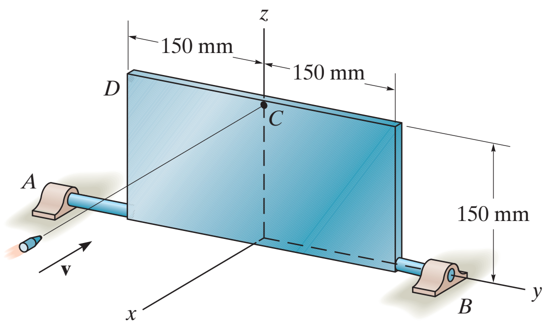
21–34. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the angular momentum of the satellite about point A at this instant.

21–35. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.



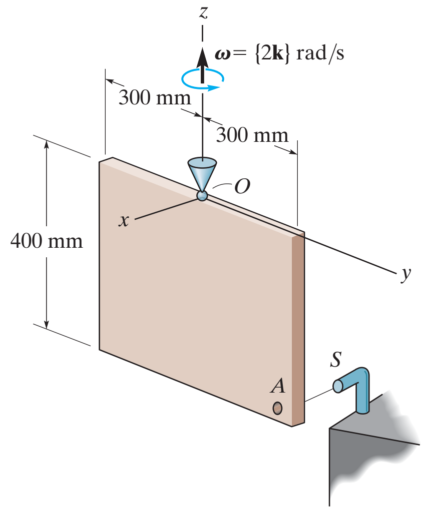
Probs. 21–34/35

***21–36.** The 15-kg rectangular plate is free to rotate about the y axis because of the bearing supports at A and B. When the plate is balanced in the vertical plane, a 3-g bullet is fired into it, perpendicular to its surface, with a velocity $\mathbf{v} = \{-2000\mathbf{i}\}$ m/s. Compute the angular velocity of the plate at the instant it has rotated 180°. If the bullet strikes corner D with the same velocity $\mathbf{v}$, instead of at C, does the angular velocity remain the same? Why or why not?



Prob. 21–36

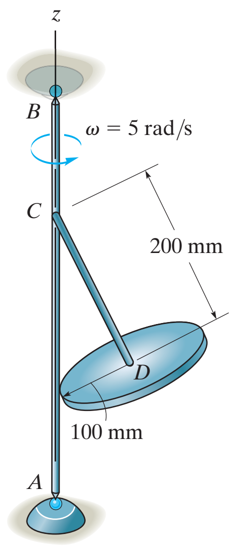
21–37. The 5-kg thin plate is suspended at O using a ball-and-socket joint. It is rotating with a constant angular velocity $\boldsymbol{\omega} = \{2\mathbf{k}\}$ rad/s when the corner A strikes the hook at S, which provides a permanent connection. Determine the angular velocity of the plate immediately after impact.



Prob. 21–37

21–38. Determine the kinetic energy of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.

21–39. Determine the angular momentum $\mathbf{H}_z$ of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.



Probs. 21–38/39

*21.4 Equations of Motion

Having become familiar with the techniques used to describe both the inertial properties and the angular momentum of a body, we can now write the equations which describe the motion of the body in their most useful forms.

Equations of Translational Motion. The *translational motion* of a body is defined in terms of the acceleration of the body's mass center, which is measured from an inertial *X, Y, Z* reference. The equation of translational motion for the body can be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_G \tag{21–18}$$

or by the three scalar equations

$$\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma F_z &= m(a_G)_z \end{aligned} \tag{21–19}$$

Here, $\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}$ represents the sum of all the external forces acting on the body.

Equations of Rotational Motion. In Sec. 15.6, we developed Eq. 15–17, namely,

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{21–20}$$

which states that the sum of the moments of all the external forces acting on a system of particles (contained in a rigid body) about a fixed point O is equal to the time rate of change of the total angular momentum of the body about point O. When moments of the external forces acting on the particles are summed about the system's *mass center G*, one again obtains the same simple form of Eq. 21–20, relating the moment summation $\Sigma \mathbf{M}_G$ to the angular momentum $\mathbf{H}_G$. To show this, consider the system of particles in Fig. 21–11, where *X, Y, Z* represents an inertial frame of reference and the *x, y, z* axes, with origin at *G*, *translate* with respect to this frame. In general, *G* is *accelerating*, so by definition the translating frame is *not* an inertial reference. The angular momentum of the *i*th particle with respect to this frame is, however,

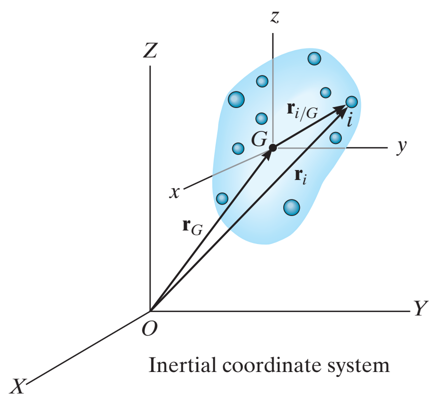


Fig. 21–11

$$(\mathbf{H}_i)_G = \mathbf{r}_{i/G} \times m_i \mathbf{v}_{i/G}$$

where $\mathbf{r}_{i/G}$ and $\mathbf{v}_{i/G}$ represent the position and velocity of the *i*th particle with respect to *G*. Taking the time derivative we have

$$(\dot{\mathbf{H}}_i)_G = \dot{\mathbf{r}}_{i/G} \times m_i \mathbf{v}_{i/G} + \mathbf{r}_{i/G} \times m_i \dot{\mathbf{v}}_{i/G}$$

By definition, $\mathbf{v}_{i/G} = \dot{\mathbf{r}}_{i/G}$. Thus, the first term on the right side is zero since the cross product of the same vectors is zero. Also, $\mathbf{a}_{i/G} = \dot{\mathbf{v}}_{i/G}$, so that

$$(\dot{\mathbf{H}}_i)_G = (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Similar expressions can be written for the other particles of the body. When the results are summed, we get

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Here $\dot{\mathbf{H}}_G$ is the time rate of change of the total angular momentum of the body computed about point G.

The relative acceleration for the ith particle is defined by the equation $\mathbf{a}_{i/G} = \mathbf{a}_i - \mathbf{a}_G$, where $\mathbf{a}_i$ and $\mathbf{a}_G$ represent, respectively, the accelerations of the ith particle and point G measured with respect to the *inertial frame of reference*. Substituting and expanding, using the distributive property of the vector cross product, yields

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_i) - (\Sigma m_i \mathbf{r}_{i/G}) \times \mathbf{a}_G$$

By definition of the mass center, the sum $(\Sigma m_i \mathbf{r}_{i/G}) = (\Sigma m_i)\bar{\mathbf{r}}$ is equal to zero, since the position vector $\bar{\mathbf{r}}$ relative to G is zero. Hence, the last term in the above equation is zero. Using the equation of motion, the product $m_i \mathbf{a}_i$ can be replaced by the resultant *external force* $\mathbf{F}_i$ acting on the ith particle. Denoting $\Sigma\mathbf{M}_G = \Sigma(\mathbf{r}_{i/G} \times \mathbf{F}_i)$, the final result can be written as

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (21\text{–}21)$$

The rotational equation of motion for the body will now be developed from either Eq. 21–20 or 21–21. In this regard, the scalar components of the angular momentum $\mathbf{H}_O$ or $\mathbf{H}_G$ are defined by Eqs. 21–10 or, if principal axes of inertia are used either at point O or G, by Eqs. 21–11. If these components are computed about x, y, z axes that are *rotating* with an angular velocity $\boldsymbol{\Omega}$ that is *different* from the body's angular velocity $\boldsymbol{\omega}$, then the time derivative $\dot{\mathbf{H}} = d\mathbf{H}/dt$, as used in Eqs. 21–20 and 21–21, must account for the rotation of the x, y, z axes as measured from the inertial X, Y, Z axes. This requires application of Eq. 20–6, in which case Eqs. 21–20 and 21–21 become

$$\begin{aligned} \Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \\ \Sigma\mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \end{aligned} \qquad (21\text{–}22)$$

Here $(\dot{\mathbf{H}})_{xyz}$ is the time rate of change of $\mathbf{H}$ measured from the x, y, z reference.

There are three ways in which one can define the motion of the x, y, z axes. Obviously, motion of this reference should be chosen so that it will yield the simplest set of moment equations for the solution of a particular problem.

x, y, z Axes Having Motion $\boldsymbol{\Omega} = \mathbf{0}$. If the body has general motion, the x, y, z axes can be chosen with origin at G, such that the axes only *translate* relative to the inertial X, Y, Z frame of reference. Doing this simplifies Eq. 21–22, since $\boldsymbol{\Omega} = \mathbf{0}$. However, the body may have a rotation $\boldsymbol{\omega}$ about these axes, and therefore the moments and products of inertia of the body would have to be expressed as *functions of time*. In most cases this would be a difficult task, so that such a choice of axes has restricted application.

x, y, z Axes Having Motion $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The x, y, z axes can be chosen such that they are *fixed in and move with the body*. The moments and products of inertia of the body relative to these axes will then be *constant* during the motion. Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, Eqs. 21–22 become

$$\begin{aligned} \Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ \Sigma\mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_G \end{aligned} \qquad (21\text{–}23)$$

We can express each of these vector equations as three scalar equations using Eqs. 21–10. Neglecting the subscripts O and G yields

$$\begin{aligned} \Sigma M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z - I_{xy}(\dot{\omega}_y - \omega_z\omega_x) \\ &\quad - I_{yz}(\omega_y^2 - \omega_z^2) - I_{zx}(\dot{\omega}_z + \omega_x\omega_y) \\ \Sigma M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) \\ &\quad - I_{zx}(\omega_z^2 - \omega_x^2) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) \\ \Sigma M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y - I_{zx}(\dot{\omega}_x - \omega_y\omega_z) \\ &\quad - I_{xy}(\omega_x^2 - \omega_y^2) - I_{yz}(\dot{\omega}_y + \omega_z\omega_x) \end{aligned} \qquad (21\text{–}24)$$

If the x, y, z axes are chosen as *principal axes of inertia*, the products of inertia are zero, $I_{xx} = I_x$, etc., and the above equations become

$$\begin{aligned} \Sigma M_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z \\ \Sigma M_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x \\ \Sigma M_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y \end{aligned} \qquad (21\text{–}25)$$

This set of equations is known historically as the *Euler equations of motion*, named after the Swiss mathematician Leonhard Euler, who first developed them. They apply *only* for moments summed about either point O or G.

When applying these equations it should be realized that $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ represent the time derivatives of the magnitudes of the *x, y, z* components of $\boldsymbol{\omega}$ as observed from *x, y, z*. To determine these components, it is first necessary to find ω_x, ω_y, ω_z when the *x, y, z* axes are oriented in a *general position* and *then* take the time derivative of the magnitude of these components, i.e., $(\dot{\boldsymbol{\omega}})_{xyz}$. However, since the *x, y, z* axes are rotating at $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then from Eq. 20–6, it should be noted that $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\omega}$. Since $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This important result indicates that the time derivative of $\boldsymbol{\omega}$ with respect to the fixed *X, Y, Z* axes, that is $\dot{\boldsymbol{\omega}}$, can also be used to obtain $(\dot{\boldsymbol{\omega}})_{xyz}$. Generally this is the easiest way to determine the result. See Example 21.5.

x, y, z Axes Having Motion $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$. To simplify the calculations for the time derivative of $\boldsymbol{\omega}$, it is often convenient to choose the *x, y, z* axes having an angular velocity $\boldsymbol{\Omega}$ which is different from the angular velocity $\boldsymbol{\omega}$ of the body. This is particularly suitable for the analysis of spinning tops and gyroscopes which are *symmetrical* about their spinning axes.* When this is the case, the moments and products of inertia remain constant about the axis of spin.

Equations 21–22 are applicable for such a set of axes. Each of these two vector equations can be reduced to a set of three scalar equations which are derived in a manner similar to Eqs. 21–25,† i.e.,

$$\begin{aligned}\Sigma M_x &= I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z\\ \Sigma M_y &= I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x\\ \Sigma M_z &= I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y\end{aligned} \tag{21–26}$$

Here Ω_x, Ω_y, Ω_z represent the *x, y, z* components of $\boldsymbol{\Omega}$, measured from the inertial frame of reference, and $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ must be determined relative to the *x, y, z* axes that have the rotation $\boldsymbol{\Omega}$. See Example 21.6.

Any one of these sets of moment equations, Eqs. 21–24, 21–25, or 21–26, represents a series of three first-order nonlinear differential equations. These equations are "coupled," since the angular-velocity components are present in all the terms. Success in determining the solution for a particular problem therefore depends upon what is unknown in these equations. Difficulty certainly arises when one attempts to solve for the unknown components of $\boldsymbol{\omega}$ when the external moments are functions of time. Further complications can arise if the moment equations are coupled to the three scalar equations of translational motion, Eqs. 21–19. This can happen because of the existence of kinematic constraints which relate the rotation of the body to the translation of its mass center, as in the case of a hoop which rolls

*A detailed discussion of such devices is given in Sec. 21.5.
†See Prob. 21–42.

without slipping. Problems that require the simultaneous solution of differential equations are generally solved using numerical methods with the aid of a computer. In many engineering problems, however, we are given information about the motion of the body and are required to determine the applied moments acting on the body. Most of these problems have direct solutions, so that there is no need to resort to computer techniques.

Procedure for Analysis

Problems involving the three-dimensional motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Draw a *free-body diagram* of the body at the instant considered and specify the x, y, z coordinate system. The origin of this reference must be located either at the body's mass center G, or at point O, considered fixed in an inertial reference frame and located either in the body or on a massless extension of the body.
- Unknown reactive force components can be shown having a positive sense of direction.
- Depending on the nature of the problem, decide what type of rotational motion $\mathbf{\Omega}$ the x, y, z coordinate system should have, i.e., $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{\Omega} = \boldsymbol{\omega}$, or $\mathbf{\Omega} \neq \boldsymbol{\omega}$. When choosing, keep in mind that the moment equations are simplified when the axes move in such a manner that they represent principal axes of inertia for the body at all times.
- Compute the necessary moments and products of inertia for the body relative to the x, y, z axes.

Kinematics.

- Determine the x, y, z components of the body's angular velocity and find the time derivatives of $\boldsymbol{\omega}$.
- Note that if $\mathbf{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore we can either find the time derivative of $\boldsymbol{\omega}$ with respect to the X, Y, Z axes, $\dot{\boldsymbol{\omega}}$, and then determine its components $\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z$, or we can find the components of $\boldsymbol{\omega}$ along the x, y, z axes, when the axes are oriented in a general position, and then take the time derivative of the magnitudes of these components, $(\dot{\boldsymbol{\omega}})_{xyz}$.

Equations of Motion.

- Apply either the two vector equations 21–18 and 21–22 or the six scalar component equations appropriate for the x, y, z coordinate axes chosen for the problem.

21

The gear shown in Fig. 21–12*a* has a mass of 10 kg and is mounted at an angle of 10° with the rotating shaft having negligible mass. If $I_z = 0.1\ \text{kg}\cdot\text{m}^2$, $I_x = I_y = 0.05\ \text{kg}\cdot\text{m}^2$, and the shaft is rotating with a constant angular velocity of $\omega = 30$ rad/s, determine the components of reaction that the thrust bearing *A* and journal bearing *B* exert on the shaft at the instant shown.

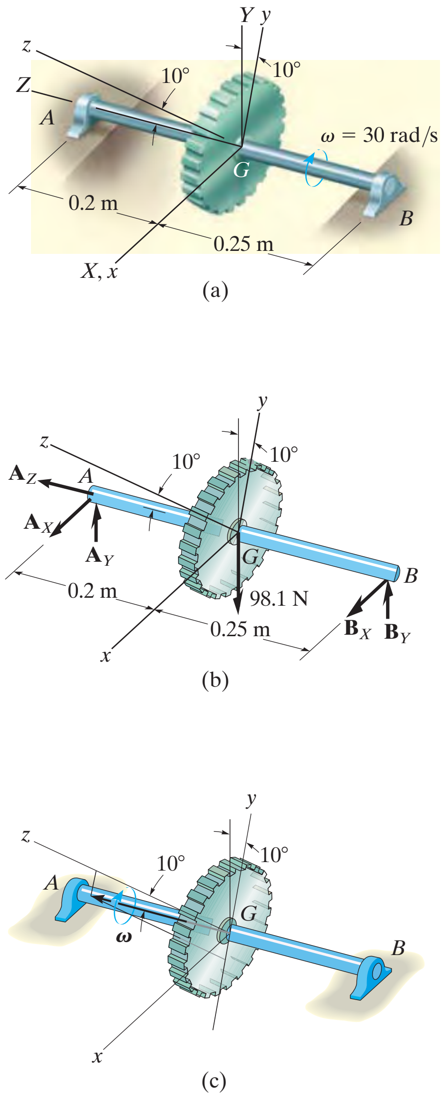


Fig. 21–12

SOLUTION

Free-Body Diagram. Fig. 21–12*b*. The origin of the *x*, *y*, *z* coordinate system is located at the gear's center of mass *G*, which is also a fixed point. The axes are fixed in and rotate with the gear so that these axes will then always represent the principal axes of inertia for the gear. Hence $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

Kinematics. As shown in Fig. 21–12*c*, the angular velocity $\boldsymbol{\omega}$ of the gear is constant in magnitude and is always directed along the axis of the shaft *AB*. Since this vector is measured from the *X*, *Y*, *Z* inertial frame of reference, for any position of the *x*, *y*, *z* axes,

$$\omega_x = 0 \quad \omega_y = -30 \sin 10^\circ \quad \omega_z = 30 \cos 10^\circ$$

These components remain constant for any general orientation of the *x*, *y*, *z* axes, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Also note that since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore, we can find these time derivatives relative to the *X*, *Y*, *Z* axes. In this regard $\boldsymbol{\omega}$ has a constant magnitude and direction (+*Z*) since $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Furthermore, since *G* is a fixed point, $(a_G)_x = (a_G)_y = (a_G)_z = 0$.

Equations of Motion. Applying Eqs. 21–25 ($\boldsymbol{\Omega} = \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-(A_Y)(0.2) + (B_Y)(0.25) = 0 - (0.05 - 0.1)(-30 \sin 10^\circ)(30 \cos 10^\circ)$$

$$-0.2A_Y + 0.25B_Y = -7.70 \qquad (1)$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$A_X(0.2) \cos 10^\circ - B_X(0.25) \cos 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \qquad (2)$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_X(0.2) \sin 10^\circ - B_X(0.25) \sin 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \text{ (check)}$$

Applying Eqs. 21–19, we have

$\Sigma F_X = m(a_G)_X;$ $\quad A_X + B_X = 0 \qquad (3)$

$\Sigma F_Y = m(a_G)_Y;$ $\quad A_Y + B_Y - 98.1 = 0 \qquad (4)$

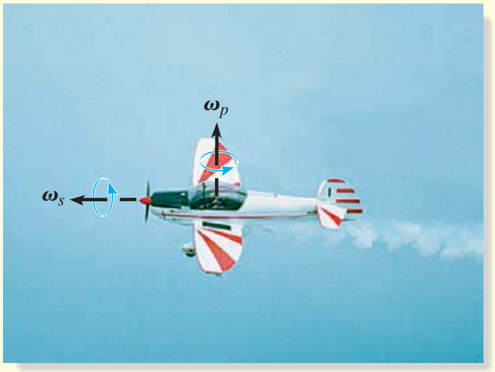
$\Sigma F_Z = m(a_G)_Z;$ $\quad A_Z = 0 \qquad$ *Ans.*

Solving Eqs. 1 through 4 simultaneously gives

$$A_X = B_X = 0 \quad A_Y = 71.6\ \text{N} \quad B_Y = 26.5\ \text{N} \qquad \textit{Ans.}$$

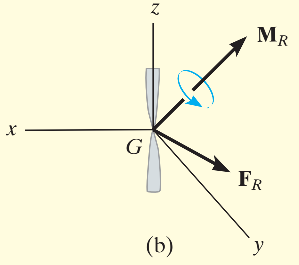
EXAMPLE 21.5

The airplane shown in Fig. 21–13*a* is in the process of making a steady *horizontal* turn at the rate of ω_p. During this motion, the propeller is spinning at the rate of ω_s. If the propeller has two blades, determine the moments which the propeller shaft exerts on the propeller at the instant the blades are in the vertical position. For simplicity, assume the blades to be a uniform slender bar having a moment of inertia I about an axis perpendicular to the blades passing through the center of the bar, and having zero moment of inertia about a longitudinal axis.

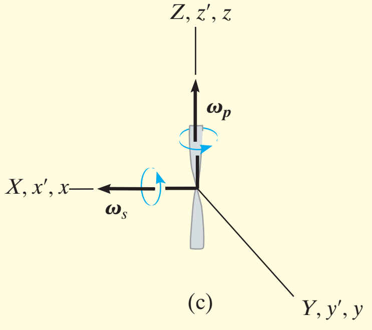


(© R.C. Hibbeler)

(a)



(b)



(c)

Fig. 21–13

SOLUTION

Free-Body Diagram. Fig. 21–13*b*. The reactions of the connecting shaft on the propeller are indicated by the resultants $\mathbf{F}_R$ and $\mathbf{M}_R$. (The propeller's weight is assumed to be negligible.) The *x, y, z* axes will be taken fixed to the propeller, since these axes always represent the principal axes of inertia for the propeller. Thus, $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The moments of inertia I_x and I_y are equal ($I_x = I_y = I$) and $I_z = 0$.

Kinematics. The angular velocity of the propeller observed from the *X, Y, Z* axes, coincident with the *x, y, z* axes, Fig. 21–13*c*, is $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \omega_s \mathbf{i} + \omega_p \mathbf{k}$, so that the *x, y, z* components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$

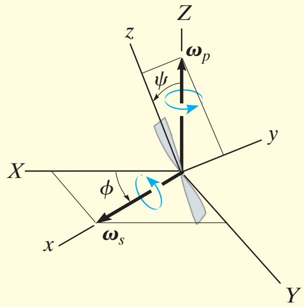
Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. To find $\dot{\boldsymbol{\omega}}$, which is the time derivative with respect to the fixed *X, Y, Z* axes, we can use Eq. 20–6 since $\boldsymbol{\omega}$ changes direction relative to *X, Y, Z*. The time rate of change of each of these components $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p$ relative to the *X, Y, Z* axes can be obtained by introducing a third coordinate system x', y', z', which has an angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$ and is coincident with the *X, Y, Z* axes at the instant shown. Thus

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}$$
$$= (\dot{\boldsymbol{\omega}}_s)_{x'y'z'} + (\dot{\boldsymbol{\omega}}_p)_{x'y'z'} + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_s + \boldsymbol{\omega}_p)$$
$$= \mathbf{0} + \mathbf{0} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p$$
$$= \mathbf{0} + \mathbf{0} + \omega_p \mathbf{k} \times \omega_s \mathbf{i} + \mathbf{0} = \omega_p \omega_s \mathbf{j}$$

Since the X, Y, Z axes are coincident with the x, y, z axes at the instant shown, the components of $\dot{\boldsymbol{\omega}}$ along x, y, z are therefore

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \omega_s \qquad \dot{\omega}_z = 0$$

These same results can also be determined by direct calculation of $(\dot{\boldsymbol{\omega}})_{xyz}$; however, this will involve a bit more work. To do this, it will be necessary to view the propeller (or the x, y, z axes) in some *general position* such as shown in Fig. 21–13*d*. Here the plane has turned through an angle ϕ (phi) and the propeller has turned through an angle ψ (psi) relative to the plane. Notice that $\boldsymbol{\omega}_p$ is always directed along the fixed Z axis and $\boldsymbol{\omega}_s$ follows the x axis. Thus the general components of $\boldsymbol{\omega}$ are



(d)

Fig. 21–13

$$\omega_x = \omega_s \qquad \omega_y = \omega_p \sin\psi \qquad \omega_z = \omega_p \cos\psi$$

Since ω_s and ω_p are constant, the time derivatives of these components become

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \cos\psi\, \dot{\psi} \qquad \omega_z = -\omega_p \sin\psi\, \dot{\psi}$$

But $\phi = \psi = 0°$ and $\dot{\psi} = \omega_s$ at the instant considered. Thus,

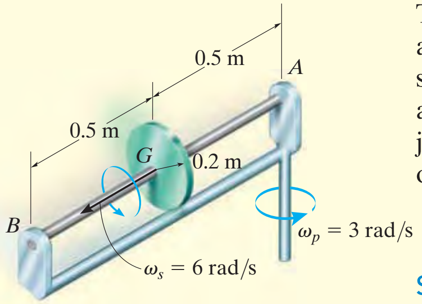
$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$
$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \omega_s \qquad \dot{\omega}_z = 0$$

which are the same results as those obtained previously.

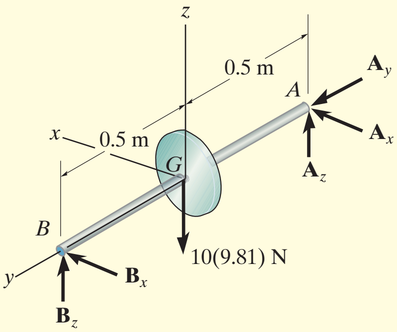
Equations of Motion. Using Eqs. 21–25, we have

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z = I(0) - (I - 0)(0)\omega_p$$
$$M_x = 0 \qquad \textit{Ans.}$$
$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x = I(\omega_p \omega_s) - (0 - I)\omega_p \omega_s$$
$$M_y = 2I\omega_p \omega_s \qquad \textit{Ans.}$$
$$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y = 0(0) - (I - I)\omega_s(0)$$
$$M_z = 0 \qquad \textit{Ans.}$$

EXAMPLE 21.6



(a)



(b)

Fig. 21–14

The 10-kg flywheel (or thin disk) shown in Fig. 21–14*a* rotates (spins) about the shaft at a constant angular velocity of $\omega_s = 6$ rad/s. At the same time, the shaft rotates (precessing) about the bearing at A with an angular velocity of $\omega_p = 3$ rad/s. If A is a thrust bearing and B is a journal bearing, determine the components of force reaction at each of these supports due to the motion.

SOLUTION I

Free-Body Diagram. Fig. 21–14*b*. The origin of the x, y, z coordinate system is located at the center of mass G of the flywheel. Here we will let these coordinates have an angular velocity of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{3\mathbf{k}\}$ rad/s. Although the wheel spins relative to these axes, the moments of inertia *remain constant*,* i.e.,

$$I_x = I_z = \tfrac{1}{4}(10\text{ kg})(0.2\text{ m})^2 = 0.1\text{ kg}\cdot\text{m}^2$$
$$I_y = \tfrac{1}{2}(10\text{ kg})(0.2\text{ m})^2 = 0.2\text{ kg}\cdot\text{m}^2$$

Kinematics. From the coincident inertial X, Y, Z frame of reference, Fig. 21–14*c*, the flywheel has an angular velocity of $\boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, so that

$$\omega_x = 0 \quad \omega_y = 6\text{ rad/s} \quad \omega_z = 3\text{ rad/s}$$

The time derivative of $\boldsymbol{\omega}$ must be determined relative to the x, y, z axes. In this case both $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_s$ do not change their magnitude or direction, and so

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

Equations of Motion. Applying Eqs. 21–26 ($\boldsymbol{\Omega} \neq \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$-A_z(0.5) + B_z(0.5) = 0 - (0.2)(3)(6) + 0 = -3.6$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$0 = 0 - 0 + 0$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$
$$A_x(0.5) - B_x(0.5) = 0 - 0 + 0$$

*This would not be true for the propeller in Example 21.5.

Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_x + B_x = 0$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_y = -10(0.5)(3)^2$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_z + B_z - 10(9.81) = 0$$

Solving these equations, we obtain

$$A_x = 0 \quad A_y = -45.0\text{ N} \quad A_z = 52.6\text{ N} \qquad \textit{Ans.}$$
$$B_x = 0 \qquad\qquad B_z = 45.4\text{ N} \qquad \textit{Ans.}$$

NOTE: If the precession $\boldsymbol{\omega}_p$ had not occurred, the z component of force at A and B would be equal to 49.05 N. In this case, however, the difference in these components is caused by the "gyroscopic moment" created whenever a spinning body precesses about another axis. We will study this effect in detail in the next section.

SOLUTION II

This example can also be solved using Euler's equations of motion, Eqs. 21–25. In this case $\boldsymbol{\Omega} = \boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, and the time derivative $(\dot{\boldsymbol{\omega}})_{xyz}$ can be conveniently obtained with reference to the fixed X, Y, Z axes since $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This calculation can be performed by choosing x', y', z' axes to have an angular velocity of $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$, Fig. 21–14$c$, so that

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega} = \mathbf{0} + 3\mathbf{k} \times (6\mathbf{j} + 3\mathbf{k}) = \{-18\mathbf{i}\}\text{ rad/s}^2$$

$$\dot{\omega}_x = -18\text{ rad/s} \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

The moment equations then become

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-A_z(0.5) + B_z(0.5) = 0.1(-18) - (0.2 - 0.1)(6)(3) = -3.6$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$0 = 0 - 0$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_x(0.5) - B_x(0.5) = 0 - 0$$

The solution then proceeds as before.

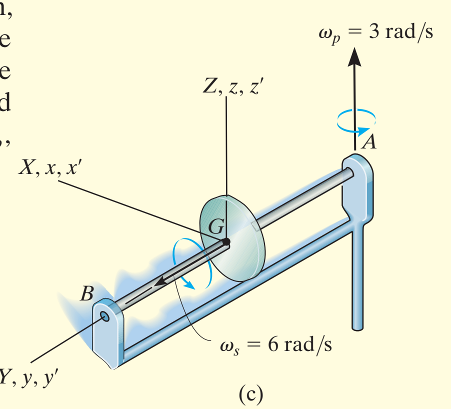


Fig. 21–14

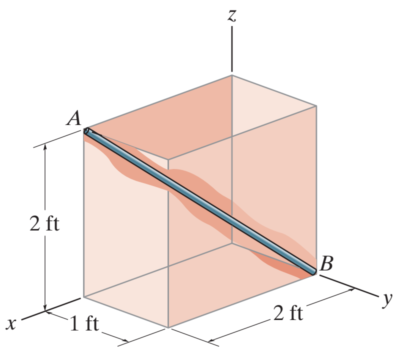
PROBLEMS

***21–40.** Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *not constant* with respect to time.

21–41. Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *constant* with respect to time.

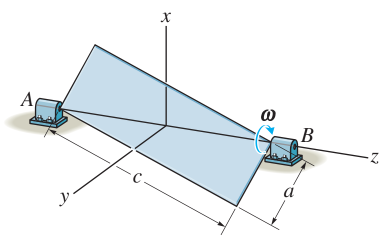
21–42. Derive the Euler equations of motion for $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, i.e., Eqs. 21–26.

21–43. The 4-lb bar rests along the smooth corners of an open box. At the instant shown, the box has a velocity $\mathbf{v} = \{3\mathbf{j}\}$ ft/s and an acceleration $\mathbf{a} = \{-6\mathbf{j}\}$ ft/s^2. Determine the x, y, z components of force which the corners exert on the bar.



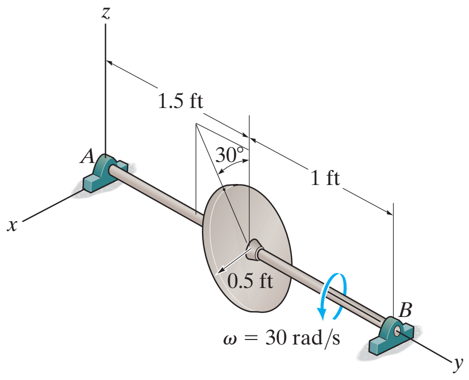
Prob. 21–43

***21–44.** The uniform plate has a mass of $m = 2$ kg and is given a rotation of $\omega = 4$ rad/s about its bearings at A and B. If $a = 0.2$ m and $c = 0.3$ m, determine the vertical reactions at the instant shown. Use the x, y, z axes shown and note that $I_{zx} = -\left(\frac{mac}{12}\right)\left(\frac{c^2 - a^2}{c^2 + a^2}\right)$.



Prob. 21–44

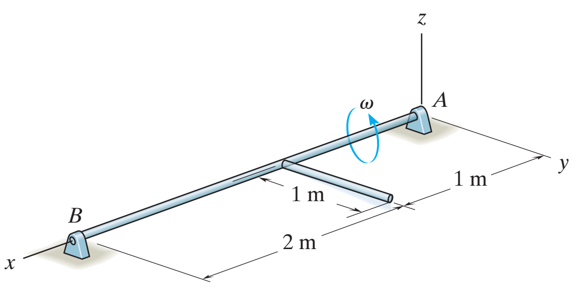
21–45. If the shaft AB is rotating with a constant angular velocity of $\omega = 30$ rad/s, determine the X, Y, Z components of reaction at the thrust bearing A and journal bearing B at the instant shown. The disk has a weight of 15 lb. Neglect the weight of the shaft AB.



Prob. 21–45

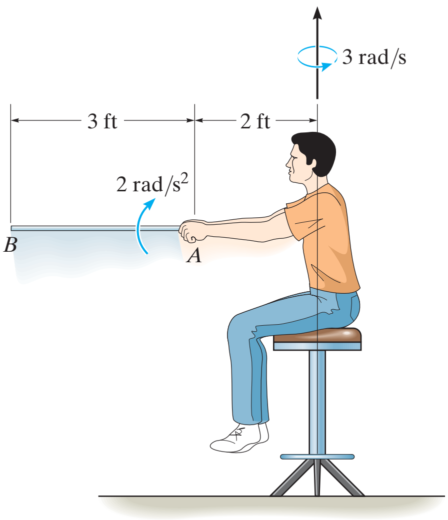
21–46. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft is rotating in the direction shown at $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings when the assembly is in the position shown. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.

21–47. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft A is subjected to a couple moment $\mathbf{M} = \{40\mathbf{i}\}$ N·m, and at the instant shown the shaft has an angular velocity of $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings of the assembly at this instant. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.



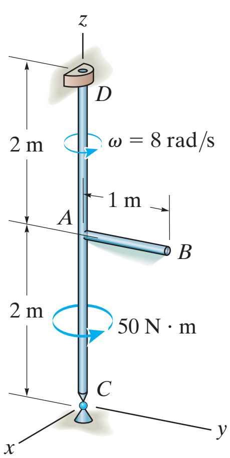
Probs. 21–46/47

***21–48.** The man sits on a swivel chair which is rotating with a constant angular velocity of 3 rad/s. He holds the uniform 5-lb rod AB horizontal. He suddenly gives it an angular acceleration of 2 rad/s^2, measured relative to him, as shown. Determine the required force and moment components at the grip, A, necessary to do this. Establish axes at the rod's center of mass G, with $+z$ upward, and $+y$ directed along the axis of the rod toward A.



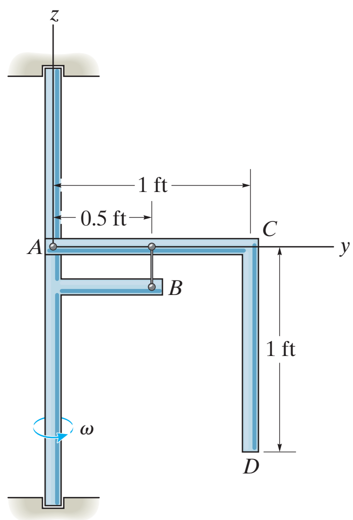
Prob. 21–48

21–49. The rod assembly is supported by a ball-and-socket joint at C and a journal bearing at D, which develops only x and y force reactions. The rods have a mass of 0.75 kg/m. Determine the angular acceleration of the rods and the components of reaction at the supports at the instant $\omega = 8$ rad/s as shown.



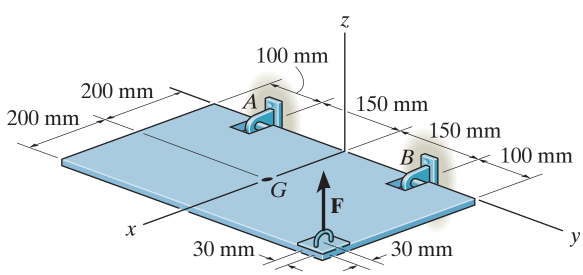
Prob. 21–49

21–50. The bent uniform rod ACD has a weight of 5 lb/ft and is supported at A by a pin and at B by a cord. If the vertical shaft rotates with a constant angular velocity $\omega = 20$ rad/s, determine the x, y, z components of force and moment developed at A and the tension in the cord.



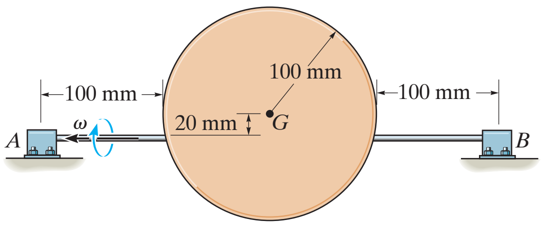
Prob. 21–50

21–51. The uniform hatch door, having a mass of 15 kg and a mass center at G, is supported in the horizontal plane by bearings at A and B. If a vertical force $F = 300$ N is applied to the door as shown, determine the components of reaction at the bearings and the angular acceleration of the door. The bearing at A will resist a component of force in the y direction, whereas the bearing at B will not. For the calculation, assume the door to be a thin plate and neglect the size of each bearing. The door is originally at rest.



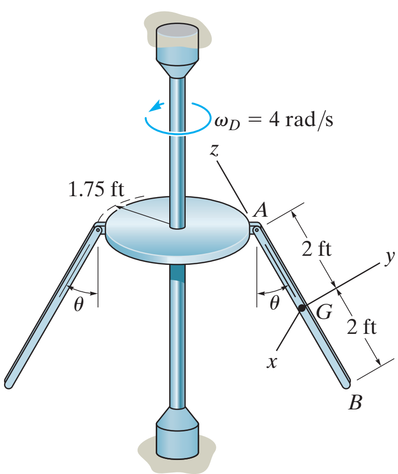
Prob. 21–51

***21–52.** The 5-kg circular disk is mounted off center on a shaft which is supported by bearings at A and B. If the shaft is rotating at a constant rate of $\omega = 10 \text{ rad/s}$, determine the vertical reactions at the bearings when the disk is in the position shown.



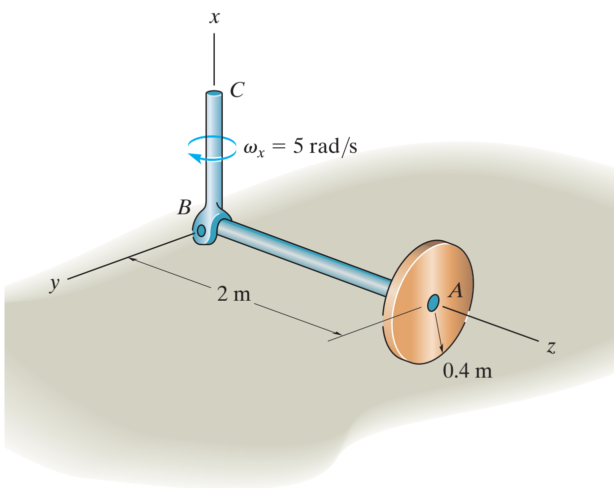
Prob. 21–52

21–53. Two uniform rods, each having a weight of 10 lb, are pin connected to the edge of a rotating disk. If the disk has a constant angular velocity $\omega_D = 4 \text{ rad/s}$, determine the angle θ made by each rod during the motion, and the components of the force and moment developed at the pin A. *Suggestion*: Use the x, y, z axes oriented as shown.



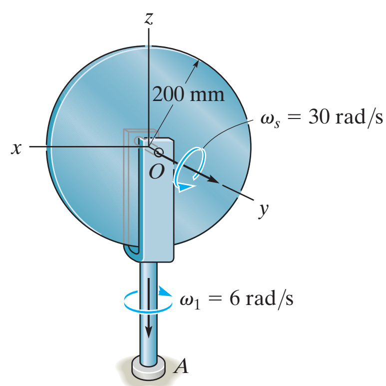
Prob. 21–53

21–54. The 10-kg disk turns around the shaft AB, while the shaft rotates about BC at a constant rate of $\omega_x = 5 \text{ rad/s}$. If the disk does not slip, determine the normal and frictional force it exerts on the ground. Neglect the mass of shaft AB.



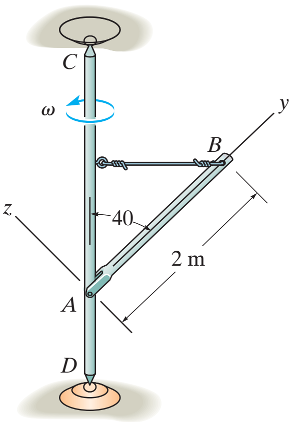
Prob. 21–54

21–55. The 20-kg disk is spinning on its axle at $\omega_s = 30 \text{ rad/s}$, while the forked rod is turning at $\omega_1 = 6 \text{ rad/s}$. Determine the x and z moment components the axle exerts on the disk during the motion.



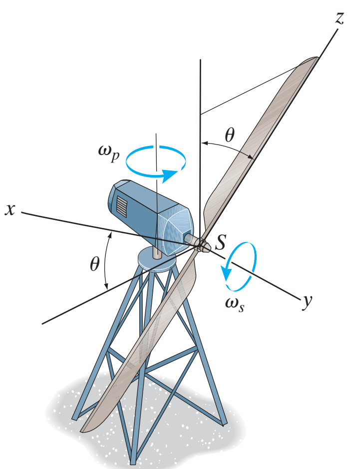
Prob. 21–55

***21–56.** The 4-kg slender rod AB is pinned at A and held at B by a cord. The axle CD is supported at its ends by ball-and-socket joints and is rotating with a constant angular velocity of 2 rad/s. Determine the tension developed in the cord and the magnitude of force developed at the pin A.



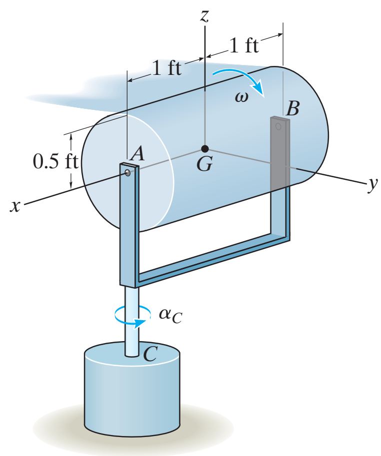
Prob. 21–56

21–57. The blades of a wind turbine spin about the shaft S with a constant angular speed of ω_s, while the frame precesses about the vertical axis with a constant angular speed of ω_p. Determine the x, y, and z components of moment that the shaft exerts on the blades as a function of θ. Consider each blade as a slender rod of mass m and length l.



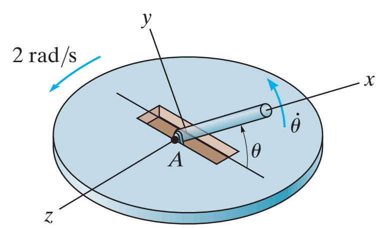
Prob. 21–57

21–58. The 15-lb cylinder is rotating about shaft AB with a constant angular speed $\omega = 4$ rad/s. If the supporting shaft at C, initially at rest, is given an angular acceleration $\alpha_C = 12$ rad/s^2, determine the components of reaction at the bearings A and B. The bearing at A cannot support a force component along the x axis, whereas the bearing at B does.

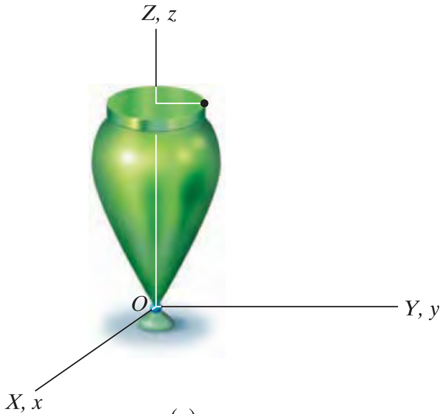


Prob. 21–58

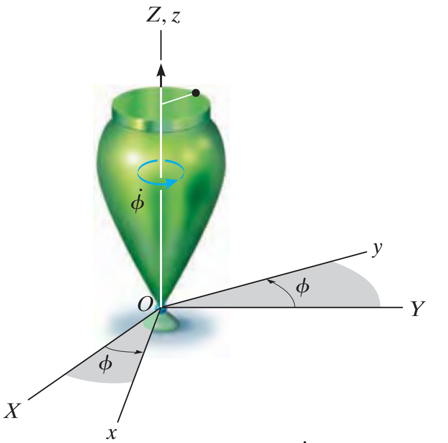
21–59. The *thin rod* has a mass of 0.8 kg and a total length of 150 mm. It is rotating about its midpoint at a constant rate $\dot{\theta} = 6$ rad/s, while the table to which its axle A is fastened is rotating at 2 rad/s. Determine the x, y, z moment components which the axle exerts on the rod when the rod is in any position θ.



Prob. 21–59



(a)



Precession $\dot{\phi}$

(b)

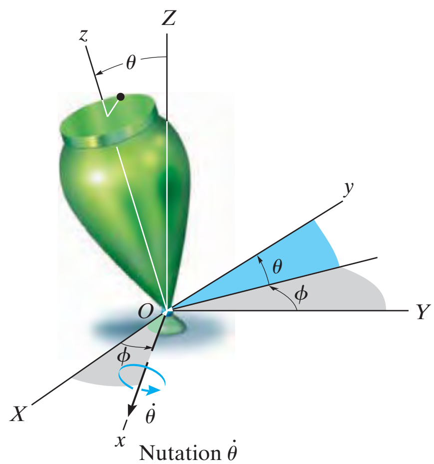
*21.5 Gyroscopic Motion

In this section we will develop the equations defining the motion of a body (top) which is symmetrical with respect to an axis and rotating about a fixed point. These equations also apply to the motion of a particularly interesting device, the gyroscope.

The body's motion will be analyzed using *Euler angles* ϕ, θ, ψ (phi, theta, psi). To illustrate how they define the position of a body, consider the top shown in Fig. 21–15*a*. To define its final position, Fig. 21–15*d*, a second set of *x, y, z* axes is fixed in the top. Starting with the *X, Y, Z* and *x, y, z* axes in coincidence, Fig. 21–15*a*, the final position of the top can be determined using the following three steps:

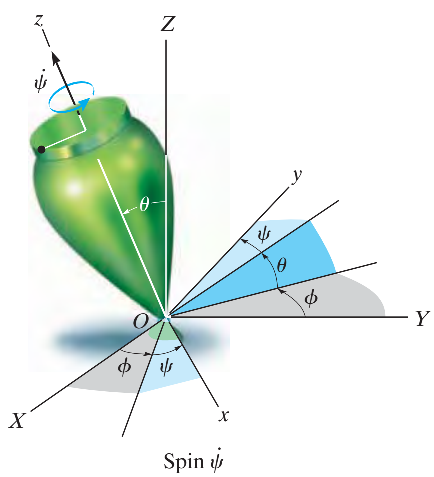
1. Rotate the top about the *Z* (or *z*) axis through an angle ϕ $(0 \leq \phi < 2\pi)$, Fig. 21–15*b*.
2. Rotate the top about the *x* axis through an angle θ $(0 \leq \theta \leq \pi)$, Fig. 21–15*c*.
3. Rotate the top about the *z* axis through an angle ψ $(0 \leq \psi < 2\pi)$ to obtain the final position, Fig. 21–15*d*.

The sequence of these three angles, ϕ, θ, then ψ, must be maintained, since finite rotations are *not vectors* (see Fig. 20–1). Although this is the case, the differential rotations $d\boldsymbol{\phi}$, $d\boldsymbol{\theta}$, and $d\boldsymbol{\psi}$ are vectors, and thus the angular velocity $\boldsymbol{\omega}$ of the top can be expressed in terms of the time derivatives of the Euler angles. The angular-velocity components $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are known as the *precession, nutation*, and *spin*, respectively.



Nutation $\dot{\theta}$

(c)



Spin $\dot{\psi}$

(d)

Fig. 21–15

Their positive directions are shown in Fig. 21–16. It is seen that these vectors are not all perpendicular to one another; however, $\boldsymbol{\omega}$ of the top can still be expressed in terms of these three components.

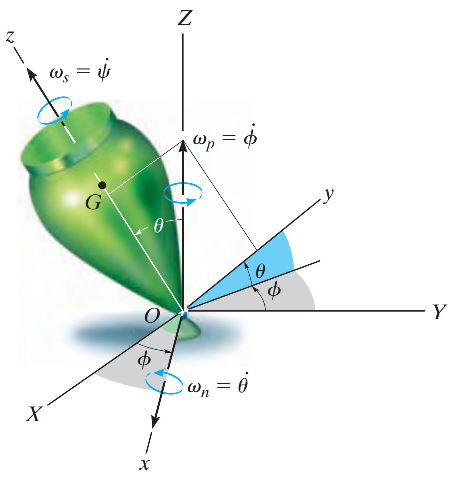


Fig. 21–16

Since the body (top) is symmetric with respect to the z or spin axis, there is no need to attach the x, y, z axes to the top since the inertial properties of the top will remain constant with respect to this frame during the motion. Therefore $\boldsymbol{\Omega} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n$, Fig. 21–16. Hence, the angular velocity of the body is

$$\begin{aligned}\boldsymbol{\omega} &= \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \\ &= \dot{\theta}\mathbf{i} + (\dot{\phi} \sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k} \qquad (21\text{–}27)\end{aligned}$$

And the angular velocity of the axes is

$$\begin{aligned}\boldsymbol{\Omega} &= \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k} \\ &= \dot{\theta}\mathbf{i} + (\dot{\phi} \sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta)\mathbf{k} \qquad (21\text{–}28)\end{aligned}$$

Have the x, y, z axes represent principal axes of inertia for the top, and so the moments of inertia will be represented as $I_{xx} = I_{yy} = I$ and $I_{zz} = I_z$. Since $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, Eqs. 21–26 are used to establish the rotational equations of motion. Substituting into these equations the respective angular-velocity components defined by Eqs. 21–27 and 21–28, their corresponding time derivatives, and the moment of inertia components, yields

$$\begin{aligned}\Sigma M_x &= I(\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta) + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}) \\ \Sigma M_y &= I(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \qquad (21\text{–}29) \\ \Sigma M_z &= I_z(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta)\end{aligned}$$

Each moment summation applies only at the fixed point O or the center of mass G of the body. Since the equations represent a coupled set of nonlinear second-order differential equations, in general a closed-form solution may not be obtained. Instead, the Euler angles ϕ, θ, and ψ may be obtained graphically as functions of time using numerical analysis and computer techniques.

A special case, however, does exist for which simplification of Eqs. 21–29 is possible. Commonly referred to as *steady precession*, it occurs when the nutation angle θ, precession $\dot{\phi}$, and spin $\dot{\psi}$ all remain *constant*. Equations 21–29 then reduce to the form

$$\boxed{\Sigma M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})} \qquad (21\text{–}30)$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

Equation 21–30 can be further simplified by noting that, from Eq. 21–27, $\omega_z = \dot{\phi}\cos\theta + \dot{\psi}$, so that

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}(\sin\theta)\omega_z$$

or

$$\Sigma M_x = \dot{\phi}\sin\theta(I_z\omega_z - I\dot{\phi}\cos\theta) \qquad (21\text{–}31)$$

It is interesting to note what effects the spin $\dot{\psi}$ has on the moment about the x axis. To show this, consider the spinning rotor in Fig. 21–17. Here $\theta = 90°$, in which case Eq. 21–30 reduces to the form

$$\Sigma M_x = I_z\dot{\phi}\dot{\psi}$$

or

$$\Sigma M_x = I_z\Omega_y\omega_z \qquad (21\text{–}32)$$

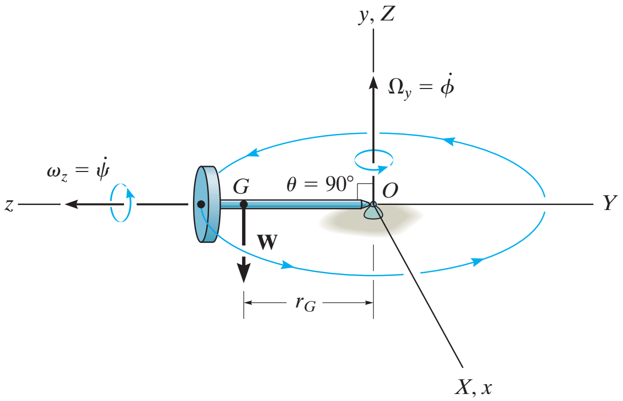


Fig. 21–17

From the figure it can be seen that $\mathbf{\Omega}_y$ and $\boldsymbol{\omega}_z$ act along their respective *positive axes* and therefore are mutually perpendicular. Instinctively, one would expect the rotor to fall down under the influence of gravity! However, this is not the case at all, provided the product $I_z\Omega_y\omega_z$ is correctly chosen to counterbalance the moment $\Sigma M_x = Wr_G$ of the rotor's weight about O. This unusual phenomenon of rigid-body motion is often referred to as the *gyroscopic effect*.

Perhaps a more intriguing demonstration of the gyroscopic effect comes from studying the action of a *gyroscope*, frequently referred to as a *gyro*. A gyro is a rotor which spins at a very high rate about its axis of symmetry. This rate of spin is considerably greater than its precessional rate of rotation about the vertical axis. Hence, for all practical purposes, the angular momentum of the gyro can be assumed directed along its axis of spin. Thus, for the gyro rotor shown in Fig. 21–18, $\omega_z \gg \Omega_y$, and the magnitude of the angular momentum about point O, as determined from Eqs. 21–11, reduces to the form $H_O = I_z\omega_z$. Since both the magnitude and direction of $\mathbf{H}_O$ are constant as observed from x, y, z, direct application of Eq. 21–22 yields

$$\Sigma \mathbf{M}_x = \boldsymbol{\Omega}_y \times \mathbf{H}_O \qquad (21\text{–}33)$$

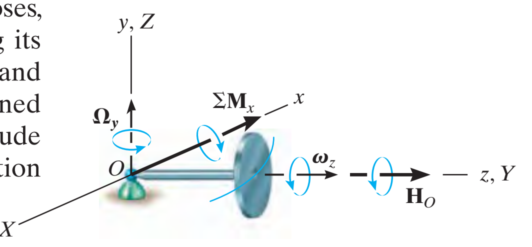
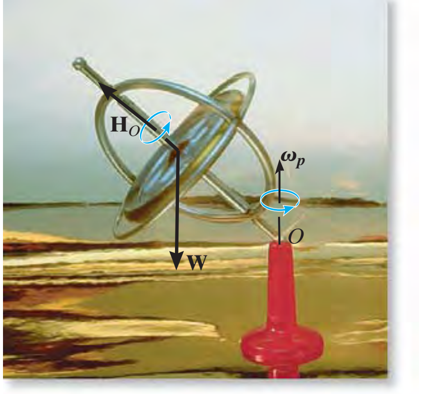


Fig. 21–18

Using the right-hand rule applied to the cross product, it can be seen that $\boldsymbol{\Omega}_y$ always swings $\mathbf{H}_O$ (or $\boldsymbol{\omega}_z$) toward the sense of $\Sigma\mathbf{M}_x$. In effect, the *change in direction* of the gyro's angular momentum, $d\mathbf{H}_O$, is equivalent to the angular impulse caused by the gyro's weight about O, i.e., $d\mathbf{H}_O = \Sigma\mathbf{M}_x\,dt$, Eq. 21–20. Also, since $H_O = I_z\omega_z$ and $\Sigma\mathbf{M}_x$, $\boldsymbol{\Omega}_y$, and $\mathbf{H}_O$ are mutually perpendicular, Eq. 21–33 reduces to Eq. 21–32.

When a gyro is mounted in gimbal rings, Fig. 21–19, it becomes *free* of external moments applied to its base. Thus, in theory, its angular momentum $\mathbf{H}$ will never precess but, instead, maintain its same fixed orientation along the axis of spin when the base is rotated. This type of gyroscope is called a *free gyro* and is useful as a gyrocompass when the spin axis of the gyro is directed north. In reality, the gimbal mechanism is never completely free of friction, so such a device is useful only for the local navigation of ships and aircraft. The gyroscopic effect is also useful as a means of stabilizing both the rolling motion of ships at sea and the trajectories of missiles and projectiles. Furthermore, this effect is of significant importance in the design of shafts and bearings for rotors which are subjected to forced precessions.



The spinning of the gyro within the frame of this toy gyroscope produces angular momentum $\mathbf{H}_O$, which is changing direction as the frame precesses $\boldsymbol{\omega}_p$ about the vertical axis. The gyroscope will not fall down since the moment of its weight $\mathbf{W}$ about the support is balanced by the change in the direction of $\mathbf{H}_O$. (© R.C. Hibbeler)

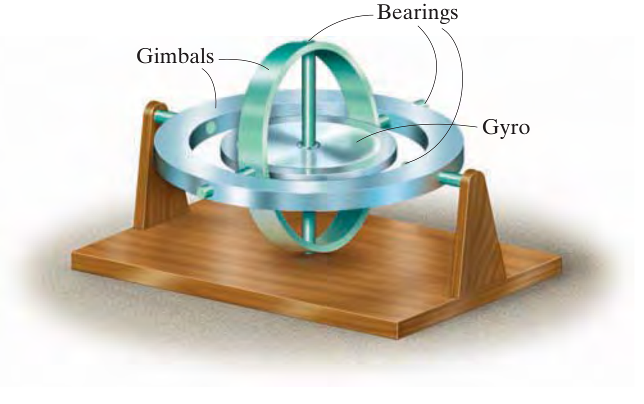


Fig. 21–19

EXAMPLE 21.7

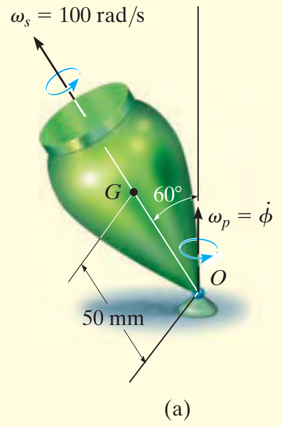
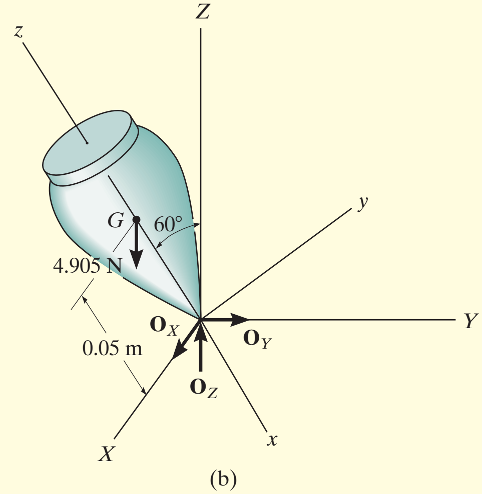


Fig. 21–20

The top shown in Fig. 21–20*a* has a mass of 0.5 kg and is precessing about the vertical axis at a constant angle of $\theta = 60°$. If it spins with an angular velocity $\omega_s = 100$ rad/s, determine the precession $\boldsymbol{\omega}_p$. Assume that the axial and transverse moments of inertia of the top are $0.45(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $1.20(10^{-3})\ \text{kg}\cdot\text{m}^2$, respectively, measured with respect to the fixed point O.



SOLUTION

Equation 21–30 will be used for the solution since the motion is *steady precession*. As shown on the free-body diagram, Fig. 21–20*b*, the coordinate axes are established in the usual manner, that is, with the positive z axis in the direction of spin, the positive Z axis in the direction of precession, and the positive x axis in the direction of the moment ΣM_x (refer to Fig. 21–16). Thus,

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$4.905\text{ N}(0.05\text{ m})\sin 60° = -[1.20(10^{-3})\text{ kg}\cdot\text{m}^2\,\dot{\phi}^2]\sin 60°\cos 60° + [0.45(10^{-3})\text{ kg}\cdot\text{m}^2]\dot{\phi}\sin 60°(\dot{\phi}\cos 60° + 100\text{ rad/s})$$

or

$$\dot{\phi}^2 - 120.0\dot{\phi} + 654.0 = 0 \qquad (1)$$

Solving this quadratic equation for the precession gives

$$\dot{\phi} = 114\text{ rad/s} \quad \text{(high precession)} \qquad \textit{Ans.}$$

and

$$\dot{\phi} = 5.72\text{ rad/s} \quad \text{(low precession)} \qquad \textit{Ans.}$$

NOTE: In reality, low precession of the top would generally be observed, since high precession would require a larger kinetic energy.

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The 1-kg disk shown in Fig. 21–21*a* spins about its axis with a constant angular velocity $\omega_D = 70\ \text{rad/s}$. The block at B has a mass of 2 kg, and by adjusting its position s one can change the precession of the disk about its supporting pivot at O while the shaft remains horizontal. Determine the position s that will enable the disk to have a constant precession $\omega_p = 0.5\ \text{rad/s}$ about the pivot. Neglect the weight of the shaft.

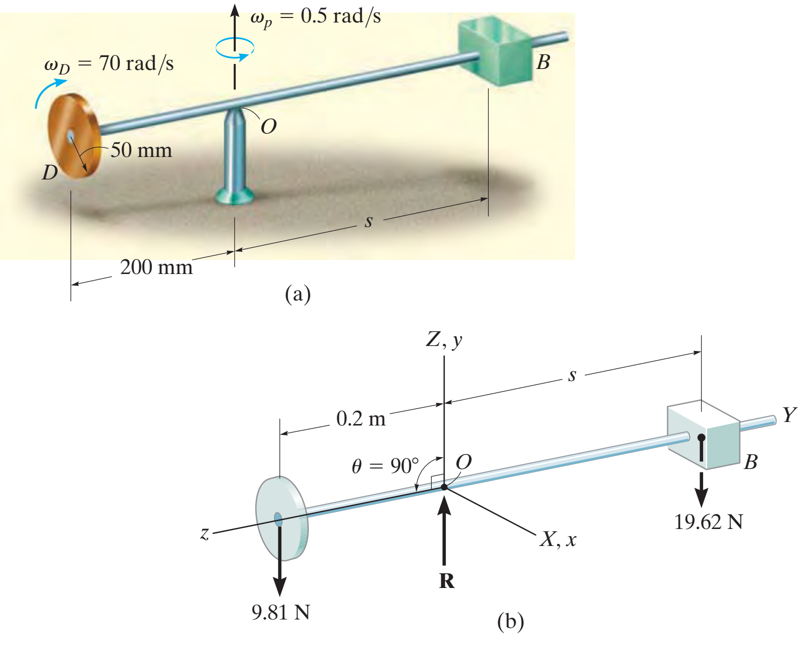

Fig. 21–21

SOLUTION

The free-body diagram of the assembly is shown in Fig. 21–21*b*. The origin for both the x, y, z and X, Y, Z coordinate systems is located at the fixed point O. In the conventional sense, the Z axis is chosen along the axis of precession, and the z axis is along the axis of spin, so that $\theta = 90°$. Since the precession is *steady*, Eq. 21–32 can be used for the solution.

$$\Sigma M_x = I_z \Omega_y \omega_z$$

Substituting the required data gives

$$(9.81\ \text{N})\,(0.2\ \text{m}) - (19.62\ \text{N})s = \left[\tfrac{1}{2}(1\ \text{kg})(0.05\ \text{m})^2\right]0.5\ \text{rad/s}(-70\ \text{rad/s})$$

$$s = 0.102\ \text{m} = 102\ \text{mm} \qquad \textit{Ans.}$$

21.6 Torque-Free Motion

When the only external force acting on a body is caused by gravity, the general motion of the body is referred to as *torque-free motion*. This type of motion is characteristic of planets, artificial satellites, and projectiles—provided air friction is neglected.

In order to describe the characteristics of this motion, the distribution of the body's mass will be assumed *axisymmetric*. The satellite shown in Fig. 21–22 is an example of such a body, where the z axis represents an axis of symmetry. The origin of the x, y, z coordinates is located at the mass center G, such that $I_{zz} = I_z$ and $I_{xx} = I_{yy} = I$. Since gravity is the only external force present, the summation of moments about the mass center is zero. From Eq. 21–21, this requires the angular momentum of the body to be constant, i.e.,

$$\mathbf{H}_G = \text{constant}$$

At the instant considered, it will be assumed that the inertial frame of reference is oriented so that the positive Z axis is directed along $\mathbf{H}_G$ and the y axis lies in the plane formed by the z and Z axes, Fig. 21–22. The Euler angle formed between Z and z is θ, and therefore, with this choice of axes the angular momentum can be expressed as

$$\mathbf{H}_G = H_G \sin\theta\,\mathbf{j} + H_G \cos\theta\,\mathbf{k}$$

Furthermore, using Eqs. 21–11, we have

$$\mathbf{H}_G = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

Equating the respective **i**, **j**, and **k** components of the above two equations yields

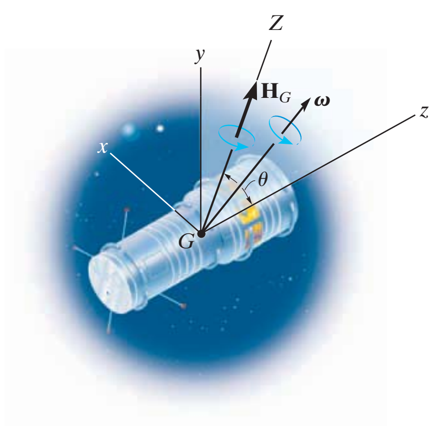

Fig. 21–22

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$$\omega_x = 0 \quad \omega_y = \frac{H_G \sin\theta}{I} \quad \omega_z = \frac{H_G \cos\theta}{I_z} \tag{21–34}$$

or

$$\boldsymbol{\omega} = \frac{H_G \sin\theta}{I}\mathbf{j} + \frac{H_G \cos\theta}{I_z}\mathbf{k} \tag{21–35}$$

In a similar manner, equating the respective **i**, **j**, **k** components of Eq. 21–27 to those of Eq. 21–34, we obtain

$$\dot{\theta} = 0$$

$$\dot{\phi}\sin\theta = \frac{H_G \sin\theta}{I}$$

$$\dot{\phi}\cos\theta + \dot{\psi} = \frac{H_G \cos\theta}{I_z}$$

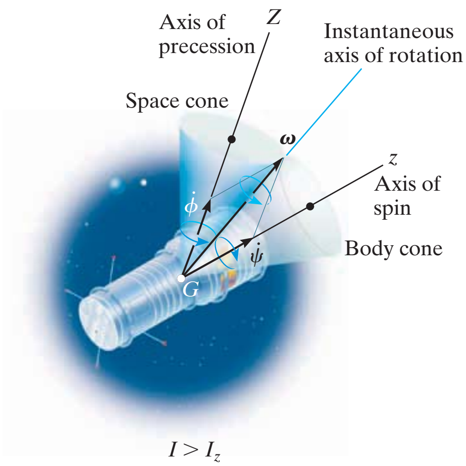
Solving, we get

$$\begin{aligned} \theta &= \text{constant} \\ \dot{\phi} &= \frac{H_G}{I} \\ \dot{\psi} &= \frac{I - I_z}{I I_z} H_G \cos\theta \end{aligned} \tag{21–36}$$

Thus, for torque-free motion of an axisymmetrical body, the angle θ formed between the angular-momentum vector and the spin of the body remains constant. Furthermore, the angular momentum $\mathbf{H}_G$, precession $\dot{\phi}$, and spin $\dot{\psi}$ for the body remain constant at all times during the motion.

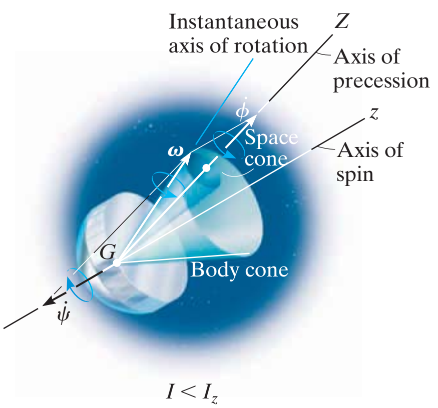
Eliminating H_G from the second and third of Eqs. 21–36 yields the following relation between the spin and precession:

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta \tag{21–37}$$



$I > I_z$

(a)



$I < I_z$

(b)

Fig. 21–23

These two components of angular motion can be studied by using the body and space cone models introduced in Sec. 20.1. The *space cone* defining the precession is fixed from rotating, since the precession has a fixed direction, while the outer surface of the *body cone* rolls on the space cone's outer surface. Try to imagine this motion in Fig. 21–23*a*. The interior angle of each cone is chosen such that the resultant angular velocity of the body is directed along the line of contact of the two cones. This line of contact represents the instantaneous axis of rotation for the body cone, and hence the angular velocity of both the body cone and the body must be directed along this line. Since the spin is a function of the moments of inertia I and I_z of the body, Eq. 21–36, the cone model in Fig. 21–23*a* is satisfactory for describing the motion, provided $I > I_z$. Torque-free motion which meets these requirements is called *regular precession*. If $I < I_z$, the spin is negative and the precession positive. This motion is represented by the satellite motion shown in Fig. 21–23*b* ($I < I_z$). The cone model can again be used to represent the motion; however, to preserve the correct vector addition of spin and precession to obtain the angular velocity $\boldsymbol{\omega}$, the inside surface of the body cone must roll on the outside surface of the (fixed) space cone. This motion is referred to as *retrograde precession*.

Satellites are often given a spin before they are launched. If their angular momentum is not collinear with the axis of spin, they will exhibit precession. In the photo on the left, regular precession will occur since $I > I_z$, and in the photo on the right, retrograde precession will occur since $I < I_z$.

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21

The motion of a football is observed using a slow-motion projector. From the film, the spin of the football is seen to be directed 30° from the horizontal, as shown in Fig. 21–24*a*. Also, the football is precessing about the vertical axis at a rate $\dot{\phi} = 3$ rad/s. If the ratio of the axial to transverse moments of inertia of the football is $\frac{1}{3}$, measured with respect to the center of mass, determine the magnitude of the football's spin and its angular velocity. Neglect the effect of air resistance.

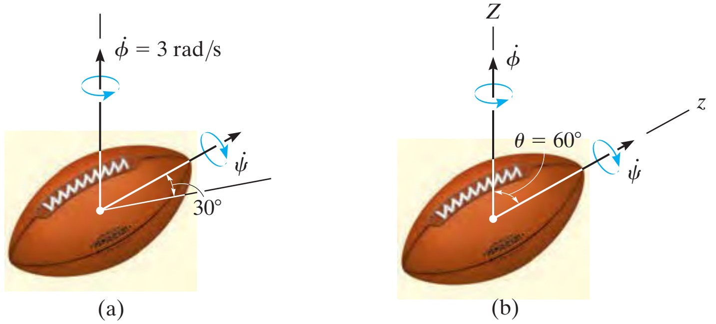


Fig. 21–24

SOLUTION

Since the weight of the football is the only force acting, the motion is torque-free. In the conventional sense, if the z axis is established along the axis of spin and the Z axis along the precession axis, as shown in Fig. 21–24*b*, then the angle $\theta = 60°$. Applying Eq. 21–37, the spin is

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta = \frac{I - \frac{1}{3}I}{\frac{1}{3}I}(3)\cos 60°$$
$$= 3 \text{ rad/s} \qquad \textit{Ans.}$$

Using Eqs. 21–34, where $H_G = \dot{\phi}I$ (Eq. 21–36), we have

$$\omega_x = 0$$

$$\omega_y = \frac{H_G \sin\theta}{I} = \frac{3I\sin 60°}{I} = 2.60 \text{ rad/s}$$

$$\omega_z = \frac{H_G\cos\theta}{I_z} = \frac{3I\cos 60°}{\frac{1}{3}I} = 4.50 \text{ rad/s}$$

Thus,

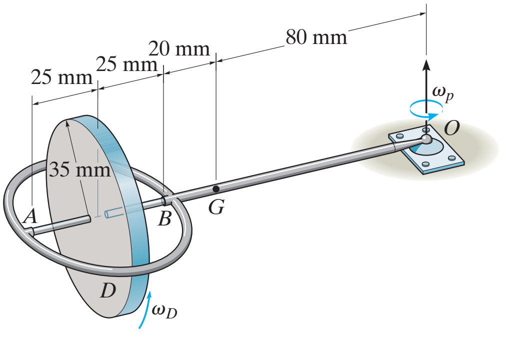
$$\omega = \sqrt{(\omega_x)^2 + (\omega_y)^2 + (\omega_z)^2}$$
$$= \sqrt{(0)^2 + (2.60)^2 + (4.50)^2}$$
$$= 5.20 \text{ rad/s} \qquad \textit{Ans.}$$

PROBLEMS

***21–60.** Show that the angular velocity of a body, in terms of Euler angles ϕ, θ, and ψ, can be expressed as $\boldsymbol{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{i} + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k}$, where **i**, **j**, and **k** are directed along the x, y, z axes as shown in Fig. 21–15*d*.

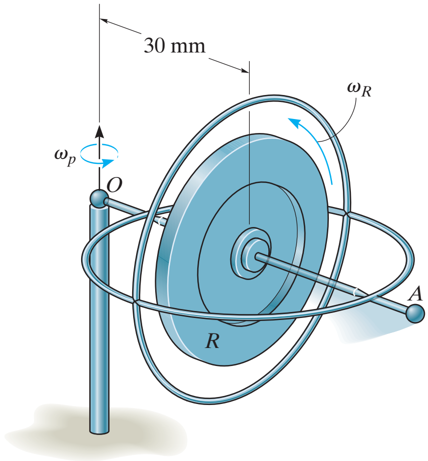
21–61. A thin rod is initially coincident with the Z axis when it is given three rotations defined by the Euler angles $\phi = 30^\circ$, $\theta = 45^\circ$, and $\psi = 60^\circ$. If these rotations are given in the order stated, determine the coordinate direction angles α, β, γ of the axis of the rod with respect to the X, Y, and Z axes. Are these directions the same for any order of the rotations? Why?

21–62. The gyroscope consists of a uniform 450-g disk D which is attached to the axle AB of negligible mass. The supporting frame has a mass of 180 g and a center of mass at G. If the disk is rotating about the axle at $\omega_D = 90$ rad/s, determine the constant angular velocity ω_p at which the frame precesses about the pivot point O. The frame moves in the horizontal plane.



Prob. 21–62

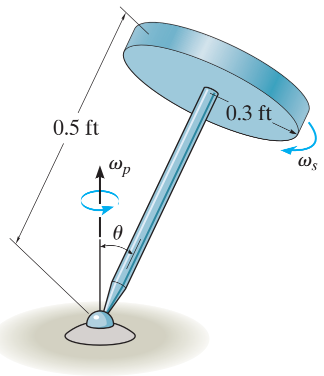
21–63. The toy gyroscope consists of a rotor R which is attached to the frame of negligible mass. If it is observed that the frame is precessing about the pivot point O at $\omega_p = 2$ rad/s, determine the angular velocity ω_R of the rotor. The stem OA moves in the horizontal plane. The rotor has a mass of 200 g and a radius of gyration $k_{OA} = 20$ mm about OA.



Prob. 21–63

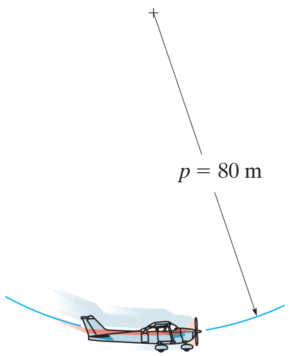
***21–64.** The top consists of a thin disk that has a weight of 8 lb and a radius of 0.3 ft. The rod has a negligible mass and a length of 0.5 ft. If the top is spinning with an angular velocity $\omega_s = 300$ rad/s, determine the steady-state precessional angular velocity ω_p of the rod when $\theta = 40^\circ$.

21–65. Solve Prob. 21–64 when $\theta = 90^\circ$.



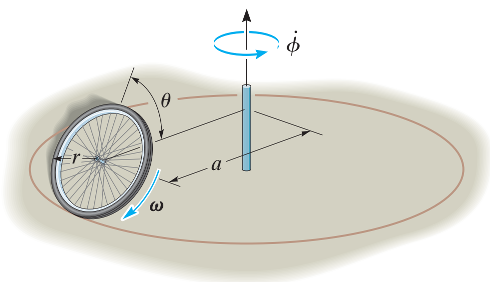
Probs. 21–64/65

21–66. The propeller on a single-engine airplane has a mass of 15 kg and a centroidal radius of gyration of 0.3 m computed about the axis of spin. When viewed from the front of the airplane, the propeller is turning clockwise at 350 rad/s about the spin axis. If the airplane enters a vertical curve having a radius of 80 m and is traveling at 200 km/h, determine the gyroscopic bending moment which the propeller exerts on the bearings of the engine when the airplane is in its lowest position.



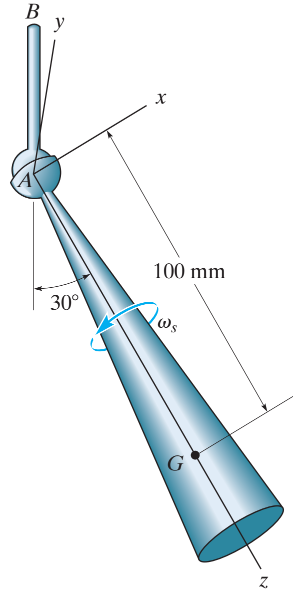
Prob. 21–66

21–67. A wheel of mass m and radius r rolls with constant spin $\boldsymbol{\omega}$ about a circular path having a radius a. If the angle of inclination is θ, determine the rate of precession. Treat the wheel as a thin ring. No slipping occurs.



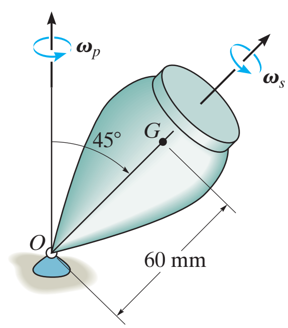
Prob. 21–67

***21–68.** The conical top has a mass of 0.8 kg, and the moments of inertia are $I_x = I_y = 3.5(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $I_z = 0.8(10^{-3})\ \text{kg}\cdot\text{m}^2$. If it spins freely in the ball-and socket joint at A with an angular velocity $\omega_s = 750$ rad/s, compute the precession of the top about the axis of the shaft AB.



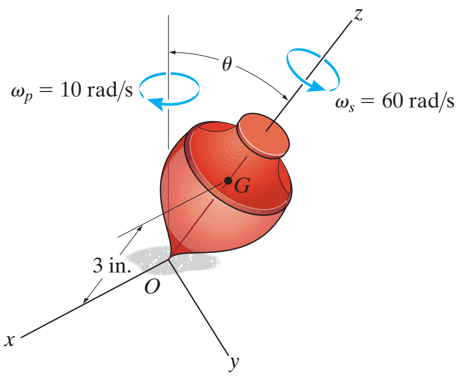
Prob. 21–68

21–69. The top has a mass of 90 g, a center of mass at G, and a radius of gyration $k = 18$ mm about its axis of symmetry. About any transverse axis acting through point O the radius of gyration is $k_t = 35$ mm. If the top is connected to a ball-and-socket joint at O and the precession is $\omega_p = 0.5$ rad/s, determine the spin $\boldsymbol{\omega}_s$.



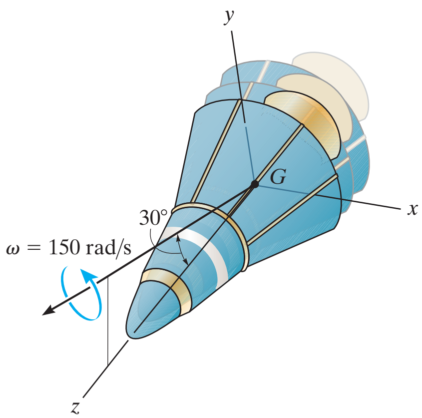
Prob. 21–69

21–70. The 1-lb top has a center of gravity at point G. If it spins about its axis of symmetry and precesses about the vertical axis at constant rates of $\omega_s = 60$ rad/s and $\omega_p = 10$ rad/s, respectively, determine the steady state angle θ. The radius of gyration of the top about the z axis is $k_z = 1$ in., and about the x and y axes it is $k_x = k_y = 4$ in.



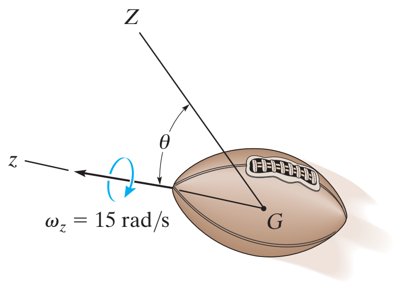
Prob. 21–70

21–71. The space capsule has a mass of 2 Mg, center of mass at G, and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 2.75$ m and $k_x = k_y = 5.5$ m, respectively. If the capsule has the angular velocity shown, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Indicate whether the precession is regular or retrograde. Also, draw the space cone and body cone for the motion.



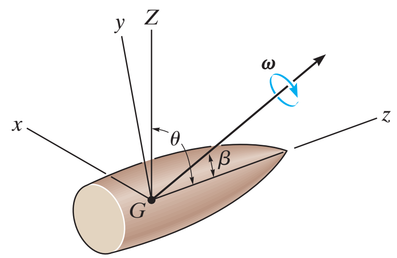
Prob. 21–71

***21–72.** The 0.25 kg football is spinning at $\omega_z = 15$ rad/s as shown. If $\theta = 40°$, determine the precession about the z axis. The radius of gyration about the spin axis is $k_z = 0.042$ m, and about a transverse axis is $k_y = 0.13$ m.



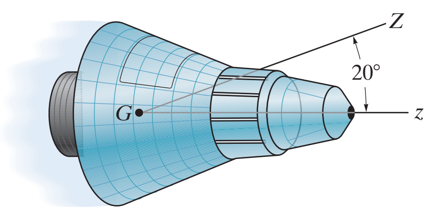
Prob. 21–72

21–73. The projectile shown is subjected to torque-free motion. The transverse and axial moments of inertia are I and I_z, respectively. If θ represents the angle between the precessional axis Z and the axis of symmetry z, and β is the angle between the angular velocity $\boldsymbol{\omega}$ and the z axis, show that β and θ are related by the equation $\tan\theta = (I/I_z)\tan\beta$.



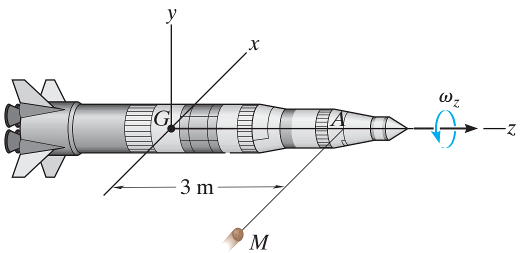
Prob. 21–73

21–74. The radius of gyration about an axis passing through the axis of symmetry of the 1.6-Mg space capsule is $k_z = 1.2$ m and about any transverse axis passing through the center of mass G, $k_t = 1.8$ m. If the capsule has a known steady-state precession of two revolutions per hour about the Z axis, determine the rate of spin about the z axis.



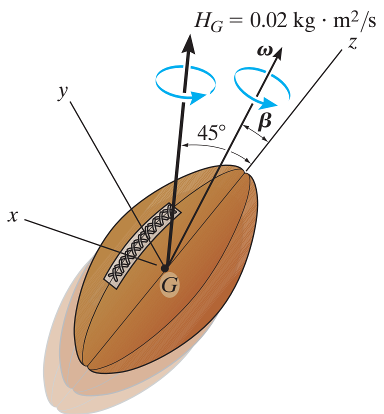
Prob. 21–74

21–75. The rocket has a mass of 4 Mg and radii of gyration $k_z = 0.85$ m and $k_x = k_y = 2.3$ m. It is initially spinning about the z axis at $\omega_z = 0.05$ rad/s when a meteoroid M strikes it at A and creates an impulse $\mathbf{I} = \{300\mathbf{i}\}$ N · s. Determine the axis of precession after the impact.



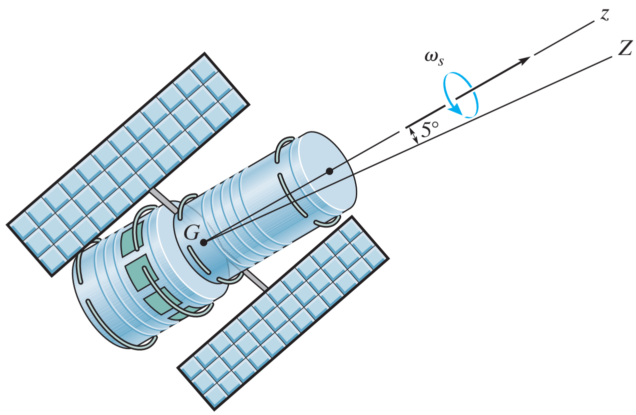
Prob. 21–75

***21–76.** The football has a mass of 450 g and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 30$ mm and $k_x = k_y = 50$ mm, respectively. If the football has an angular momentum of $H_G = 0.02$ kg · m²/s, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Also, find the angle β that the angular velocity vector makes with the z axis.



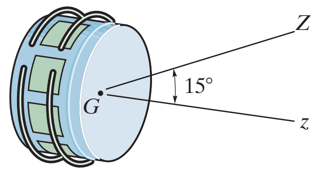
Prob. 21–76

21–77. The satellite has a mass of 1.8 Mg, and about axes passing through the mass center G the axial and transverse radii of gyration are $k_z = 0.8$ m and $k_t = 1.2$ m, respectively. If it is spinning at $\omega_s = 6$ rad/s when it is launched, determine its angular momentum. Precession occurs about the Z axis.



Prob. 21–77

21–78. The radius of gyration about an axis passing through the axis of symmetry of the 1.2-Mg satellite is $k_z = 1.4$ m, and about any transverse axis passing through the center of mass G, $k_t = 2.20$ m. If the satellite has a known spin of 2700 rev/h about the z axis, determine the steady-state precession about the z axis.



Prob. 21–78

CHAPTER REVIEW

Moments and Products of Inertia

A body has six components of inertia for any specified x, y, z axes. Three of these are moments of inertia about each of the axes, I_{xx}, I_{yy}, I_{zz}, and three are products of inertia, each defined from two orthogonal planes, I_{xy}, I_{yz}, I_{xz}. If either one or both of these planes are planes of symmetry, then the product of inertia with respect to these planes will be zero.

$$I_{xx} = \int_m r_x^2\,dm = \int_m (y^2 + z^2)\,dm \qquad I_{xy} = I_{yx} = \int_m xy\,dm$$

$$I_{yy} = \int_m r_y^2\,dm = \int_m (x^2 + z^2)\,dm \qquad I_{yz} = I_{zy} = \int_m yz\,dm$$

$$I_{zz} = \int_m r_z^2\,dm = \int_m (x^2 + y^2)\,dm \qquad I_{xz} = I_{zx} = \int_m xz\,dm$$

The moments and products of inertia can be determined by direct integration or by using tabulated values. If these quantities are to be determined with respect to axes or planes that do not pass through the mass center, then parallel-axis and parallel-plane theorems must be used.

Provided the six components of inertia are known, then the moment of inertia about any axis can be determined using the inertia transformation equation.

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x$$

Principal Moments of Inertia

At any point on or off the body, the x, y, z axes can be oriented so that the products of inertia will be zero. The resulting moments of inertia are called the principal moments of inertia. In general, one will be a maximum and the other a minimum.

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Principle of Impulse and Momentum

The angular momentum for a body can be determined about any arbitrary point A.

Once the linear and angular momentum for the body have been formulated, then the principle of impulse and momentum can be used to solve problems that involve force, velocity, and time.

$$m(\mathbf{v}_G)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{F}\,dt = m(\mathbf{v}_G)_2 \qquad (\mathbf{H}_O)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$$

$$\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\,dm$$

Fixed Point O

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\,dm$$

Center of Mass

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G$$

Arbitrary Point

where

$$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$

$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

Principle of Work and Energy

The kinetic energy for a body is usually determined relative to a fixed point or the body's mass center.

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Fixed Point

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Center of Mass

These formulations can be used with the principle of work and energy to solve problems that involve force, velocity, and displacement.

$$T_1 + \Sigma U_{1-2} = T_2$$

Equations of Motion

There are three scalar equations of translational motion for a rigid body that moves in three dimensions.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma F_z = m(a_G)_z$$

The three scalar equations of rotational motion depend upon the motion of the x, y, z reference. Most often, these axes are oriented so that they are principal axes of inertia. If the axes are fixed in and move with the body so that $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then the equations are referred to as the Euler equations of motion.

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}$$

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$

$$\boldsymbol{\Omega} \neq \boldsymbol{\omega}$$

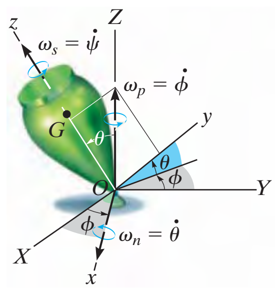
A free-body diagram should always accompany the application of the equations of motion.

Gyroscopic Motion

The angular motion of a gyroscope is best described using the three Euler angles ϕ, θ, and ψ. The angular velocity components are called the precession $\dot{\phi}$, the nutation $\dot{\theta}$, and the spin $\dot{\psi}$.

If $\dot{\theta} = 0$ and $\dot{\phi}$ and $\dot{\psi}$ are constant, then the motion is referred to as steady precession.

It is the spin of a gyro rotor that is responsible for holding a rotor from falling downward, and instead causing it to precess about a vertical axis. This phenomenon is called the gyroscopic effect.



$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$\Sigma M_y = 0, \ \Sigma M_z = 0$$

Torque-Free Motion

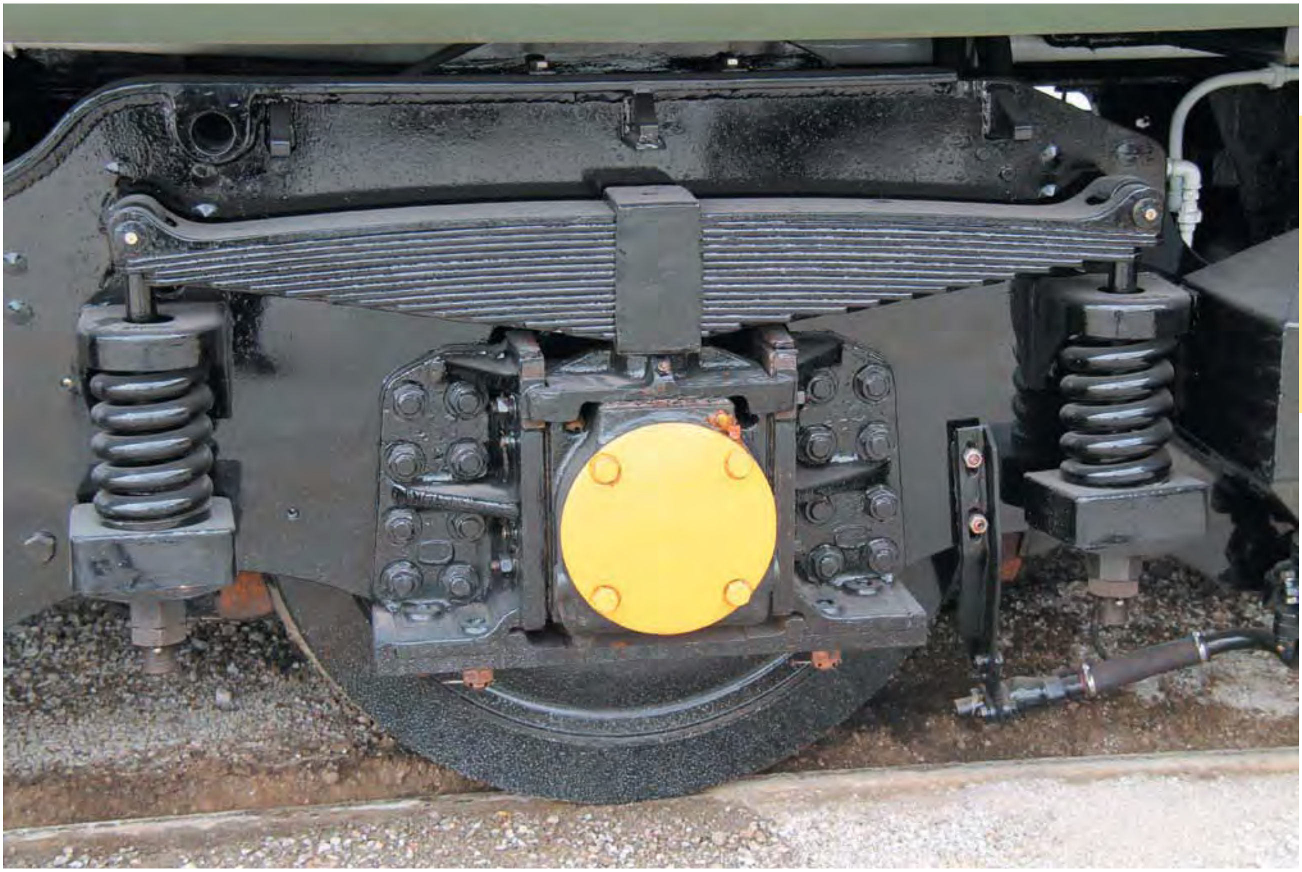
A body that is only subjected to a gravitational force will have no moments on it about its mass center, and so the motion is described as torque-free motion. The angular momentum for the body about its mass center will remain constant. This causes the body to have both a spin and a precession. The motion depends upon the magnitude of the moment of inertia of a symmetric body about the spin axis, I_z, versus that about a perpendicular axis, I.

$$\theta = \text{constant}$$

$$\dot{\phi} = \frac{H_G}{I}$$

$$\dot{\psi} = \frac{I - I_z}{I I_z} H_G \cos\theta$$

Chapter 22



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The analysis of vibrations plays an important role in the study of the behavior of structures subjected to earthquakes.

Vibrations

CHAPTER OBJECTIVES

- To discuss undamped one-degree-of-freedom vibration of a rigid body using the equation of motion and energy methods.
- To study the analysis of undamped forced vibration and viscous damped forced vibration.

*22.1 Undamped Free Vibration

A *vibration* is the oscillating motion of a body or system of connected bodies displaced from a position of equilibrium. In general, there are two types of vibration, free and forced. *Free vibration* occurs when the motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or the vibration of an elastic rod. *Forced vibration* is caused by an external periodic or intermittent force applied to the system. Both of these types of vibration can either be damped or undamped. *Undamped* vibrations exclude frictional effects in the analysis. Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually *damped*.

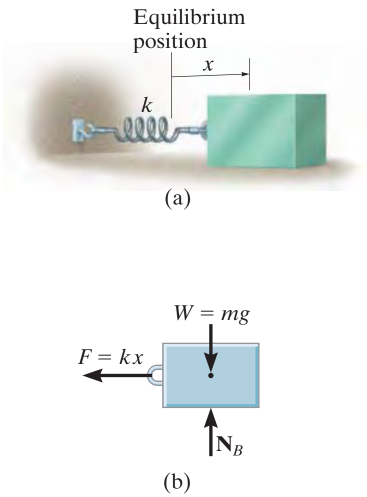


Fig. 22–1

The simplest type of vibrating motion is undamped free vibration, represented by the block and spring model shown in Fig. 22–1*a*. Vibrating motion occurs when the block is released from a displaced position x so that the spring pulls on the block. The block will attain a velocity such that it will proceed to move out of equilibrium when $x = 0$, and provided the supporting surface is smooth, the block will oscillate back and forth.

The time-dependent path of motion of the block can be determined by applying the equation of motion to the block when it is in the displaced position x. The free-body diagram is shown in Fig. 22–1*b*. The elastic restoring force $F = kx$ is always directed toward the equilibrium position, whereas the acceleration **a** is assumed to act in the direction of *positive displacement*. Since $a = d^2x/dt^2 = \ddot{x}$, we have

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -kx = m\ddot{x}$$

Note that the acceleration is proportional to the block's displacement. Motion described in this manner is called *simple harmonic motion*. Rearranging the terms into a "standard form" gives

$$\ddot{x} + \omega_n^2 x = 0 \tag{22–1}$$

The constant ω_n, generally reported in rad/s, is called the *natural frequency*, and in this case

$$\omega_n = \sqrt{\frac{k}{m}} \tag{22–2}$$

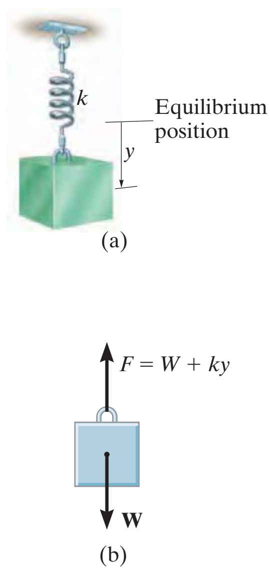


Fig. 22–2

Equation 22–1 can also be obtained by considering the block to be suspended so that the displacement y is measured from the block's *equilibrium position*, Fig. 22–2*a*. When the block is in equilibrium, the spring exerts an upward force of $F = W = mg$ on the block. Hence, when the block is displaced a distance y downward from this position, the magnitude of the spring force is $F = W + ky$, Fig. 22–2*b*. Applying the equation of motion gives

$$+\downarrow \Sigma F_y = ma_y; \qquad -W - ky + W = m\ddot{y}$$

or

$$\ddot{y} + \omega_n^2 y = 0$$

which is the same form as Eq. 22–1 and ω_n is defined by Eq. 22–2.

Equation 22–1 is a homogeneous, second-order, linear, differential equation with constant coefficients. It can be shown, using the methods of differential equations, that the general solution is

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{22–3}$$

Here A and B represent two constants of integration. The block's velocity and acceleration are determined by taking successive time derivatives, which yields

$$v = \dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t \tag{22–4}$$

$$a = \ddot{x} = -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t \tag{22–5}$$

When Eqs. 22–3 and 22–5 are substituted into Eq. 22–1, the differential equation will be satisfied, showing that Eq. 22–3 is indeed the solution to Eq. 22–1.

The integration constants in Eq. 22–3 are generally determined from the initial conditions of the problem. For example, suppose that the block in Fig. 22–1*a* has been displaced a distance x_1 to the right from its equilibrium position and given an initial (positive) velocity $\mathbf{v}_1$ directed to the right. Substituting $x = x_1$ when $t = 0$ into Eq. 22–3 yields $B = x_1$. And since $v = v_1$ when $t = 0$, using Eq. 22–4 we obtain $A = v_1/\omega_n$. If these values are substituted into Eq. 22–3, the equation describing the motion becomes

$$x = \frac{v_1}{\omega_n} \sin \omega_n t + x_1 \cos \omega_n t \tag{22–6}$$

Equation 22–3 may also be expressed in terms of simple sinusoidal motion. To show this, let

$$A = C \cos \phi \tag{22–7}$$

and

$$B = C \sin \phi \tag{22–8}$$

where C and ϕ are new constants to be determined in place of A and B. Substituting into Eq. 22–3 yields

$$x = C \cos \phi \sin \omega_n t + C \sin \phi \cos \omega_n t$$

And since $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$, then

$$x = C \sin(\omega_n t + \phi) \tag{22–9}$$

If this equation is plotted on an x versus $\omega_n t$ axis, the graph shown in Fig. 22–3 is obtained. The maximum displacement of the block from its equilibrium position is defined as the *amplitude* of vibration. From either the figure or Eq. 22–9 the amplitude is C. The angle ϕ is called the *phase angle* since it represents the amount by which the curve is displaced from the origin when $t = 0$. We can relate these two constants to A and B using Eqs. 22–7 and 22–8. Squaring and adding these two equations, the amplitude becomes

$$C = \sqrt{A^2 + B^2} \tag{22–10}$$

If Eq. 22–8 is divided by Eq. 22–7, the phase angle is then

$$\phi = \tan^{-1}\frac{B}{A} \tag{22–11}$$

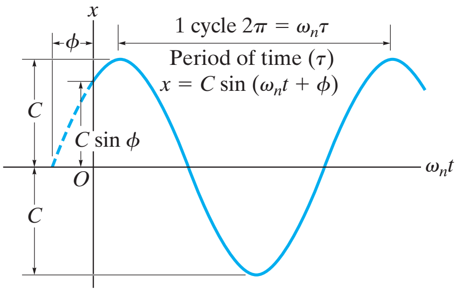


Fig. 22–3

Note that the sine curve, Eq. 22–9, completes one *cycle* in time $t = \tau$ (tau) when $\omega_n\tau = 2\pi$, or

$$\tau = \frac{2\pi}{\omega_n} \tag{22–12}$$

This time interval is called a *period*, Fig. 22–3. Using Eq. 22–2, the period can also be represented as

$$\tau = 2\pi\sqrt{\frac{m}{k}} \tag{22–13}$$

Finally, the *frequency* f is defined as the number of cycles completed per unit of time, which is the reciprocal of the period; that is,

$$f = \frac{1}{\tau} = \frac{\omega_n}{2\pi} \tag{22–14}$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{22–15}$$

The frequency is expressed in cycles/s. This ratio of units is called a *hertz* (Hz), where 1 Hz = 1 cycle/s = 2π rad/s.

When a body or system of connected bodies is given an initial displacement from its equilibrium position and released, it will vibrate with the *natural frequency*, ω_n. Provided the system has a single degree of freedom, that is, it requires only one coordinate to specify completely the position of the system at any time, then the vibrating motion will have the same characteristics as the simple harmonic motion of the block and spring just presented. Consequently, the motion is described by a differential equation of the same "standard form" as Eq. 22–1, i.e.,

$$\ddot{x} + \omega_n^2 x = 0 \tag{22–16}$$

Hence, if the natural frequency ω_n is known, the period of vibration τ, frequency f, and other vibrating characteristics can be established using Eqs. 22–3 through 22–15.

Important Points

- Free vibration occurs when the motion is maintained by gravitational or elastic restoring forces.
- The amplitude is the maximum displacement of the body.
- The period is the time required to complete one cycle.
- The frequency is the number of cycles completed per unit of time, where 1 Hz = 1 cycle/s.
- Only one position coordinate is needed to describe the location of a one-degree-of-freedom system.

Procedure for Analysis

As in the case of the block and spring, the natural frequency ω_n of a body or system of connected bodies having a single degree of freedom can be determined using the following procedure:

Free-Body Diagram.

- Draw the free-body diagram of the body when the body is displaced a *small amount* from its equilibrium position.
- Locate the body with respect to its equilibrium position by using an appropriate *inertial coordinate* q. The acceleration of the body's mass center $\mathbf{a}_G$ or the body's angular acceleration $\boldsymbol{\alpha}$ should have an assumed sense of direction which is in the *positive direction* of the position coordinate.
- If the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then it may be beneficial to also draw the kinetic diagram since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$, and thereby makes it convenient for visualizing the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equation of Motion.

- Apply the equation of motion to relate the elastic or gravitational *restoring* forces and couple moments acting on the body to the body's accelerated motion.

Kinematics.

- Using kinematics, express the body's accelerated motion in terms of the second time derivative of the position coordinate, $\ddot{q}$.
- Substitute the result into the equation of motion and determine ω_n by rearranging the terms so that the resulting equation is in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE 22.1

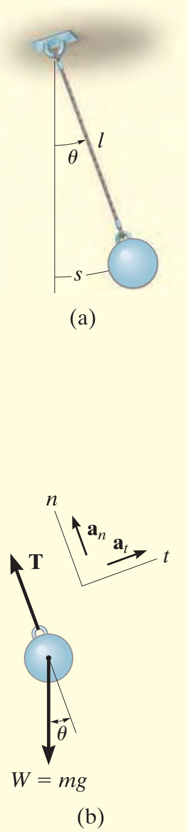


Fig. 22–4

Determine the period of oscillation for the simple pendulum shown in Fig. 22–4*a*. The bob has a mass m and is attached to a cord of length l. Neglect the size of the bob.

SOLUTION

Free-Body Diagram. Motion of the system will be related to the position coordinate $(q =)\ \theta$, Fig. 22–4*b*. When the bob is displaced by a small angle θ, the *restoring force* acting on the bob is created by the tangential component of its weight, $mg \sin \theta$. Furthermore, $\mathbf{a}_t$ acts in the direction of *increasing* s (or θ).

Equation of Motion. Applying the equation of motion in the *tangential direction*, since it involves the restoring force, yields

$$+\nearrow \Sigma F_t = ma_t; \qquad -mg \sin \theta = ma_t \tag{1}$$

Kinematics. $a_t = d^2s/dt^2 = \ddot{s}$. Furthermore, s can be related to θ by the equation $s = l\theta$, so that $a_t = l\ddot{\theta}$. Hence, Eq. 1 reduces to

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \tag{2}$$

The solution of this equation involves the use of an elliptic integral. For *small displacements*, however, $\sin \theta \approx \theta$, in which case

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{3}$$

Comparing this equation with Eq. 22–16 ($\ddot{x} + \omega_n^2 x = 0$), it is seen that $\omega_n = \sqrt{g/l}$. From Eq. 22–12, the period of time required for the bob to make one complete swing is therefore

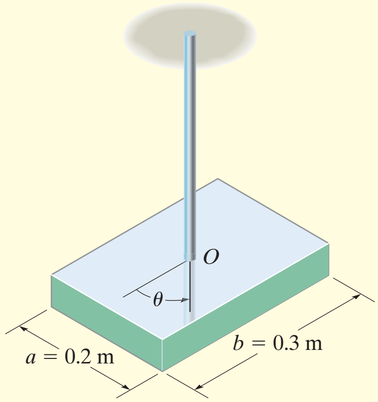
$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}} \qquad \textit{Ans.}$$

This interesting result, originally discovered by Galileo Galilei through experiment, indicates that the period depends only on the length of the cord and not on the mass of the pendulum bob or the angle θ.

NOTE: The solution of Eq. 3 is given by Eq. 22–3, where $\omega_n = \sqrt{g/l}$ and θ is substituted for x. Like the block and spring, the constants A and B in this problem can be determined if, for example, one knows the displacement and velocity of the bob at a given instant.

EXAMPLE 22.2

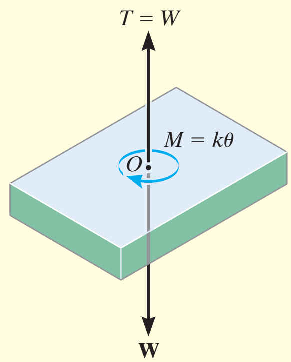
The 10-kg rectangular plate shown in Fig. 22–5*a* is suspended at its center from a rod having a torsional stiffness $k = 1.5\ \text{N}\cdot\text{m/rad}$. Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



(a)

SOLUTION

Free-Body Diagram. Fig. 22–5*b*. Since the plate is displaced in its own plane, the torsional *restoring* moment created by the rod is $M = k\theta$. This moment acts in the direction opposite to the angular displacement θ. The angular acceleration $\ddot{\theta}$ acts in the direction of *positive* θ.



(b)

Fig. 22–5

Equation of Motion.

$$\Sigma M_O = I_O\alpha; \qquad -k\theta = I_O\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{k}{I_O}\theta = 0$$

Since this equation is in the "standard form," the natural frequency is $\omega_n = \sqrt{k/I_O}$.

From the table on the inside back cover, the moment of inertia of the plate about an axis coincident with the rod is $I_O = \frac{1}{12}m(a^2 + b^2)$. Hence,

$$I_O = \frac{1}{12}(10\ \text{kg})\left[(0.2\ \text{m})^2 + (0.3\ \text{m})^2\right] = 0.1083\ \text{kg}\cdot\text{m}^2$$

The natural period of vibration is therefore,

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{k}} = 2\pi\sqrt{\frac{0.1083}{1.5}} = 1.69\ \text{s} \qquad \textit{Ans.}$$

EXAMPLE 22.3

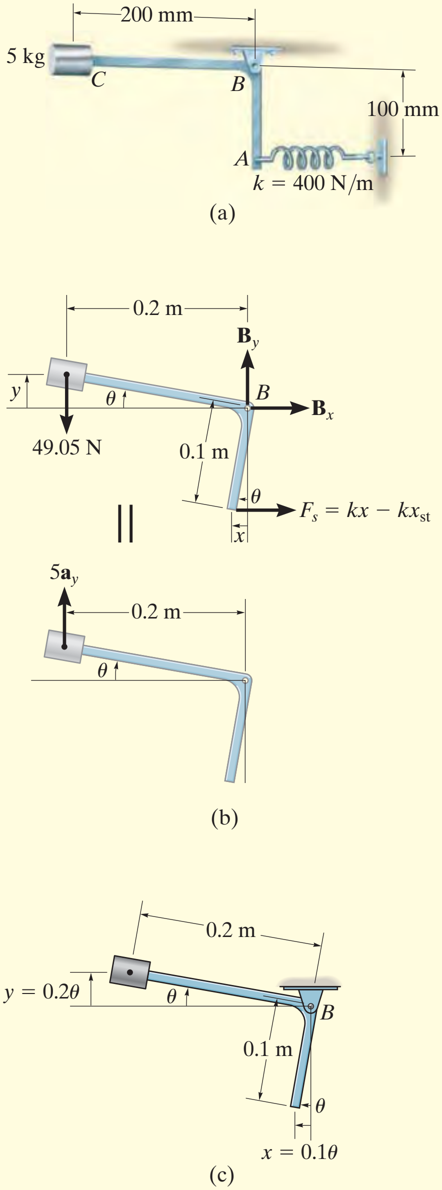


Fig. 22–6

The bent rod shown in Fig. 22–6*a* has a negligible mass and supports a 5-kg collar at its end. If the rod is in the equilibrium position shown, determine the natural period of vibration for the system.

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–6*b*. Here the rod is displaced by a small angle θ from the equilibrium position. Since the spring is subjected to an initial compression of x_{st} for equilibrium, then when the displacement $x > x_{st}$ the spring exerts a force of $F_s = kx - kx_{st}$ on the rod. To obtain the "standard form," Eq. 22–16, $5\mathbf{a}_y$ must act *upward*, which is in accordance with positive θ displacement.

Equation of Motion. Moments will be summed about point B to eliminate the unknown reaction at this point. Since θ is small,

$$\circlearrowleft + \Sigma M_B = \Sigma(\mathcal{M}_k)_B;$$

$$kx(0.1 \text{ m}) - kx_{st}(0.1 \text{ m}) + 49.05 \text{ N}(0.2 \text{ m}) = -(5 \text{ kg})a_y(0.2 \text{ m})$$

The second term on the left side, $-kx_{st}(0.1 \text{ m})$, represents the moment created by the spring force which is necessary to hold the collar in *equilibrium*, i.e., at $x = 0$. Since this moment is equal and opposite to the moment 49.05 N(0.2 m) created by the weight of the collar, these two terms cancel in the above equation, so that

$$kx(0.1) = -5a_y(0.2) \qquad (1)$$

Kinematics. The deformation of the spring and the position of the collar can be related to the angle θ, Fig. 22–6*c*. Since θ is small, $x = (0.1 \text{ m})\theta$ and $y = (0.2 \text{ m})\theta$. Therefore, $a_y = \ddot{y} = 0.2\ddot{\theta}$. Substituting into Eq. 1 yields

$$400(0.1\theta)\,0.1 = -5(0.2\ddot{\theta})0.2$$

Rewriting this equation in the "standard form" gives

$$\ddot{\theta} + 20\theta = 0$$

Compared with $\ddot{x} + \omega_n^2 x = 0$ (Eq. 22–16), we have

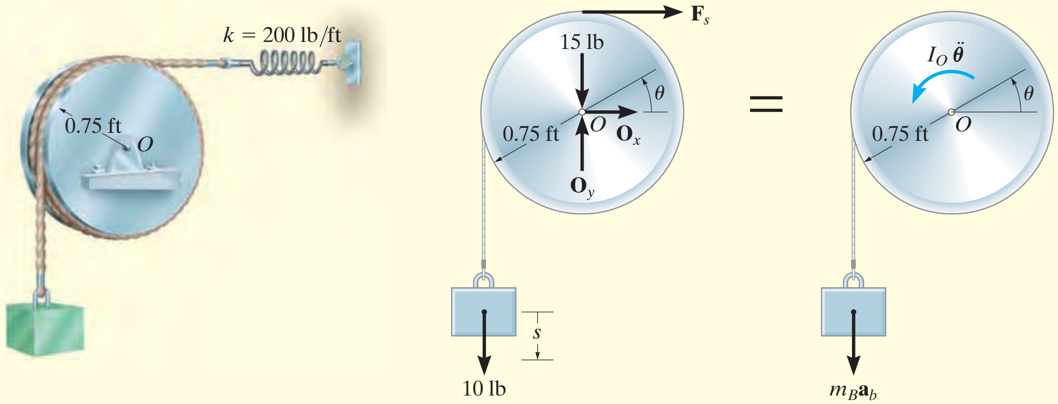
$$\omega_n^2 = 20 \qquad \omega_n = 4.47 \text{ rad/s}$$

The natural period of vibration is therefore

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.47} = 1.40 \text{ s} \qquad \textit{Ans.}$$

EXAMPLE 22.4

A 10-lb block is suspended from a cord that passes over a 15-lb disk, as shown in Fig. 22–7*a*. The spring has a stiffness $k = 200$ lb/ft. Determine the natural period of vibration for the system.



(a) (b)

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–7*b*. The *system* consists of the disk, which undergoes a rotation defined by the angle θ, and the block, which translates by an amount s. The vector $I_O\ddot{\boldsymbol{\theta}}$ acts in the direction of *positive* θ, and consequently $m_B\mathbf{a}_b$ acts downward in the direction of *positive* s.

Equation of Motion. Summing moments about point O to eliminate the reactions $\mathbf{O}_x$ and $\mathbf{O}_y$, realizing that $I_O = \frac{1}{2}mr^2$, yields

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O;$$

$$10 \text{ lb}(0.75 \text{ ft}) - F_s(0.75 \text{ ft})$$

$$= \frac{1}{2}\left(\frac{15 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(0.75 \text{ ft})^2\,\ddot{\theta} + \left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)a_b(0.75 \text{ ft}) \quad (1)$$

Kinematics. As shown on the kinematic diagram in Fig. 22–7*c*, a small positive displacement θ of the disk causes the block to lower by an amount $s = 0.75\theta$; hence, $a_b = \ddot{s} = 0.75\ddot{\theta}$. When $\theta = 0°$, the spring force required for *equilibrium* of the disk is 10 lb, acting to the right. For position θ, the spring force is $F_s = (200 \text{ lb/ft})(0.75\theta \text{ ft}) + 10$ lb. Substituting these results into Eq. 1 and simplifying yields

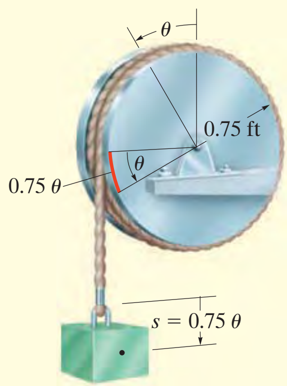
$$\ddot{\theta} + 368\theta = 0$$

Hence,

$$\omega_n^2 = 368 \qquad \omega_n = 19.18 \text{ rad/s}$$

Therefore, the natural period of vibration is

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{19.18} = 0.328 \text{ s} \qquad \textit{Ans.}$$



(c)

Fig. 22–7

22

PROBLEMS

22–1. A spring is stretched 175 mm by an 8-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 1.50 m/s, determine the differential equation which describes the motion. Assume that positive displacement is downward. Also, determine the position of the block when $t = 0.22$ s.

22–2. A spring has a stiffness of 800 N/m. If a 2-kg block is attached to the spring, pushed 50 mm above its equilibrium position, and released from rest, determine the equation that describes the block's motion. Assume that positive displacement is downward.

22–3. A spring is stretched 200 mm by a 15-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 0.75 m/s, determine the equation which describes the motion. What is the phase angle? Assume that positive displacement is downward.

***22–4.** When a 20-lb weight is suspended from a spring, the spring is stretched a distance of 4 in. Determine the natural frequency and the period of vibration for a 10-lb weight attached to the same spring.

22–5. When a 3-kg block is suspended from a spring, the spring is stretched a distance of 60 mm. Determine the natural frequency and the period of vibration for a 0.2-kg block attached to the same spring.

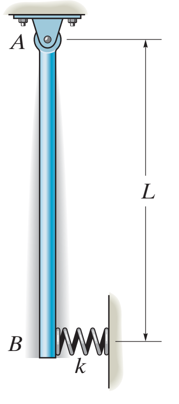
22–6. An 8-kg block is suspended from a spring having a stiffness $k = 80$ N/m. If the block is given an upward velocity of 0.4 m/s when it is 90 mm above its equilibrium position, determine the equation which describes the motion and the maximum upward displacement of the block measured from the equilibrium position. Assume that positive displacement is measured downward.

22–7. A 2-lb weight is suspended from a spring having a stiffness $k = 2$ lb/in. If the weight is pushed 1 in. upward from its equilibrium position and then released from rest, determine the equation which describes the motion. What is the amplitude and the natural frequency of the vibration?

***22–8.** A 6-lb weight is suspended from a spring having a stiffness $k = 3$ lb/in. If the weight is given an upward velocity of 20 ft/s when it is 2 in. above its equilibrium position, determine the equation which describes the motion and the maximum upward displacement of the weight, measured from the equilibrium position. Assume positive displacement is downward.

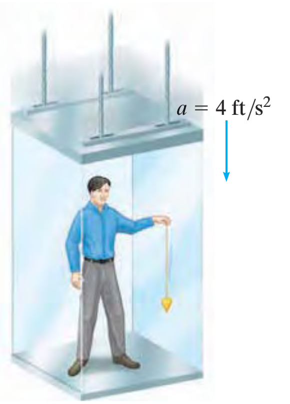
22–9. A 3-kg block is suspended from a spring having a stiffness of $k = 200$ N/m. If the block is pushed 50 mm upward from its equilibrium position and then released from rest, determine the equation that describes the motion. What are the amplitude and the natural frequency of the vibration? Assume that positive displacement is downward.

22–10. The uniform rod of mass m is supported by a pin at A and a spring at B. If B is given a small sideward displacement and released, determine the natural period of vibration.



Prob. 22–10

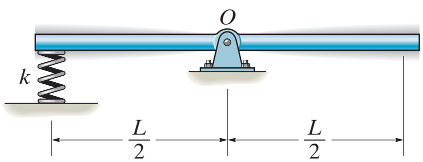
22–11. While standing in an elevator, the man holds a pendulum which consists of an 18-in. cord and a 0.5-lb bob. If the elevator is descending with an acceleration $a = 4$ ft/s^2, determine the natural period of vibration for small amplitudes of swing.



Prob. 22–11

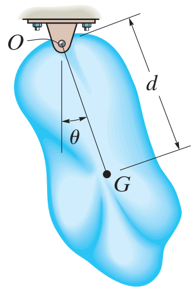
22

***22–12.** Determine the natural period of vibration of the uniform bar of mass m when it is displaced downward slightly and released.



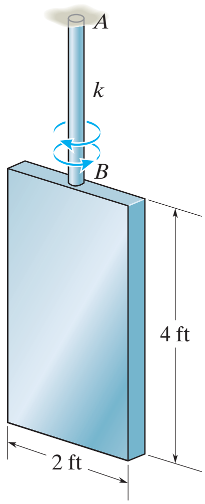
Prob. 22–12

22–13. The body of arbitrary shape has a mass m, mass center at G, and a radius of gyration about G of k_G. If it is displaced a slight amount θ from its equilibrium position and released, determine the natural period of vibration.



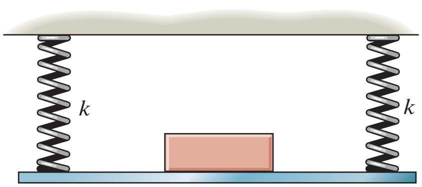
Prob. 22–13

22–14. The 20-lb rectangular plate has a natural period of vibration $\tau = 0.3$ s, as it oscillates around the axis of rod AB. Determine the torsional stiffness k, measured in lb · ft/rad, of the rod. Neglect the mass of the rod.



Prob. 22–14

22–15. A platform, having an unknown mass, is supported by *four* springs, each having the same stiffness k. When nothing is on the platform, the period of vertical vibration is measured as 2.35 s; whereas if a 3-kg block is supported on the platform, the period of vertical vibration is 5.23 s. Determine the mass of a block placed on the (empty) platform which causes the platform to vibrate vertically with a period of 5.62 s. What is the stiffness k of each of the springs?

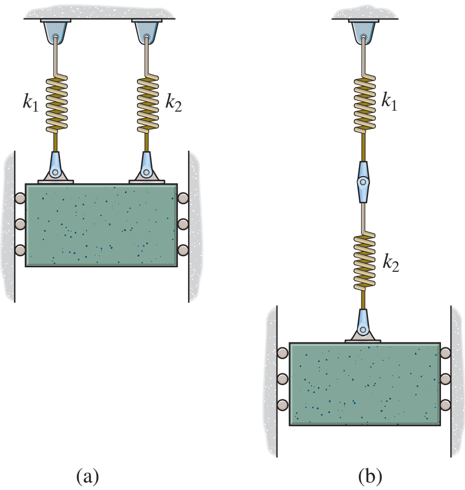


Prob. 22–15

22

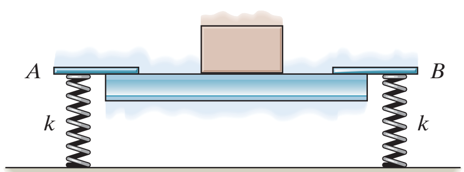
***22–16.** A block of mass m is suspended from two springs having a stiffness of k_1 and k_2, arranged a) parallel to each other, and b) as a series. Determine the equivalent stiffness of a single spring with the same oscillation characteristics and the period of oscillation for each case.

22–17. The 15-kg block is suspended from two springs having a different stiffness and arranged a) parallel to each other, and b) as a series. If the natural periods of oscillation of the parallel system and series system are observed to be 0.5 s and 1.5 s, respectively, determine the spring stiffnesses k_1 and k_2.



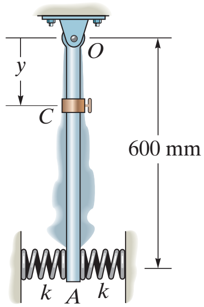
Probs. 22–16/17

22–18. The uniform beam is supported at its ends by two springs A and B, each having the same stiffness k. When nothing is supported on the beam, it has a period of vertical vibration of 0.83 s. If a 50-kg mass is placed at its center, the period of vertical vibration is 1.52 s. Compute the stiffness of each spring and the mass of the beam.



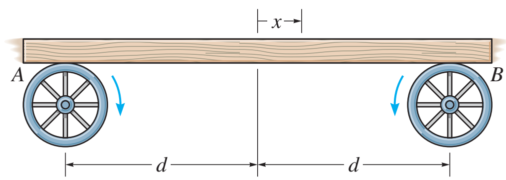
Prob. 22–18

22–19. The slender rod has a mass of 0.2 kg and is supported at O by a pin and at its end A by two springs, each having a stiffness $k = 4\ \text{N/m}$. The period of vibration of the rod can be set by fixing the 0.5-kg collar C to the rod at an appropriate location along its length. If the springs are originally unstretched when the rod is vertical, determine the position y of the collar so that the natural period of vibration becomes $\tau = 1$ s. Neglect the size of the collar.



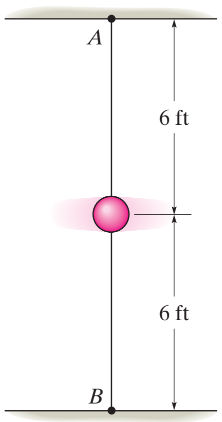
Prob. 22–19

***22–20.** A uniform board is supported on two wheels which rotate in opposite directions at a constant angular speed. If the coefficient of kinetic friction between the wheels and board is μ, determine the frequency of vibration of the board if it is displaced slightly, a distance x from the midpoint between the wheels, and released.



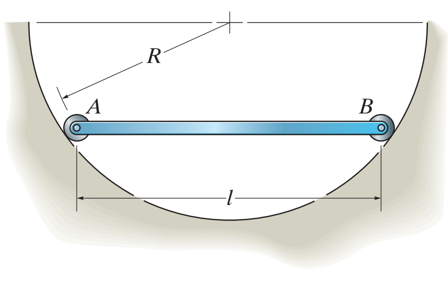
Prob. 22–20

22–21. If the wire AB is subjected to a tension of 20 lb, determine the equation which describes the motion when the 5-lb weight is displaced 2 in. horizontally and released from rest.



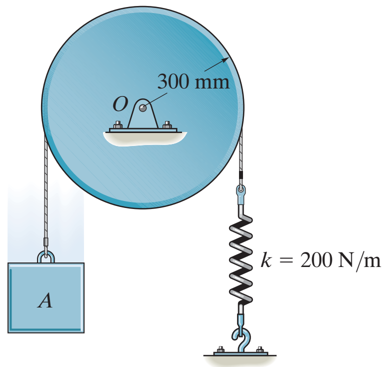
Prob. 22–21

22–22. The bar has a length l and mass m. It is supported at its ends by rollers of negligible mass. If it is given a small displacement and released, determine the natural frequency of vibration.



Prob. 22–22

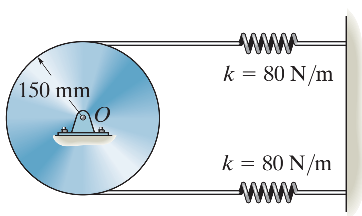
22–23. The 20-kg disk, is pinned at its mass center O and supports the 4-kg block A. If the belt which passes over the disk is not allowed to slip at its contacting surface, determine the natural period of vibration of the system.



Prob. 22–23

***22–24.** The 10-kg disk is pin connected at its mass center. Determine the natural period of vibration of the disk if the springs have sufficient tension in them to prevent the cord from slipping on the disk as it oscillates. *Hint:* Assume that the initial stretch in each spring is δ_O.

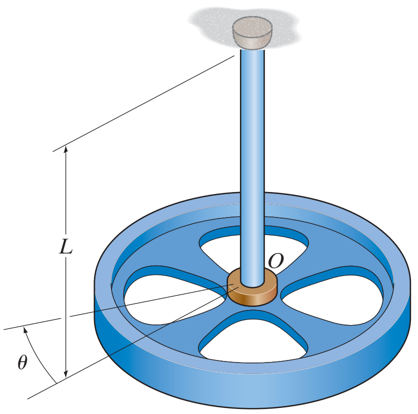
22–25. If the disk in Prob. 22–24 has a mass of 10 kg, determine the natural frequency of vibration. *Hint:* Assume that the initial stretch in each spring is δ_O.



Probs. 22–24/25

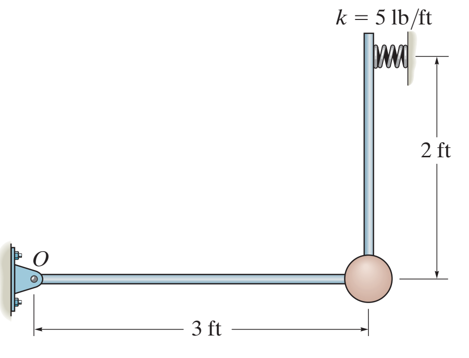
22

22–26. A flywheel of mass m, which has a radius of gyration about its center of mass of k_O, is suspended from a circular shaft that has a torsional resistance of $M = C\theta$. If the flywheel is given a small angular displacement of θ and released, determine the natural period of oscillation.



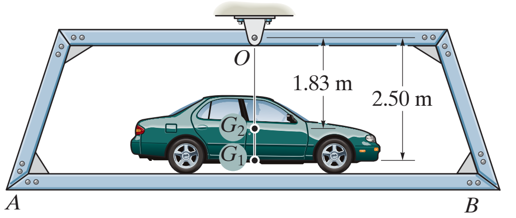
Prob. 22–26

22–27. The 6-lb weight is attached to the rods of negligible mass. Determine the natural frequency of vibration of the weight when it is displaced slightly from the equilibrium position and released.



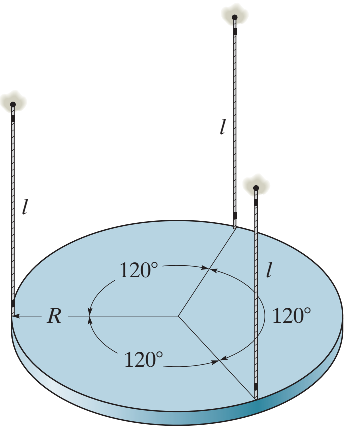
Prob. 22–27

***22–28.** The platform AB when empty has a mass of 400 kg, center of mass at G_1, and natural period of oscillation $\tau_1 = 2.38$ s. If a car, having a mass of 1.2 Mg and center of mass at G_2, is placed on the platform, the natural period of oscillation becomes $\tau_2 = 3.16$ s. Determine the moment of inertia of the car about an axis passing through G_2.



Prob. 22–28

22–29. The plate of mass m is supported by three symmetrically placed cords of length l as shown. If the plate is given a slight rotation about a vertical axis through its center and released, determine the natural period of oscillation.



Prob. 22–29

22

*22.2 Energy Methods

The simple harmonic motion of a body, discussed in the previous section, is due only to gravitational and elastic restoring forces acting on the body. Since these forces are *conservative*, it is also possible to use the conservation of energy equation to obtain the body's natural frequency or period of vibration. To show how to do this, consider again the block and spring model in Fig. 22–8. When the block is displaced x from the equilibrium position, the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$ and the potential energy is $V = \frac{1}{2}kx^2$. Since energy is conserved, it is necessary that

$$T + V = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \qquad (22\text{–}17)$$

The differential equation describing the *accelerated motion* of the block can be obtained by *differentiating* this equation with respect to time; i.e.,

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\dot{x}(m\ddot{x} + kx) = 0$$

Since the velocity $\dot{x}$ is not *always* zero in a vibrating system,

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m}$$

which is the same as Eq. 22–1.

If the conservation of energy equation is written for a *system of connected bodies*, the natural frequency or the equation of motion can also be determined by time differentiation. It is *not necessary* to dismember the system to account for the internal forces because they do no work.

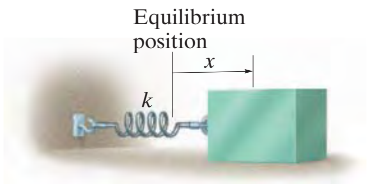


Fig. 22–8

The suspension of a railroad car consists of a set of springs which are mounted between the frame of the car and the wheel truck. This will give the car a natural frequency of vibration which can be determined. (© R.C. Hibbeler)

Procedure for Analysis

The natural frequency ω_n of a body or system of connected bodies can be determined by applying the conservation of energy equation using the following procedure.

Energy Equation.

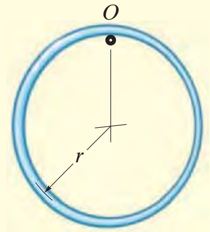
- Draw the body when it is displaced by a *small amount* from its equilibrium position and define the location of the body from its equilibrium position by an appropriate position coordinate q.
- Formulate the conservation of energy for the body, $T + V =$ constant, in terms of the position coordinate.
- In general, the kinetic energy must account for both the body's translational and rotational motion, $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$, Eq. 18–2.
- The potential energy is the sum of the gravitational and elastic potential energies of the body, $V = V_g + V_e$, Eq. 18–17. In particular, V_g should be measured from a datum for which $q = 0$ (equilibrium position).

Time Derivative.

- Take the time derivative of the energy equation using the chain rule of calculus and factor out the common terms. The resulting differential equation represents the equation of motion for the system. The natural frequency of ω_n is obtained after rearranging the terms in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE 22.5

The thin hoop shown in Fig. 22–9*a* is supported by the peg at O. Determine the natural period of oscillation for small amplitudes of swing. The hoop has a mass m.



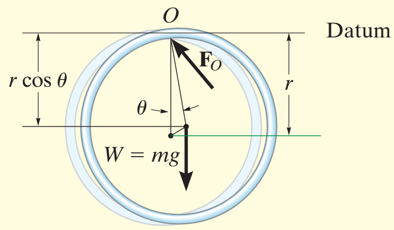
(a)

SOLUTION

Energy Equation. A diagram of the hoop when it is displaced a small amount $(q =)\,\theta$ from the equilibrium position is shown in Fig. 22–9*b*. Using the table on the inside back cover and the parallel-axis theorem to determine I_O, the kinetic energy is

$$T = \tfrac{1}{2}I_O\omega_n^2 = \tfrac{1}{2}[mr^2 + mr^2]\dot{\theta}^2 = mr^2\dot{\theta}^2$$

If a horizontal datum is placed through point O, then in the displaced position, the potential energy is



(b)

Fig. 22–9

$$V = -mg(r\cos\theta)$$

The total energy in the system is

$$T + V = mr^2\dot{\theta}^2 - mgr\cos\theta$$

Time Derivative.

$$mr^2(2\dot{\theta})\ddot{\theta} + mgr(\sin\theta)\dot{\theta} = 0$$

$$mr\dot{\theta}(2r\ddot{\theta} + g\sin\theta) = 0$$

Since $\dot{\theta}$ is not always equal to zero, from the terms in parentheses,

$$\ddot{\theta} + \frac{g}{2r}\sin\theta = 0$$

For small angle θ, $\sin\theta \approx \theta$.

$$\ddot{\theta} + \frac{g}{2r}\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{2r}}$$

so that

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2r}{g}} \qquad \textit{Ans.}$$

EXAMPLE 22.6

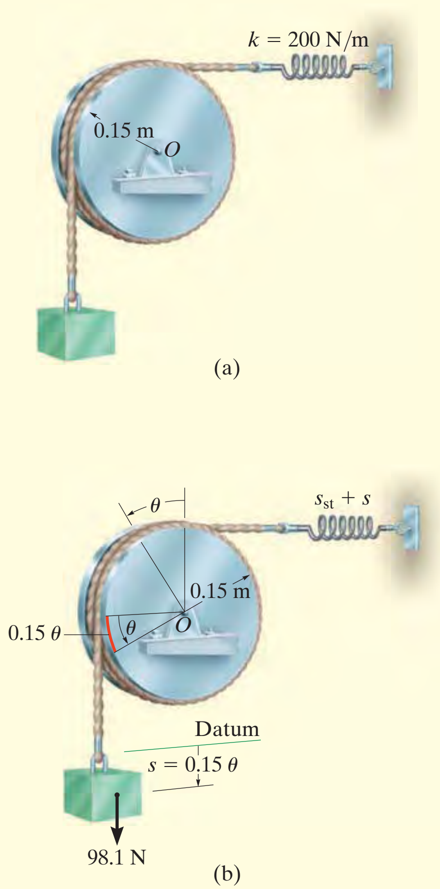


Fig. 22–10

A 10-kg block is suspended from a cord wrapped around a 5-kg disk, as shown in Fig. 22–10*a*. If the spring has a stiffness $k = 200\ \text{N/m}$, determine the natural period of vibration for the system.

SOLUTION

Energy Equation. A diagram of the block and disk when they are displaced by respective amounts s and θ from the equilibrium position is shown in Fig. 22–10*b*. Since $s = (0.15\ \text{m})\theta$, then $v_b \approx \dot{s} = (0.15\ \text{m})\dot{\theta}$. Thus, the kinetic energy of the system is

$$\begin{aligned} T &= \tfrac{1}{2}m_b v_b^2 + \tfrac{1}{2}I_O\omega_d^2 \\ &= \tfrac{1}{2}(10\ \text{kg})[(0.15\ \text{m})\dot{\theta}]^2 + \tfrac{1}{2}\left[\tfrac{1}{2}(5\ \text{kg})(0.15\ \text{m})^2\right](\dot{\theta})^2 \\ &= 0.1406(\dot{\theta})^2 \end{aligned}$$

Establishing the datum at the equilibrium position of the block and realizing that the spring stretches s_{st} for equilibrium, the potential energy is

$$\begin{aligned} V &= \tfrac{1}{2}k(s_{st} + s)^2 - Ws \\ &= \tfrac{1}{2}(200\ \text{N/m})[s_{st} + (0.15\ \text{m})\theta]^2 - 98.1\ \text{N}[(0.15\ \text{m})\theta] \end{aligned}$$

The total energy for the system is therefore,

$$T + V = 0.1406(\dot{\theta})^2 + 100(s_{st} + 0.15\theta)^2 - 14.715\theta$$

Time Derivative.

$$0.28125(\dot{\theta})\ddot{\theta} + 200(s_{st} + 0.15\theta)0.15\dot{\theta} - 14.72\dot{\theta} = 0$$

Since $s_{st} = 98.1/200 = 0.4905\ \text{m}$, the above equation reduces to the "standard form"

$$\ddot{\theta} + 16\theta = 0$$

so that

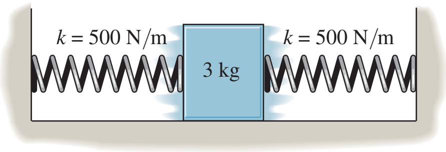
$$\omega_n = \sqrt{16} = 4\ \text{rad/s}$$

Thus,

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4} = 1.57\ \text{s} \qquad \textit{Ans.}$$

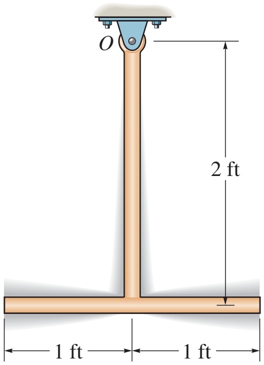
PROBLEMS

22–30. Determine the differential equation of motion of the 3-kg block when it is displaced slightly and released. The surface is smooth and the springs are originally unstretched.



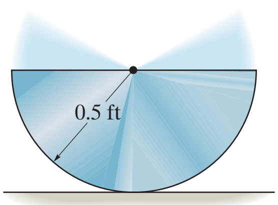
Prob. 22–30

22–31. Determine the natural period of vibration of the pendulum. Consider the two rods to be slender, each having a weight of 8 lb/ft.



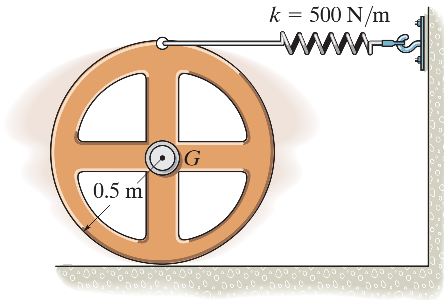
Prob. 22–31

***22–32.** Determine the natural period of vibration of the 10-lb semicircular disk.



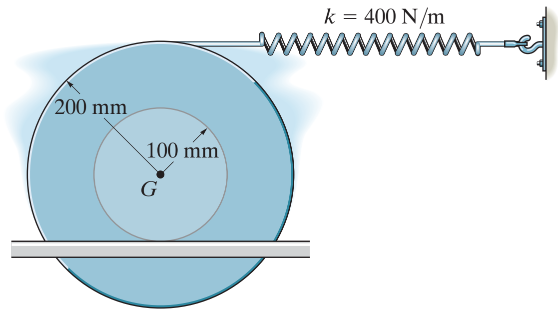
Prob. 22–32

22–33. If the 20-kg wheel is displaced a small amount and released, determine the natural period of vibration. The radius of gyration of the wheel is $k_G = 0.36$ m. The wheel rolls without slipping.



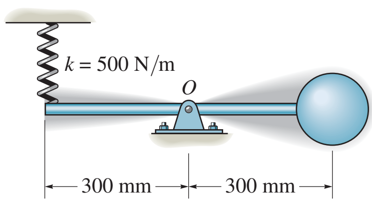
Prob. 22–33

22–34. Determine the differential equation of motion of the 3-kg spool. Assume that it does not slip at the surface of contact as it oscillates. The radius of gyration of the spool about its center of mass is $k_G = 125$ mm.



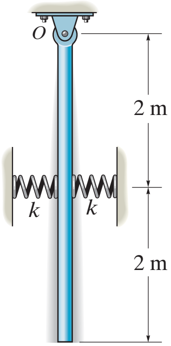
Prob. 22–34

22–35. Determine the natural period of vibration of the 3-kg sphere. Neglect the mass of the rod and the size of the sphere.



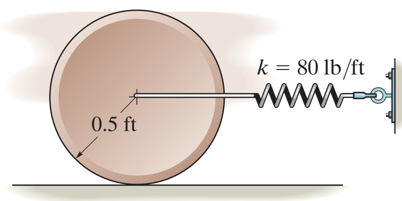
Prob. 22–35

***22–36.** If the lower end of the 6-kg slender rod is displaced a small amount and released from rest, determine the natural frequency of vibration. Each spring has a stiffness of $k = 200\ \text{N/m}$ and is unstretched when the rod is hanging vertically.



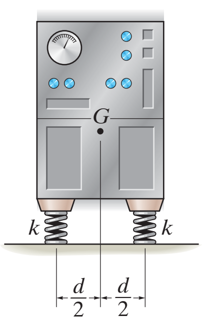
Prob. 22–36

22–37. The disk has a weight of 30 lb and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counterclockwise 0.2 rad, determine the equation which describes its oscillatory motion and the natural period when it is released.



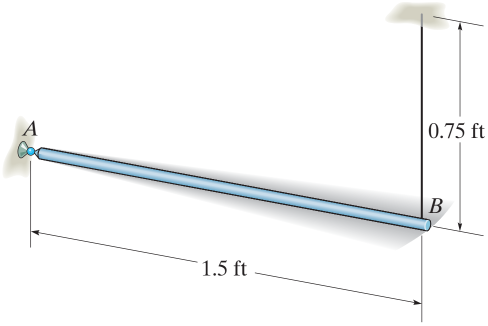
Prob. 22–37

22–38. The machine has a mass m and is uniformly supported by *four* springs, each having a stiffness k. Determine the natural period of vertical vibration.



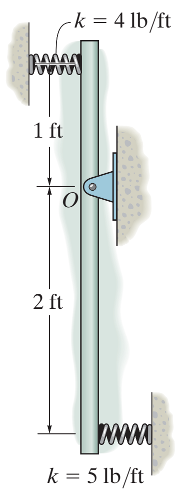
Prob. 22–38

22–39. The slender rod has a weight of 4 lb/ft. If it is supported in the horizontal plane by a ball-and-socket joint at A and a cable at B, determine the natural frequency of vibration when the end B is given a small horizontal displacement and then released.



Prob. 22–39

***22–40.** If the slender rod has a weight of 5 lb, determine the natural frequency of vibration. The springs are originally unstretched.



Prob. 22–40

*22.3 Undamped Forced Vibration

Undamped forced vibration is considered to be one of the most important types of vibrating motion in engineering. Its principles can be used to describe the motion of many types of machines and structures.

Periodic Force. The block and spring shown in Fig. 22–11*a* provide a convenient model which represents the vibrational characteristics of a system subjected to a periodic force $F = F_0 \sin \omega_0 t$. This force has an amplitude of F_0 and a *forcing frequency* ω_0. The free-body diagram for the block when it is displaced a distance x is shown in Fig. 22–11*b*. Applying the equation of motion, we have

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad F_0 \sin \omega_0 t - kx = m\ddot{x}$$

or

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_0 t \tag{22–18}$$

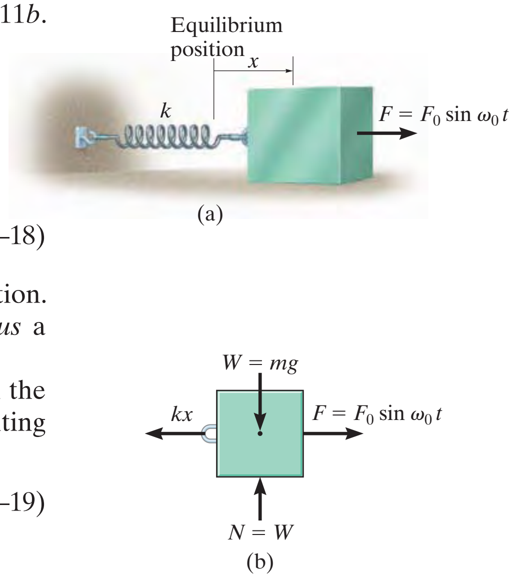


Fig. 22–11

This equation is a nonhomogeneous second-order differential equation. The general solution consists of a complementary solution, x_c, *plus* a particular solution, x_p.

The *complementary solution* is determined by setting the term on the right side of Eq. 22–18 equal to zero and solving the resulting homogeneous equation. The solution is defined by Eq. 22–9, i.e.,

$$x_c = C \sin(\omega_n t + \phi) \tag{22–19}$$

where ω_n is the natural frequency, $\omega_n = \sqrt{k/m}$, Eq. 22–2.

Since the motion is periodic, the *particular solution* of Eq. 22–18 can be determined by assuming a solution of the form

$$x_p = X \sin \omega_0 t \tag{22–20}$$

where X is a constant. Taking the second time derivative and substituting into Eq. 22–18 yields

$$-X\omega_0^2 \sin \omega_0 t + \frac{k}{m}(X \sin \omega_0 t) = \frac{F_0}{m}\sin \omega_0 t$$

Factoring out $\sin \omega_0 t$ and solving for X gives

$$X = \frac{F_0/m}{(k/m) - \omega_0^2} = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \tag{22–21}$$

Substituting into Eq. 22–20, we obtain the particular solution

$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}\sin \omega_0 t \tag{22–22}$$

Shaker tables provide forced vibration and are used to separate out granular materials. (© R.C. Hibbeler)

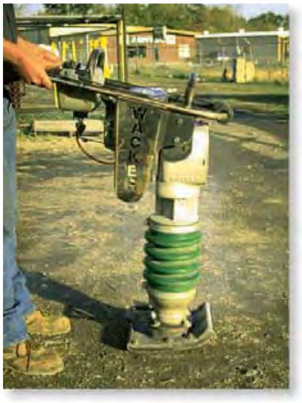
22

The *general solution* is therefore the sum of two sine functions having different frequencies.

$$x = x_c + x_p = C\sin(\omega_n t + \phi) + \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}\sin\omega_0 t \qquad (22\text{–}23)$$

The *complementary solution* x_c defines the *free vibration*, which depends on the natural frequency $\omega_n = \sqrt{k/m}$ and the constants C and ϕ. The *particular solution* x_p describes the *forced vibration* of the block caused by the applied force $F = F_0 \sin \omega_0 t$. Since all vibrating systems are subject to *friction*, the free vibration, x_c, will in time dampen out. For this reason the free vibration is referred to as *transient*, and the forced vibration is called *steady-state*, since it is the only vibration that remains.

From Eq. 22–21 it is seen that the *amplitude* of forced or steady-state vibration depends on the *frequency ratio* ω_0/ω_n. If the *magnification factor* MF is defined as the ratio of the amplitude of steady-state vibration, X, to the static deflection, F_0/k, which would be produced by the amplitude of the periodic force F_0, then, from Eq. 22–21,



The soil compactor operates by forced vibration developed by an internal motor. It is important that the forcing frequency not be close to the natural frequency of vibration of the compactor, which can be determined when the motor is turned off; otherwise resonance will occur and the machine will become uncontrollable. (© R.C. Hibbeler)

$$\text{MF} = \frac{X}{F_0/k} = \frac{1}{1 - (\omega_0/\omega_n)^2} \tag{22–24}$$

This equation is graphed in Fig. 22–12. Note that if the force or displacement is applied with a frequency close to the natural frequency of the system, i.e., $\omega_0/\omega_n \approx 1$, the amplitude of vibration of the block becomes extremely large. This occurs because the force **F** is applied to the block so that it always follows the motion of the block. This condition is called *resonance*, and in practice, resonating vibrations can cause tremendous stress and rapid failure of parts.*

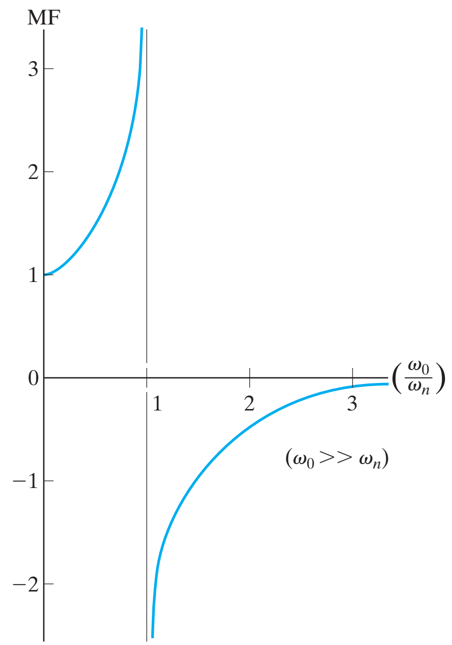


Fig. 22–12

Periodic Support Displacement. Forced vibrations can also arise from the periodic excitation of the support of a system. The model shown in Fig. 22–13*a* represents the periodic vibration of a block which is caused by harmonic movement $\delta = \delta_0 \sin \omega_0 t$ of the support. The free-body diagram for the block in this case is shown in Fig. 22–13*b*. The displacement δ of the support is measured from the point of zero displacement, i.e., when the radial line OA coincides with OB. Therefore, general deformation of the spring is $(x - \delta_0 \sin \omega_0 t)$. Applying the equation of motion yields

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -k(x - \delta_0 \sin \omega_0 t) = m\ddot{x}$$

or

$$\ddot{x} + \frac{k}{m}x = \frac{k\delta_0}{m}\sin \omega_0 t \tag{22–25}$$

By comparison, this equation is identical to the form of Eq. 22–18, *provided* F_0 is *replaced* by $k\delta_0$. If this substitution is made into the solutions defined by Eqs. 22–21 to 22–23, the results are appropriate for describing the motion of the block when subjected to the support displacement $\delta = \delta_0 \sin \omega_0 t$.

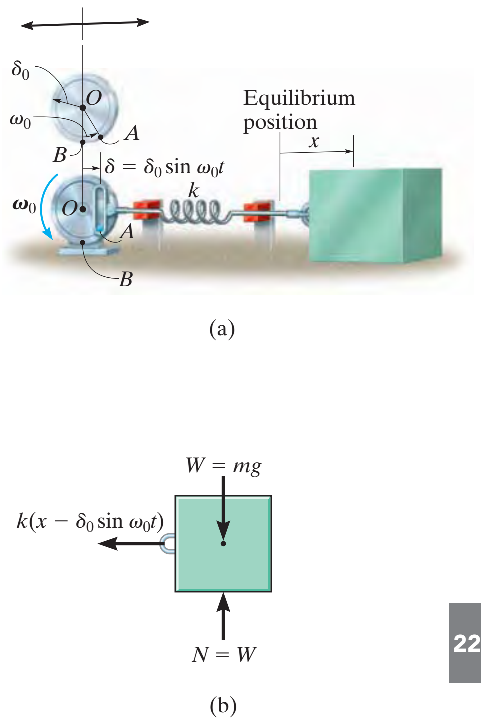


Fig. 22–13

*A swing has a natural period of vibration, as determined in Example 22.1. If someone pushes on the swing only when it reaches its highest point, neglecting drag or wind resistance, resonance will occur since the natural and forcing frequencies are the same.

22

EXAMPLE 22.7

The instrument shown in Fig. 22–14 is rigidly attached to a platform P, which in turn is supported by *four* springs, each having a stiffness $k = 800\ \text{N/m}$. If the floor is subjected to a vertical displacement $\delta = 10\sin(8t)$ mm, where t is in seconds, determine the amplitude of steady-state vibration. What is the frequency of the floor vibration required to cause resonance? The instrument and platform have a total mass of 20 kg.

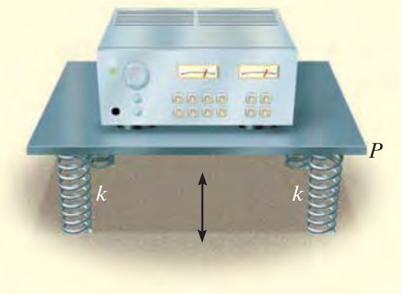


Fig. 22–14

SOLUTION

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4(800\ \text{N/m})}{20\ \text{kg}}} = 12.65\ \text{rad/s}$$

The amplitude of steady-state vibration is found using Eq. 22–21, with $k\delta_0$ replacing F_0.

$$X = \frac{\delta_0}{1 - (\omega_0/\omega_n)^2} = \frac{10}{1 - [(8\ \text{rad/s})/(12.65\ \text{rad/s})]^2} = 16.7\ \text{mm} \qquad \textit{Ans.}$$

Resonance will occur when the amplitude of vibration X caused by the floor displacement approaches infinity. This requires

$$\omega_0 = \omega_n = 12.6\ \text{rad/s} \qquad \textit{Ans.}$$

*22.4 Viscous Damped Free Vibration

The vibration analysis considered thus far has not included the effects of friction or damping in the system, and as a result, the solutions obtained are only in close agreement with the actual motion. Since all vibrations die out in time, the presence of damping forces should be included in the analysis.

In many cases damping is attributed to the resistance created by the substance, such as water, oil, or air, in which the system vibrates. Provided the body moves slowly through this substance, the resistance to motion is directly proportional to the body's speed. The type of force developed under these conditions is called a *viscous damping force*. The magnitude of this force is expressed by an equation of the form

$$F = c\dot{x} \qquad (22\text{–}26)$$

where the constant c is called the *coefficient of viscous damping* and has units of $\text{N}\cdot\text{s/m}$ or $\text{lb}\cdot\text{s/ft}$.

The vibrating motion of a body or system having viscous damping can be characterized by the block and spring shown in Fig. 22–15*a*. The effect of damping is provided by the *dashpot* connected to the block on the right side. Damping occurs when the piston P moves to the right or left within the enclosed cylinder. The cylinder contains a fluid, and the motion of the piston is retarded since the fluid must flow around or through a small hole in the piston. The dashpot is assumed to have a coefficient of viscous damping c.

If the block is displaced a distance x from its equilibrium position, the resulting free-body diagram is shown in Fig. 22–15*b*. Both the spring and damping force oppose the forward motion of the block, so that applying the equation of motion yields

$$\overset{+}{\rightarrow}\Sigma F_x = ma_x; \qquad -kx - c\dot{x} = m\ddot{x}$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (22\text{–}27)$$

This linear, second-order, homogeneous, differential equation has a solution of the form

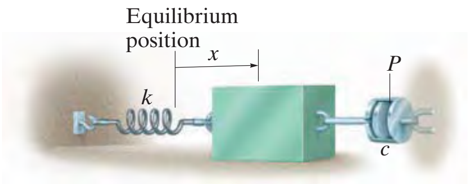
$$x = e^{\lambda t}$$

where e is the base of the natural logarithm and λ (lambda) is a constant. The value of λ can be obtained by substituting this solution and its time derivatives into Eq. 22–27, which yields

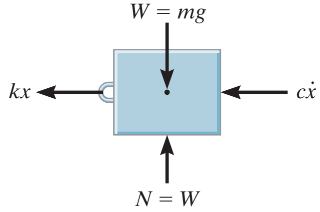
$$m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + ke^{\lambda t} = 0$$

or

$$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0$$



(a)



(b)

Fig. 22–15

Since $e^{\lambda t}$ can never be zero, a solution is possible provided

$$m\lambda^2 + c\lambda + k = 0$$

Hence, by the quadratic formula, the two values of λ are

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$
$$\lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (22\text{–}28)$$

The general solution of Eq. 22–27 is therefore a combination of exponentials which involves both of these roots. There are three possible combinations of λ_1 and λ_2 which must be considered. Before discussing these combinations, however, we will first define the *critical damping coefficient* c_c as the value of c which makes the radical in Eqs. 22–28 equal to zero; i.e.,

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \quad (22\text{–}29)$$

Overdamped System. When $c > c_c$, the roots λ_1 and λ_2 are both real. The general solution of Eq. 22–27 can then be written as

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \quad (22\text{–}30)$$

Motion corresponding to this solution is *nonvibrating*. The effect of damping is so strong that when the block is displaced and released, it simply creeps back to its original position without oscillating. The system is said to be *overdamped*.

Critically Damped System. If $c = c_c$, then $\lambda_1 = \lambda_2 = -c_c/2m = -\omega_n$. This situation is known as *critical damping*, since it represents a condition where c has the smallest value necessary to cause the system to be nonvibrating. Using the methods of differential equations, it can be shown that the solution to Eq. 22–27 for critical damping is

$$x = (A + Bt)e^{-\omega_n t} \quad (22\text{–}31)$$

Underdamped System. Most often $c < c_c$, in which case the system is referred to as *underdamped*. In this case the roots λ_1 and λ_2 are complex numbers, and it can be shown that the general solution of Eq. 22–27 can be written as

$$x = D[e^{-(c/2m)t}\sin(\omega_d t + \phi)] \tag{22–32}$$

where D and ϕ are constants generally determined from the initial conditions of the problem. The constant ω_d is called the *damped natural frequency* of the system. It has a value of

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1 - \left(\frac{c}{c_c}\right)^2} \tag{22–33}$$

where the ratio c/c_c is called the *damping factor*.

The graph of Eq. 22–32 is shown in Fig. 22–16. The initial limit of motion, D, diminishes with each cycle of vibration, since motion is confined within the bounds of the exponential curve. Using the damped natural frequency ω_d, the period of damped vibration can be written as

$$\tau_d = \frac{2\pi}{\omega_d} \tag{22–34}$$

Since $\omega_d < \omega_n$, Eq. 22–33, the period of damped vibration, τ_d, will be greater than that of free vibration, $\tau = 2\pi/\omega_n$.

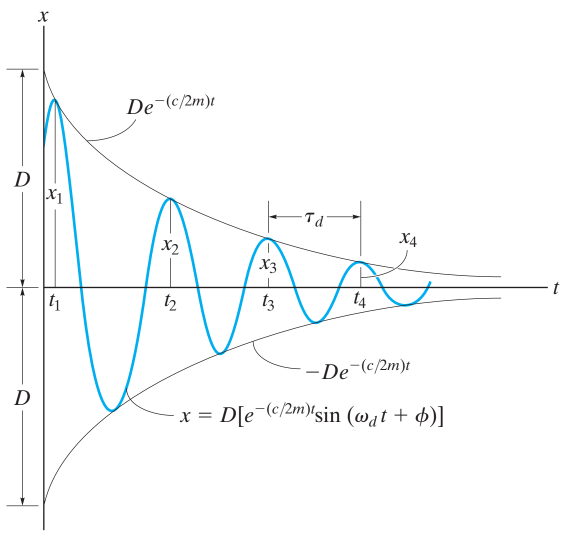


Fig. 22–16

*22.5 Viscous Damped Forced Vibration

The most general case of single-degree-of-freedom vibrating motion occurs when the system includes the effects of forced motion and induced damping. The analysis of this particular type of vibration is of practical value when applied to systems having significant damping characteristics.

If a dashpot is attached to the block and spring shown in Fig. 22–11*a*, the differential equation which describes the motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_0 t \qquad (22\text{–}35)$$

A similar equation can be written for a block and spring having a periodic support displacement, Fig. 22–13*a*, which includes the effects of damping. In that case, however, F_0 is replaced by $k\delta_0$. Since Eq. 22–35 is nonhomogeneous, the general solution is the sum of a complementary solution, x_c, and a particular solution, x_p. The complementary solution is determined by setting the right side of Eq. 22–35 equal to zero and solving the homogeneous equation, which is equivalent to Eq. 22–27. The solution is therefore given by Eq. 22–30, 22–31, or 22–32, depending on the values of λ_1 and λ_2. Because all systems are subjected to friction, then this solution will dampen out with time. Only the particular solution, which describes the *steady-state vibration* of the system, will remain. Since the applied forcing function is harmonic, the steady-state motion will also be harmonic. Consequently, the particular solution will be of the form

$$X_P = X' \sin(\omega_0 t - \phi') \qquad (22\text{–}36)$$

The constants X' and ϕ' are determined by taking the first and second time derivatives and substituting them into Eq. 22–35, which after simplification yields

$$-X'm\omega_0^2 \sin(\omega_0 t - \phi') + X'c\omega_0 \cos(\omega_0 t - \phi') + X'k \sin(\omega_0 t - \phi') = F_0 \sin \omega_0 t$$

Since this equation holds for all time, the constant coefficients can be obtained by setting $\omega_0 t - \phi' = 0$ and $\omega_0 t - \phi' = \pi/2$, which causes the above equation to become

22

$$X'c\omega_0 = F_0 \sin \phi'$$

$$-X'm\omega_0^2 + X'k = F_0 \cos \phi'$$

The amplitude is obtained by squaring these equations, adding the results, and using the identity $\sin^2\phi' + \cos^2\phi' = 1$, which gives

$$X' = \frac{F_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}} \tag{22–37}$$

Dividing the first equation by the second gives

$$\phi' = \tan^{-1}\left[\frac{c\omega_0}{k - m\omega_0^2}\right] \tag{22–38}$$

Since $\omega_n = \sqrt{k/m}$ and $c_c = 2m\omega_n$, then the above equations can also be written as

$$X' = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}$$

$$\phi' = \tan^{-1}\left[\frac{2(c/c_c)(\omega_0/\omega_n)}{1 - (\omega_0/\omega_n)^2}\right] \tag{22–39}$$

The angle ϕ' represents the phase difference between the applied force and the resulting steady-state vibration of the damped system.

The *magnification factor* MF has been defined in Sec. 22.3 as the ratio of the amplitude of deflection caused by the forced vibration to the deflection caused by a static force F_0. Thus,

$$\text{MF} = \frac{X'}{F_0/k} = \frac{1}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}} \tag{22–40}$$

The MF is plotted in Fig. 22–17 versus the frequency ratio ω_0/ω_n for various values of the damping factor c/c_c. It can be seen from this graph that the magnification of the amplitude increases as the damping factor decreases. Resonance obviously occurs only when the damping factor is zero and the frequency ratio equals 1.

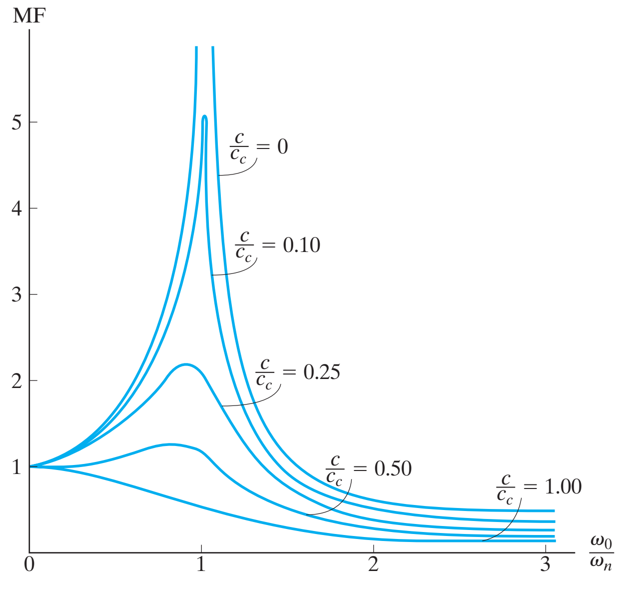


Fig. 22–17

EXAMPLE 22.8

The 30-kg electric motor shown in Fig. 22–18 is confined to move vertically, and is supported by *four* springs, each spring having a stiffness of 200 N/m. If the rotor is unbalanced such that its effect is equivalent to a 4-kg mass located 60 mm from the axis of rotation, determine the amplitude of vibration when the rotor is turning at $\omega_0 = 10$ rad/s. The damping factor is $c/c_c = 0.15$.

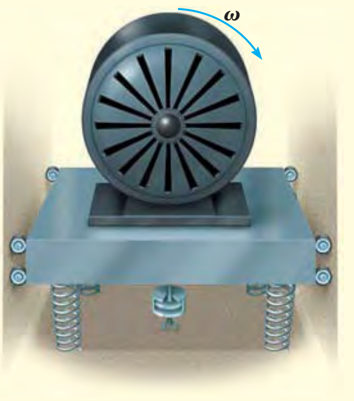


Fig. 22–18

SOLUTION

The periodic force which causes the motor to vibrate is the centrifugal force due to the unbalanced rotor. This force has a constant magnitude of

$$F_0 = ma_n = mr\omega_0^2 = 4\text{ kg}(0.06\text{ m})(10\text{ rad/s})^2 = 24\text{ N}$$

The stiffness of the entire system of four springs is $k = 4(200\text{ N/m}) = 800\text{ N/m}$. Therefore, the natural frequency of vibration is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{800\text{ N/m}}{30\text{ kg}}} = 5.164\text{ rad/s}$$

Since the damping factor is known, the steady-state amplitude can be determined from the first of Eqs. 22–39, i.e.,

$$X' = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}$$

$$= \frac{24/800}{\sqrt{[1 - (10/5.164)^2]^2 + [2(0.15)(10/5.164)]^2}}$$

$$= 0.0107\text{ m} = 10.7\text{ mm} \qquad \textit{Ans.}$$

*22.6 Electrical Circuit Analogs

The characteristics of a vibrating mechanical system can be represented by an electric circuit. Consider the circuit shown in Fig. 22–19*a*, which consists of an inductor L, a resistor R, and a capacitor C. When a voltage $E(t)$ is applied, it causes a current of magnitude i to flow through the circuit. As the current flows past the inductor the voltage drop is $L(di/dt)$, when it flows across the resistor the drop is Ri, and when it arrives at the capacitor the drop is $(1/C)\int i\,dt$. Since current cannot flow past a capacitor, it is only possible to measure the charge q acting on the capacitor. The charge can, however, be related to the current by the equation $i = dq/dt$. Thus, the voltage drops which occur across the inductor, resistor, and capacitor become $L\,d^2q/dt^2$, $R\,dq/dt$, and q/C, respectively. According to Kirchhoff's voltage law, the applied voltage balances the sum of the voltage drops around the circuit. Therefore,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \qquad (22\text{–}41)$$

Consider now the model of a single-degree-of-freedom mechanical system, Fig. 22–19*b*, which is subjected to both a general forcing function $F(t)$ and damping. The equation of motion for this system was established in the previous section and can be written as

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \qquad (22\text{–}42)$$

By comparison, it is seen that Eqs. 22–41 and 22–42 have the same form, and hence mathematically the procedure of analyzing an electric circuit is the same as that of analyzing a vibrating mechanical system. The analogs between the two equations are given in Table 22–1.

This analogy has important application to experimental work, for it is much easier to simulate the vibration of a complex mechanical system using an electric circuit, which can be constructed on an analog computer, than to make an equivalent mechanical spring-and-dashpot model.

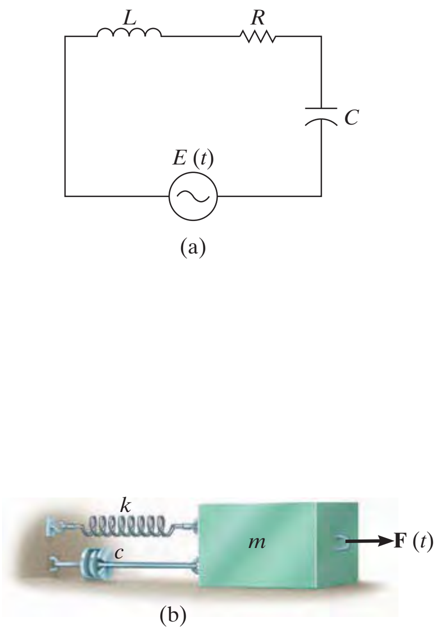


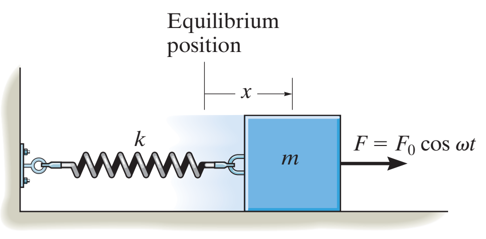
Fig. 22–19

TABLE 22–1
Electrical–Mechanical Analogs

Electrical		Mechanical	
Electric charge	q	Displacement	x
Electric current	i	Velocity	dx/dt
Voltage	$E(t)$	Applied force	$F(t)$
Inductance	L	Mass	m
Resistance	R	Viscous damping coefficient	c
Reciprocal of capacitance	$1/C$	Spring stiffness	k

PROBLEMS

22–41. If the block-and-spring model is subjected to the periodic force $F = F_0 \cos \omega t$, show that the differential equation of motion is $\ddot{x} + (k/m)x = (F_0/m)\cos \omega t$, where x is measured from the equilibrium position of the block. What is the general solution of this equation?



Prob. 22–41

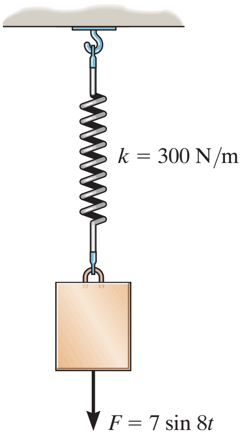
22–42. A block which has a mass m is suspended from a spring having a stiffness k. If an impressed downward vertical force $F = F_O$ acts on the weight, determine the equation which describes the position of the block as a function of time.

22–43. A 4-lb weight is attached to a spring having a stiffness $k = 10$ lb/ft. The weight is drawn downward a distance of 4 in. and released from rest. If the support moves with a vertical displacement $\delta = (0.5 \sin 4t)$ in., where t is in seconds, determine the equation which describes the position of the weight as a function of time.

***22–44.** A 4-kg block is suspended from a spring that has a stiffness of $k = 600$ N/m. The block is drawn downward 50 mm from the equilibrium position and released from rest when $t = 0$. If the support moves with an impressed displacement of $\delta = (10 \sin 4t)$ mm, where t is in seconds, determine the equation that describes the vertical motion of the block. Assume positive displacement is downward.

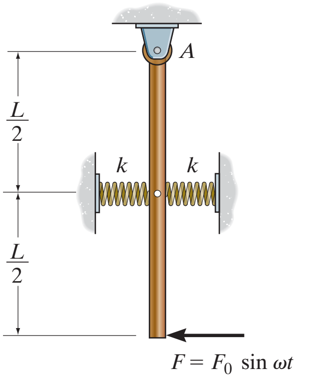
22–45. Use a block-and-spring model like that shown in Fig. 22–13*a*, but suspended from a vertical position and subjected to a periodic support displacement $\delta = \delta_0 \sin \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

22–46. A 5-kg block is suspended from a spring having a stiffness of 300 N/m. If the block is acted upon by a vertical force $F = (7 \sin 8t)$ N, where t is in seconds, determine the equation which describes the motion of the block when it is pulled down 100 mm from the equilibrium position and released from rest at $t = 0$. Assume that positive displacement is downward.



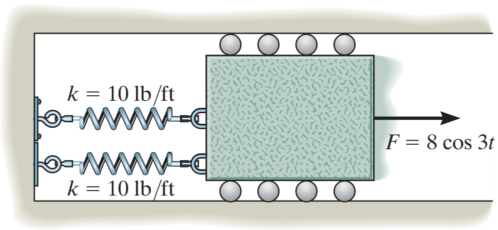
Prob. 22–46

22–47. The uniform rod has a mass of m. If it is acted upon by a periodic force of $F = F_0 \sin \omega t$, determine the amplitude of the steady-state vibration.



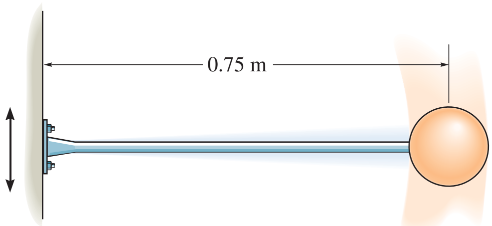
Prob. 22–47

*22–48. The 30-lb block is attached to two springs having a stiffness of 10 lb/ft. A periodic force $F = (8 \cos 3t)$ lb, where t is in seconds, is applied to the block. Determine the maximum speed of the block after frictional forces cause the free vibrations to dampen out.



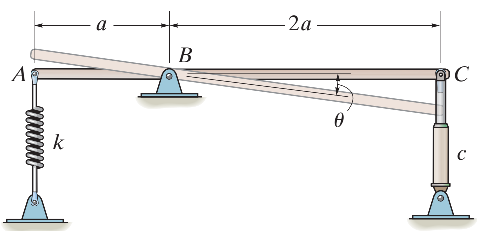
Prob. 22–48

22–49. The light elastic rod supports a 4-kg sphere. When an 18-N vertical force is applied to the sphere, the rod deflects 14 mm. If the wall oscillates with harmonic frequency of 2 Hz and has an amplitude of 15 mm, determine the amplitude of vibration for the sphere.



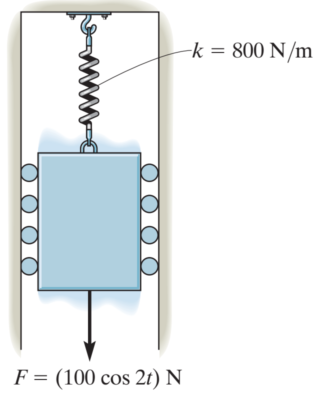
Prob. 22–49

22–50. Find the differential equation for small oscillations in terms of θ for the uniform rod of mass m. Also show that if $c < \sqrt{mk}/2$, then the system remains underdamped. The rod is in a horizontal position when it is in equilibrium.



Prob. 22–50

22–51. The 40-kg block is attached to a spring having a stiffness of 800 N/m. A force $F = (100 \cos 2t)$ N, where t is in seconds is applied to the block. Determine the maximum speed of the block for the steady-state vibration.



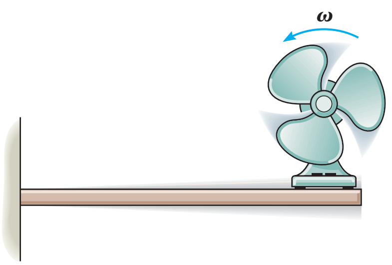
Prob. 22–51

*22–52. Using a block-and-spring model, like that shown in Fig. 22–13a, but suspended from a vertical position and subjected to a periodic support displacement of $\delta = \delta_0 \cos \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

22–53. The fan has a mass of 25 kg and is fixed to the end of a horizontal beam that has a negligible mass. The fan blade is mounted eccentrically on the shaft such that it is equivalent to an unbalanced 3.5-kg mass located 100 mm from the axis of rotation. If the static deflection of the beam is 50 mm as a result of the weight of the fan, determine the angular velocity of the fan blade at which resonance will occur. *Hint*: See the first part of Example 22.8.

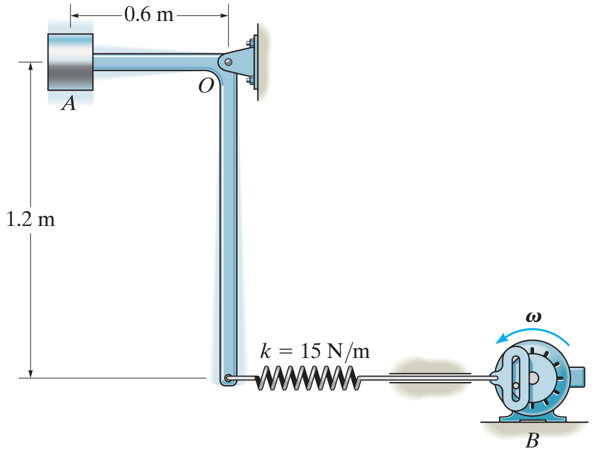
22–54. In Prob. 22–53, determine the amplitude of steady-state vibration of the fan if its angular velocity is 10 rad/s.

22–55. What will be the amplitude of steady-state vibration of the fan in Prob. 22–53 if the angular velocity of the fan blade is 18 rad/s? *Hint*: See the first part of Example 22.8.



Probs. 22–53/54/55

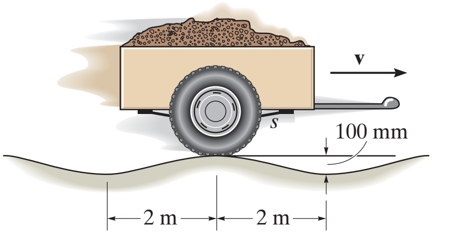
*22–56. The small block at A has a mass of 4 kg and is mounted on the bent rod having negligible mass. If the rotor at B causes a harmonic movement $\delta_B = (0.1 \cos 15t)$ m, where t is in seconds, determine the steady-state amplitude of vibration of the block.



Prob. 22–56

*22–60. The 450-kg trailer is pulled with a constant speed over the surface of a bumpy road, which may be approximated by a cosine curve having an amplitude of 50 mm and wave length of 4 m. If the two springs s which support the trailer each have a stiffness of 800 N/m, determine the speed v which will cause the greatest vibration (resonance) of the trailer. Neglect the weight of the wheels.

22–61. Determine the amplitude of vibration of the trailer in Prob. 22–60 if the speed $v = 15$ km/h.

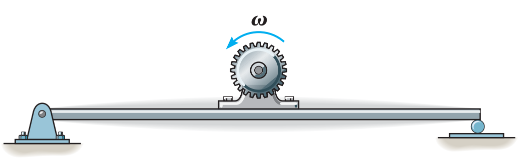


Probs. 22–60/61

22–57. The electric motor turns an eccentric flywheel which is equivalent to an unbalanced 0.25-lb weight located 10 in. from the axis of rotation. If the static deflection of the beam is 1 in. because of the weight of the motor, determine the angular velocity of the flywheel at which resonance will occur. The motor weighs 150 lb. Neglect the mass of the beam.

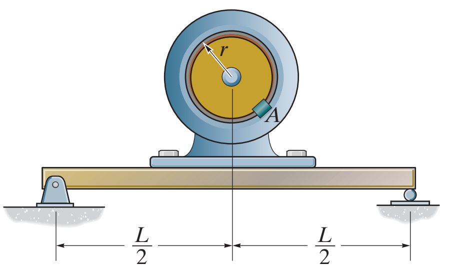
22–58. What will be the amplitude of steady-state vibration of the motor in Prob. 22–57 if the angular velocity of the flywheel is 20 rad/s?

22–59. Determine the angular velocity of the flywheel in Prob. 22–57 which will produce an amplitude of vibration of 0.25 in.



Probs. 22–57/58/59

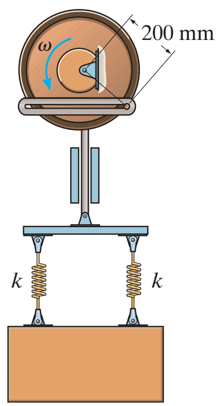
22–62. The motor of mass M is supported by a simply supported beam of negligible mass. If block A of mass m is clipped onto the rotor, which is turning at constant angular velocity of ω, determine the amplitude of the steady-state vibration. *Hint*: When the beam is subjected to a concentrated force of P at its mid-span, it deflects $\delta = PL^3/48EI$ at this point. Here E is Young's modulus of elasticity, a property of the material, and I is the moment of inertia of the beam's cross-sectional area.



Prob. 22–62

22–63. The spring system is connected to a crosshead that oscillates vertically when the wheel rotates with a constant angular velocity of $\boldsymbol{\omega}$. If the amplitude of the steady-state vibration is observed to be 400 mm, and the springs each have a stiffness of $k = 2500\ \text{N/m}$, determine the two possible values of $\boldsymbol{\omega}$ at which the wheel must rotate. The block has a mass of 50 kg.

***22–64.** The spring system is connected to a crosshead that oscillates vertically when the wheel rotates with a constant angular velocity of $\omega = 5\ \text{rad/s}$. If the amplitude of the steady-state vibration is observed to be 400 mm, determine the two possible values of the stiffness k of the springs. The block has a mass of 50 kg.



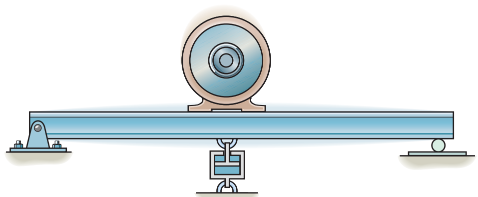
Probs. 22–63/64

22–65. A 7-lb block is suspended from a spring having a stiffness of $k = 75\ \text{lb/ft}$. The support to which the spring is attached is given simple harmonic motion which may be expressed as $\delta = (0.15 \sin 2t)$ ft, where t is in seconds. If the damping factor is $c/c_c = 0.8$, determine the phase angle ϕ of forced vibration.

22–66. Determine the magnification factor of the block, spring, and dashpot combination in Prob. 22–65.

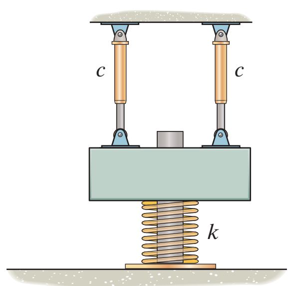
22–67. A block having a mass of 7 kg is suspended from a spring that has a stiffness $k = 600\ \text{N/m}$. If the block is given an upward velocity of 0.6 m/s from its equilibrium position at $t = 0$, determine its position as a function of time. Assume that positive displacement of the block is downward and that motion takes place in a medium which furnishes a damping force $F = (50|v|)$ N, where v is in m/s.

***22–68.** The 200-lb electric motor is fastened to the midpoint of the simply supported beam. It is found that the beam deflects 2 in. when the motor is not running. The motor turns an eccentric flywheel which is equivalent to an unbalanced weight of 1 lb located 5 in. from the axis of rotation. If the motor is turning at 100 rpm, determine the amplitude of steady-state vibration. The damping factor is $c/c_c = 0.20$. Neglect the mass of the beam.



Prob. 22–68

22–69. Two identical dashpots are arranged parallel to each other, as shown. Show that if the damping coefficient $c < \sqrt{mk}$, then the block of mass m will vibrate as an underdamped system.

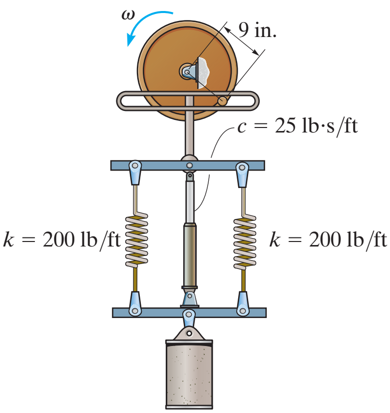


Prob. 22–69

22

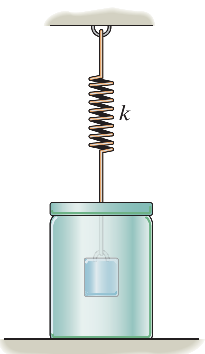
22–70. The damping factor, c/c_c, may be determined experimentally by measuring the successive amplitudes of vibrating motion of a system. If two of these maximum displacements can be approximated by x_1 and x_2, as shown in Fig. 22–16, show that $\ln(x_1/x_2) = 2\pi(c/c_c)/\sqrt{1-(c/c_c)^2}$. The quantity $\ln(x_1/x_2)$ is called the *logarithmic decrement*.

22–71. If the amplitude of the 50-lb cylinder's steady-state vibration is 6 in., determine the wheel's angular velocity ω.



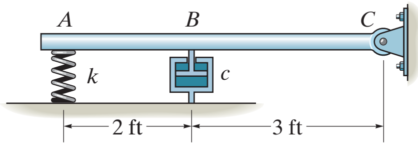
Prob. 22–71

***22–72.** The block, having a weight of 12 lb, is immersed in a liquid such that the damping force acting on the block has a magnitude of $F = (0.7|v|)$ lb, where v is in ft/s. If the block is pulled down 0.62 ft and released from rest, determine the position of the block as a function of time. The spring has a stiffness of $k = 53$ lb/ft. Assume that positive displacement is downward.



Prob. 22–72

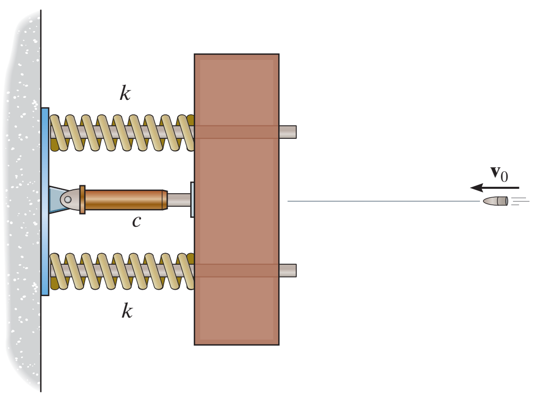
22–73. The bar has a weight of 6 lb. If the stiffness of the spring is $k = 8$ lb/ft and the dashpot has a damping coefficient $c = 60$ lb·s/ft, determine the differential equation which describes the motion in terms of the angle θ of the bar's rotation. Also, what should be the damping coefficient of the dashpot if the bar is to be critically damped?



Prob. 22–73

22–74. A bullet of mass m has a velocity of $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the vibration is to be critically damped, determine the dashpot's critical damping coefficient, and the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.

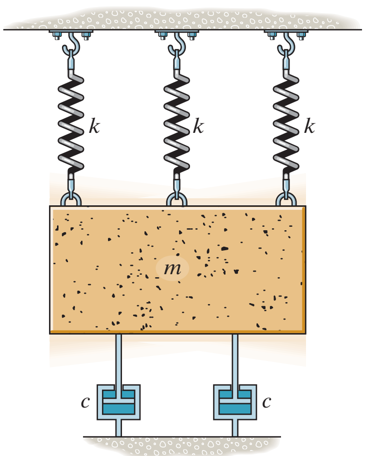
22–75. A bullet of mass m has a velocity $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the dashpot's damping coefficient is $0 < c << c_c$, determine the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.



Probs. 22–74/75

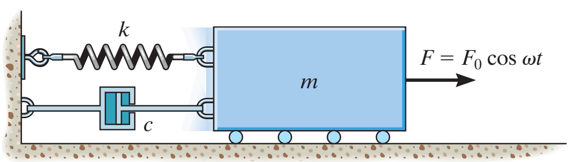
22

*22–76. Determine the differential equation of motion for the damped vibratory system shown. What type of motion occurs? Take $k = 100$ N/m, $c = 200$ N·s/m, $m = 25$ kg.



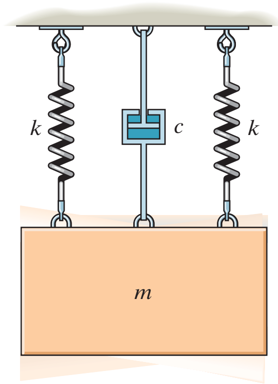
Prob. 22–76

22–77. Draw the electrical circuit that is equivalent to the mechanical system shown. Determine the differential equation which describes the charge q in the circuit.



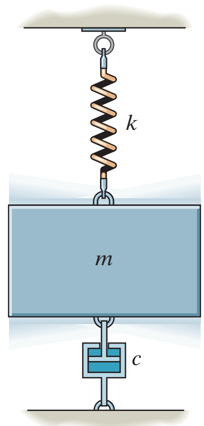
Prob. 22–77

22–78. Draw the electrical circuit that is equivalent to the mechanical system shown. What is the differential equation which describes the charge q in the circuit?



Prob. 22–78

22–79. Draw the electrical circuit that is equivalent to the mechanical system shown. Determine the differential equation which describes the charge q in the circuit.



Prob. 22–79

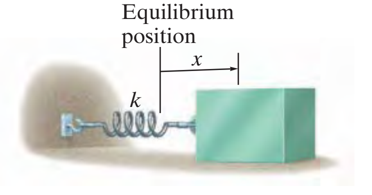
22

CHAPTER REVIEW

Undamped Free Vibration

A body has free vibration when gravitational or elastic restoring forces cause the motion. This motion is undamped when friction forces are neglected. The periodic motion of an undamped, freely vibrating body can be studied by displacing the body from the equilibrium position and then applying the equation of motion along the path.

For a one-degree-of-freedom system, the resulting differential equation can be written in terms of its natural frequency ω_n.



$$\ddot{x} + \omega_n^2 x = 0 \qquad \tau = \frac{2\pi}{\omega_n} \qquad f = \frac{1}{\tau} = \frac{\omega_n}{2\pi}$$

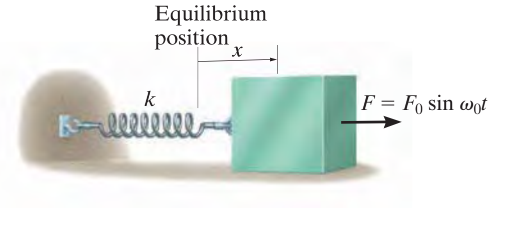
Energy Methods

Provided the restoring forces acting on the body are gravitational and elastic, then conservation of energy can also be used to determine its simple harmonic motion. To do this, the body is displaced a small amount from its equilibrium position, and an expression for its kinetic and potential energy is written. The time derivative of this equation can then be rearranged in the standard form $\ddot{x} + \omega_n^2 x = 0$.

Undamped Forced Vibration

When the equation of motion is applied to a body, which is subjected to a periodic force, or the support has a displacement with a frequency ω_0, then the solution of the differential equation consists of a complementary solution and a particular solution. The complementary solution is caused by the free vibration and can be neglected. The particular solution is caused by the forced vibration.

Resonance will occur if the natural frequency of vibration ω_n is equal to the forcing frequency ω_0. This should be avoided, since the motion will tend to become unbounded.

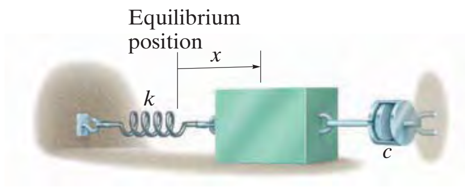


$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t$$

22

Viscous Damped Free Vibration

A viscous damping force is caused by fluid drag on the system as it vibrates. If the motion is slow, this drag force will be proportional to the velocity, that is, $F = c\dot{x}$. Here c is the coefficient of viscous damping. By comparing its value to the critical damping coefficient $c_c = 2m\omega_n$, we can specify the type of vibration that occurs. If $c > c_c$, it is an overdamped system; if $c = c_c$, it is a critically damped system; if $c < c_c$, it is an underdamped system.



Viscous Damped Forced Vibration

The most general type of vibration for a one-degree-of-freedom system occurs when the system is damped and subjected to periodic forced motion. The solution provides insight as to how the damping factor, c/c_c, and the frequency ratio, ω_0/ω_n, influence the vibration.

Resonance is avoided provided $c/c_c \neq 0$ and $\omega_0/\omega_n \neq 1$.

Electrical Circuit Analogs

The vibrating motion of a complex mechanical system can be studied by modeling it as an electrical circuit. This is possible since the differential equations that govern the behavior of each system are the same.

APPENDIX A

Mathematical Expressions

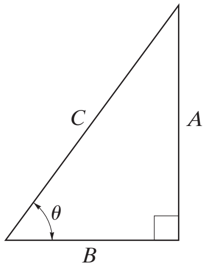
Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x}$

Trigonometric Identities



$\sin\theta = \dfrac{A}{C}$, $\csc\theta = \dfrac{C}{A}$

$\cos\theta = \dfrac{B}{C}$, $\sec\theta = \dfrac{C}{B}$

$\tan\theta = \dfrac{A}{B}$, $\cot\theta = \dfrac{B}{A}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos\theta = \pm\sqrt{\dfrac{1 + \cos 2\theta}{2}}$, $\sin\theta = \pm\sqrt{\dfrac{1 - \cos 2\theta}{2}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$

Power-Series Expansions

$\sin x = x - \dfrac{x^3}{3!} + \cdots \qquad \sinh x = x + \dfrac{x^3}{3!} + \cdots$

$\cos x = 1 - \dfrac{x^2}{2!} + \cdots \qquad \cosh x = 1 + \dfrac{x^2}{2!} + \cdots$

Derivatives

$\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$

$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{d}{dx}(\cot u) = -\csc^2 u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\sec u) = \tan u \sec u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\csc u) = -\csc u \cot u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\sin u) = \cos u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\cos u) = -\sin u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\tan u) = \sec^2 u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\sinh u) = \cosh u\dfrac{du}{dx}$

$\dfrac{d}{dx}(\cosh u) = \sinh u\dfrac{du}{dx}$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ba}}\ln\left[\frac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}\right] + C,\ ab < 0$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(bx^2+a) + C$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C,\ ab > 0$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left[\frac{a+x}{a-x}\right] + C,\ a^2 > x^2$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + C$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,\ a > 0$$

$$\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2\pm a^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C,\ a > 0$$

$$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\ln\left(x+\sqrt{x^2\pm a^2}\right)\right] + C$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{x^2\pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2\pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2\pm a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2\pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) + C,\ c < 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

A

APPENDIX

B Vector Analysis

The following discussion provides a brief review of vector analysis. A more detailed treatment of these topics is given in *Engineering Mechanics: Statics*.

Vector. A vector, **A**, is a quantity which has magnitude and direction, and adds according to the parallelogram law. As shown in Fig. B–1, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where **A** is the *resultant vector* and **B** and **C** are *component vectors*.

Unit Vector. A unit vector, $\mathbf{u}_A$, has a magnitude of one "dimensionless" unit and acts in the same direction as **A**. It is determined by dividing **A** by its magnitude A, i.e,

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} \qquad \text{(B–1)}$$

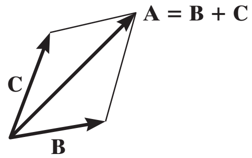


Fig. B–1

Cartesian Vector Notation. The directions of the positive *x, y, z* axes are defined by the Cartesian unit vectors **i**, **j**, **k**, respectively.

As shown in Fig. B–2, vector **A** is formulated by the addition of its *x, y, z* components as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad \text{(B–2)}$$

The *magnitude* of **A** is determined from

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \text{(B–3)}$$

The *direction* of **A** is defined in terms of its *coordinate direction angles*, α, β, γ, measured from the *tail* of **A** to the *positive x, y, z* axes, Fig. B–3. These angles are determined from the *direction cosines* which represent the **i**, **j**, **k** components of the unit vector $\mathbf{u}_A$; i.e., from Eqs. B–1 and B–2

$$\mathbf{u}_A = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \qquad \text{(B–4)}$$

so that the direction cosines are

$$\cos\alpha = \frac{A_x}{A} \quad \cos\beta = \frac{A_y}{A} \quad \cos\gamma = \frac{A_z}{A} \qquad \text{(B–5)}$$

Hence, $\mathbf{u}_A = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, and using Eq. B–3, it is seen that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad \text{(B–6)}$$

The Cross Product. The cross product of two vectors **A** and **B**, which yields the resultant vector **C**, is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad \text{(B–7)}$$

and reads **C** equals **A** "cross" **B**. The *magnitude* of **C** is

$$C = AB\sin\theta \qquad \text{(B–8)}$$

where θ is the angle made between the *tails* of **A** and **B** ($0° \le \theta \le 180°$). The *direction* of **C** is determined by the right-hand rule, whereby the fingers of the right hand are curled *from* **A** *to* **B** and the thumb points in the direction of **C**, Fig. B–4. This vector is perpendicular to the plane containing vectors **A** and **B**.

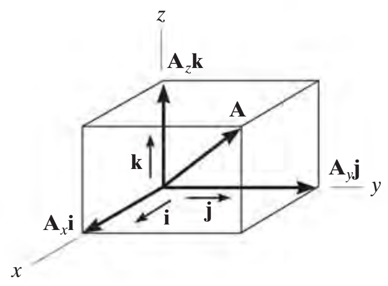


Fig. B–2

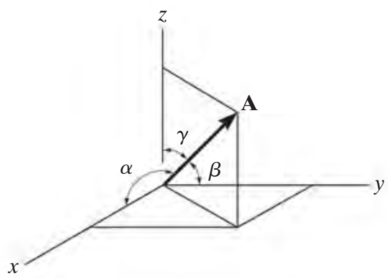


Fig. B–3

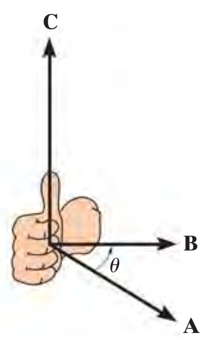


Fig. B–4

The vector cross product is *not* commutative, i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{B–9}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{D} \tag{B–10}$$

And the cross product may be multiplied by a scalar m in any manner; i.e.,

$$m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m \tag{B–11}$$

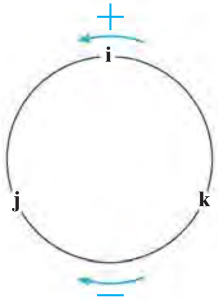


Fig. B–5

Equation B–7 can be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude is $(i)(j) \sin 90° = (1)(1)(1) = 1$, and its direction $+\mathbf{k}$ is determined from the right-hand rule, applied to $\mathbf{i} \times \mathbf{j}$, Fig. B–2. A simple scheme shown in Fig. B–5 may be helpful in obtaining this and other results when the need arises. If the circle is constructed as shown, then "crossing" two of the unit vectors in a *counterclockwise* fashion around the circle yields a *positive* third unit vector, e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Moving *clockwise*, a *negative* unit vector is obtained, e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

If $\mathbf{A}$ and $\mathbf{B}$ are expressed in Cartesian component form, then the cross product, Eq. B–7, may be evaluated by expanding the determinant

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{B–12}$$

which yields

$$\mathbf{C} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Recall that the cross product is used in statics to define the moment of a force $\mathbf{F}$ about point O, in which case

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{B–13}$$

where $\mathbf{r}$ is a position vector directed from point O to *any point* on the line of action of $\mathbf{F}$.

The Dot Product. The dot product of two vectors **A** and **B**, which yields a scalar, is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \tag{B–14}$$

and reads **A** "dot" **B**. The angle θ is formed between the *tails* of **A** and **B** ($0° \leq \theta \leq 180°$).

The dot product is commutative; i.e.,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{B–15}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{D} \tag{B–16}$$

And scalar multiplication can be performed in any manner, i.e.,

$$m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m \tag{B–17}$$

Using Eq. B–14, the dot product between any two Cartesian vectors can be determined. For example, $\mathbf{i} \cdot \mathbf{i} = (1)(1) \cos 0° = 1$ and $\mathbf{i} \cdot \mathbf{j} = (1)(1) \cos 90° = 0$.

If **A** and **B** are expressed in Cartesian component form, then the dot product, Eq. C–14, can be determined from

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \tag{B–18}$$

The dot product may be used to determine the *angle θ formed between two vectors*. From Eq. B–14,

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB}\right) \tag{B–19}$$

B

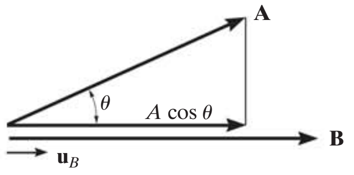


Fig. B–6

It is also possible to find the *component of a vector in a given direction* using the dot product. For example, the magnitude of the component (or projection) of vector **A** in the direction of **B**, Fig. B–6, is defined by $A\cos\theta$. From Eq. B–14, this magnitude is

$$A\cos\theta = \mathbf{A}\cdot\frac{\mathbf{B}}{B} = \mathbf{A}\cdot\mathbf{u}_B \qquad \text{(B–20)}$$

where $\mathbf{u}_B$ represents a unit vector acting in the direction of **B**, Fig. B–6.

Differentiation and Integration of Vector Functions. The rules for differentiation and integration of the sums and products of scalar functions also apply to vector functions. Consider, for example, the two vector functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$. Provided these functions are smooth and continuous for all s, then

$$\frac{d}{ds}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{ds} + \frac{d\mathbf{B}}{ds} \qquad \text{(B–21)}$$

$$\int(\mathbf{A}+\mathbf{B})\,ds = \int\mathbf{A}\,ds + \int\mathbf{B}\,ds \qquad \text{(B–22)}$$

For the cross product,

$$\frac{d}{ds}(\mathbf{A}\times\mathbf{B}) = \left(\frac{d\mathbf{A}}{ds}\times\mathbf{B}\right) + \left(\mathbf{A}\times\frac{d\mathbf{B}}{ds}\right) \qquad \text{(B–23)}$$

Similarly, for the dot product,

$$\frac{d}{ds}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{ds}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{ds} \qquad \text{(B–24)}$$

The Chain Rule

APPENDIX

C

The chain rule of calculus can be used to determine the time derivative of a composite function. For example, if y is a function of x and x is a function of t, then we can find the derivative of y with respect to t as follows

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{C–1}$$

In other words, to find $\dot{y}$ we take the ordinary derivative (dy/dx) and multiply it by the time derivative (dx/dt).

If several variables are functions of time and they are multiplied together, then the product rule $d(uv) = du\,v + u\,dv$ must be used along with the chain rule when taking the time derivatives. Here are some examples.

EXAMPLE C–1

If $y = x^3$ and $x = t^4$, find $\ddot{y}$, the second derivative of y with respect to time.

SOLUTION

Using the chain rule, Eq. C–1,

$$\dot{y} = 3x^2\dot{x}$$

To obtain the second time derivative we must use the product rule since x and $\dot{x}$ are both functions of time, and also, for $3x^2$ the chain rule must be applied. Thus, with $u = 3x^2$ and $v = \dot{x}$, we have

$$\ddot{y} = [6x\dot{x}]\dot{x} + 3x^2[\ddot{x}]$$

$$= 3x[2\dot{x}^2 + x\ddot{x}]$$

Since $x = t^4$, then $\dot{x} = 4t^3$ and $\ddot{x} = 12t^2$ so that

$$\ddot{y} = 3(t^4)[2(4t^3)^2 + t^4(12t^2)]$$

$$= 132t^{10}$$

Note that this result can also be obtained by combining the functions, then taking the time derivatives, that is,

$$y = x^3 = (t^4)^3 = t^{12}$$

$$\dot{y} = 12t^{11}$$

$$\ddot{y} = 132t^{10}$$

EXAMPLE C–2

If $y = xe^x$, find $\ddot{y}$.

SOLUTION

Since x and e^x are both functions of time the product and chain rules must be applied. Have $u = x$ and $v = e^x$.

$$\dot{y} = [\dot{x}]e^x + x[e^x\dot{x}]$$

The second time derivative also requires application of the product and chain rules. Note that the product rule applies to the three time variables in the last term, i.e., x, e^x, and $\dot{x}$.

$$\ddot{y} = \{[\ddot{x}]e^x + \dot{x}[e^x\dot{x}]\} + \{[\dot{x}]e^x\dot{x} + x[e^x\dot{x}]\dot{x} + xe^x[\ddot{x}]\}$$

$$= e^x[\ddot{x}(1 + x) + \dot{x}^2(2 + x)]$$

If $x = t^2$ then $\dot{x} = 2t$, $\ddot{x} = 2$ so that in terms in t, we have

$$\ddot{y} = e^{t^2}[2(1 + t^2) + 4t^2(2 + t^2)]$$

EXAMPLE C–3

If the path in radial coordinates is given as $r = 5\theta^2$, where θ is a known function of time, find $\ddot{r}$.

SOLUTION

First, using the chain rule then the chain and product rules where $u = 10\theta$ and $v = \dot{\theta}$, we have

$$r = 5\theta^2$$

$$\dot{r} = 10\theta\dot{\theta}$$

$$\ddot{r} = 10[(\dot{\theta})\dot{\theta} + \theta(\ddot{\theta})]$$

$$= 10\dot{\theta}^2 + 10\theta\ddot{\theta}$$

EXAMPLE C–4

If $r^2 = 6\theta^3$, find $\ddot{r}$.

SOLUTION

Here the chain and product rules are applied as follows.

$$r^2 = 6\theta^3$$

$$2r\dot{r} = 18\theta^2\dot{\theta}$$

$$2[(\dot{r})\dot{r} + r(\ddot{r})] = 18[(2\theta\dot{\theta})\dot{\theta} + \theta^2(\ddot{\theta})]$$

$$\dot{r}^2 + r\ddot{r} = 9(2\theta\dot{\theta}^2 + \theta^2\ddot{\theta})$$

To find $\ddot{r}$ at a specified value of θ which is a known function of time, we can first find $\dot{\theta}$ and $\ddot{\theta}$. Then using these values, evaluate r from the first equation, $\dot{r}$ from the second equation and $\ddot{r}$ using the last equation.

Fundamental Problems Partial Solutions And Answers

Chapter 12

F12–1. $v = v_0 + a_c t$

$10 = 35 + a_c(15)$

$a_c = -1.67 \text{ m/s}^2 = 1.67 \text{ m/s}^2 \leftarrow$ *Ans.*

F12–2. $s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$0 = 0 + 15t + \frac{1}{2}(-9.81)t^2$

$t = 3.06 \text{ s}$ *Ans.*

F12–3. $ds = v\, dt$

$\int_0^s ds = \int_0^t \left(4t - 3t^2\right) dt$

$s = \left(2t^2 - t^3\right) \text{m}$

$s = 2\left(4^2\right) - 4^3$

$= -32 \text{ m} = 32 \text{ m} \leftarrow$ *Ans.*

F12–4. $a = \frac{dv}{dt} = \frac{d}{dt}\left(0.5t^3 - 8t\right)$

$a = \left(1.5t^2 - 8\right) \text{m/s}^2$

When $t = 2$ s,

$a = 1.5\left(2^2\right) - 8 = -2 \text{ m/s}^2 = 2 \text{ m/s}^2 \leftarrow$ *Ans.*

F12–5. $v = \frac{ds}{dt} = \frac{d}{dt}(2t^2 - 8t + 6) = (4t - 8) \text{ m/s}$

$v = 0 = (4t - 8)$

$t = 2 \text{ s}$ *Ans.*

$s|_{t=0} = 2\left(0^2\right) - 8(0) + 6 = 6 \text{ m}$

$s|_{t=2} = 2\left(2^2\right) - 8(2) + 6 = -2 \text{ m}$

$s|_{t=3} = 2\left(3^2\right) - 8(3) + 6 = 0 \text{ m}$

$(\Delta s)_{\text{Tot}} = 8 \text{ m} + 2 \text{ m} = 10 \text{ m}$ *Ans.*

F12–6. $\int v\, dv = \int a\, ds$

$\int_{5 \text{ m/s}}^{v} v\, dv = \int_0^s (10 - 0.2s) ds$

$v = \left(\sqrt{20s - 0.2s^2 + 25}\right) \text{m/s}$

At $s = 10$ m,

$v = \sqrt{20(10) - 0.2(10^2) + 25}$

$= 14.3 \text{ m/s} \rightarrow$ *Ans.*

F12–7. $v = \int (4t^2 - 2)\, dt$

$v = \frac{4}{3}t^3 - 2t + C_1$

$s = \int \left(\frac{4}{3}t^3 - 2t + C_1\right) dt$

$s = \frac{1}{3}t^4 - t^2 + C_1 t + C_2$

$t = 0, s = -2, C_2 = -2$

$t = 2, s = -20, C_1 = -9.67$

$t = 4, s = 28.7 \text{ m}$ *Ans.*

F12–8. $a = v\frac{dv}{ds}$

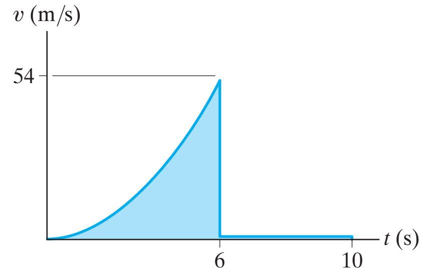
$= \left(20 - 0.05s^2\right)\left(-0.1s\right)$

At $s = 15$ m,

$a = -13.1 \text{ m/s}^2 = 13.1 \text{ m/s}^2 \leftarrow$ *Ans.*

F12–9. $v = \frac{ds}{dt} = \frac{d}{dt}\left(0.5t^3\right) = 1.5t^2$

$v = \frac{ds}{dt} = \frac{d}{dt}(108) = 0$ *Ans.*



F12–10. $ds = v\, dt$

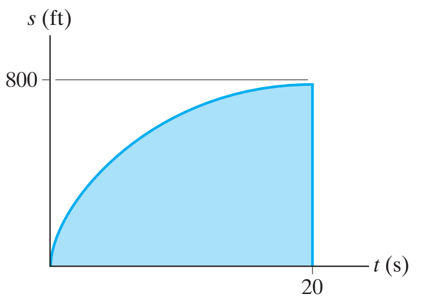
$\int_0^s ds = \int_0^t (-4t + 80)\, dt$

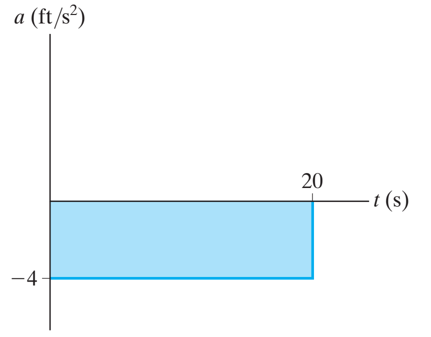
$s = -2t^2 + 80t$

$a = \frac{dv}{dt} = \frac{d}{dt}(-4t + 80) = -4 \text{ ft/s}^2 = 4 \text{ ft/s}^2 \leftarrow$

Also,

$a = \frac{\Delta v}{\Delta t} = \frac{0 - 80 \text{ ft/s}}{20 \text{ s} - 0} = -4 \text{ ft/s}^2$

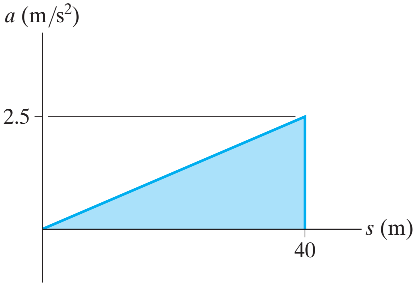




F12–11. $a\,ds = v\,dv$

$$a = v\frac{dv}{ds} = 0.25s\frac{d}{ds}(0.25s) = 0.0625s$$

$$a|_{s=40\text{ m}} = 0.0625(40\text{ m}) = 2.5\text{ m/s}^2 \rightarrow$$



F12–12. For $0 \le s \le 10$ m

$$a = s$$

$$\int_0^v v\,dv = \int_0^s s\,ds$$

$$v = s$$

at $s = 10$ m, $v = 10$ m

For $10\text{ m} \le s \le 15$

$$a = 10$$

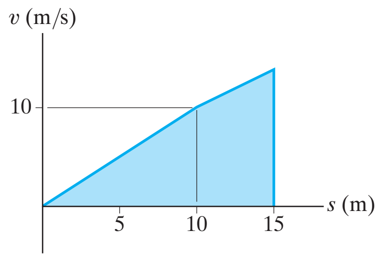
$$\int_{10}^{v} v\,dv = \int_{10}^{s} 10\,ds$$

$$\frac{1}{2}v^2 - 50 = 10s - 100$$

$$v = \sqrt{20s - 100}$$

at $s = 15$ m

$$v = 14.1\text{ m/s}$$ *Ans.*



F12–13. $0 \le t < 5$ s,

$$dv = a\,dt \qquad \int_0^v dv = \int_0^t 20\,dt$$

$$v = (20t)\text{ m/s}$$

$5\text{ s} < t \le t'$,

$$(\overset{+}{\rightarrow}) \quad dv = a\,dt \qquad \int_{100\text{ m/s}}^{v} dv = \int_{5\text{ s}}^{t} -10\,dt$$

$$v - 100 = (50 - 10t)\text{ m/s},$$

$$0 = 150 - 10t'$$

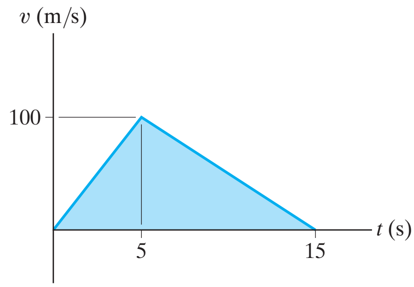
$$t' = 15\text{ s}$$

Also,

$\Delta v = 0 =$ Area under the $a{-}t$ graph

$$0 = (20\text{ m/s}^2)(5\text{ s}) + [-(10\text{ m/s})(t' - 5)\text{ s}]$$

$$t' = 15\text{ s}$$



F12–14. $0 \le t \le 5$ s,

$$ds = v\,dt \qquad \int_0^s ds = \int_0^t 30t\,dt$$

$$s|_0^s = 15t^2|_0^t$$

$$s = (15t^2)\text{ m}$$

$5\text{ s} < t \le 15$ s,

$$(\overset{+}{\rightarrow})\,ds = v\,dt; \qquad \int_{375\text{ m}}^{s} ds = \int_{5\text{ s}}^{t} (-15t + 225)dt$$

$$s = (-7.5t^2 + 225t - 562.5)\text{ m}$$

$$s = (-7.5)(15)^2 + 225(15) - 562.5\text{ m}$$

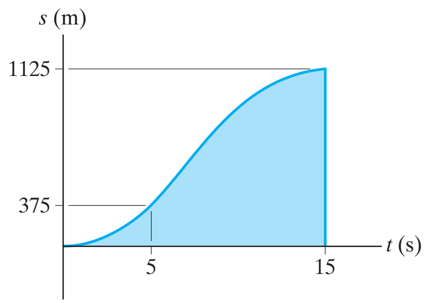
$$= 1125\text{ m}$$ *Ans.*

Also,

$\Delta s =$ Area under the $v{-}t$ graph

$$= \tfrac{1}{2}(150\text{ m/s})(15\text{ s})$$

$$= 1125\text{ m}$$ *Ans.*



F12–15. $\int_0^x dx = \int_0^t 32t\,dt$

$$x = \left(16t^2\right)\text{ m} \qquad (1)$$

$$\int_0^y dy = \int_0^t 8\,dt$$

$$t = \frac{y}{8} \qquad (2)$$

Substituting Eq. (2) into Eq. (1), get

$y = 2\sqrt{x}$ *Ans.*

F12–16. $y = 0.75(8t) = 6t$

$v_x = \dot{x} = \frac{dx}{dt} = \frac{d}{dt}(8t) = 8 \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{dy}{dt} = \frac{d}{dt}(6t) = 6 \text{ m/s} \uparrow$

The magnitude of the particle's velocity is

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8 \text{ m/s})^2 + (6 \text{ m/s})^2}$

$= 10 \text{ m/s}$ *Ans.*

F12–17. $y = (4t^2) \text{ m}$

$v_x = \dot{x} = \frac{d}{dt}\left(4t^4\right) = \left(16t^3\right) \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{d}{dt}\left(4t^2\right) = (8t) \text{ m/s} \uparrow$

When $t = 0.5$ s,

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2 \text{ m/s})^2 + (4 \text{ m/s})^2}$

$= 4.47 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = \frac{d}{dt}\left(16t^3\right) = \left(48t^2\right) \text{ m/s}^2$

$a_y = \dot{v}_y = \frac{d}{dt}(8t) = 8 \text{ m/s}^2$

When $t = 0.5$ s,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(12 \text{ m/s}^2)^2 + (8 \text{ m/s}^2)^2}$

$= 14.4 \text{ m/s}^2$ *Ans.*

F12–18. $y = 0.5x$

$\dot{y} = 0.5\dot{x}$

$v_y = t^2$

When $t = 4$ s,

$v_x = 32 \text{ m/s} \qquad v_y = 16 \text{ m/s}$

$v = \sqrt{v_x^2 + v_y^2} = 35.8 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = 4t$

$a_y = \dot{v}_y = 2t$

When $t = 4$ s,

$a_x = 16 \text{ m/s}^2 \qquad a_y = 8 \text{ m/s}^2$

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{16^2 + 8^2} = 17.9 \text{ m/s}^2$ *Ans.*

F12–19. $v_y = \dot{y} = 0.5\,x\,\dot{x} = 0.5(8)(8) = 32 \text{ m/s}$

Thus,

$v = \sqrt{v_x^2 + v_y^2} = 33.0 \text{ m/s}$ *Ans.*

$a_y = \dot{v}_y = 0.5\,\dot{x}^2 + 0.5\,x\ddot{x}$

$= 0.5(8)^2 + 0.5(8)(4)$

$= 48 \text{ m/s}^2$

$a_x = 4 \text{ m/s}^2$

Thus,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 48^2} = 48.2 \text{ m/s}^2$ *Ans.*

F12–20. $\dot{y} = 0.1x\dot{x}$

$v_y = 0.1(5)(-3) = -1.5 \text{ m/s} = 1.5 \text{ m/s} \downarrow$ *Ans.*

$\ddot{y} = 0.1[\dot{x}\dot{x} + x\ddot{x}]$

$a_y = 0.1[(-3)^2 + 5(-1.5)] = 0.15 \text{ m/s}^2 \uparrow$ *Ans.*

F12–21. $(v_B)_y^2 = (v_A)_y^2 + 2a_y(y_B - y_A)$

$0^2 = (5 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(h - 0)$

$h = 1.27 \text{ m}$ *Ans.*

F12–22. $y_C = y_A + (v_A)_y t_{AC} + \frac{1}{2} a_y t_{AC}^2$

$0 = 0 + (5 \text{ m/s})t_{AC} + \frac{1}{2}(-9.81 \text{ m/s}^2)t_{AC}^2$

$t_{AC} = 1.0194 \text{ s}$

$(v_C)_y = (v_A)_y + a_y t_{AC}$

$(v_C)_y = 5 \text{ m/s} + (-9.81 \text{ m/s}^2)(1.0194 \text{ s})$

$= -5 \text{ m/s} = 5 \text{ m/s} \downarrow$

$v_C = \sqrt{(v_C)_x^2 + (v_C)_y^2}$

$= \sqrt{(8.660 \text{ m/s})^2 + (5 \text{ m/s})^2} = 10 \text{ m/s}$ *Ans.*

$R = x_A + (v_A)_x t_{AC} = 0 + (8.660 \text{ m/s})(1.0194 \text{ s})$

$= 8.83 \text{ m}$ *Ans.*

F12–23. $s = s_0 + v_0 t$

$10 = 0 + v_A \cos 30° t$

$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$3 = 1.5 + v_A \sin 30° t + \frac{1}{2}(-9.81)t^2$

$t = 0.9334 \text{ s}, \quad v_A = 12.4 \text{ m/s}$ *Ans.*

F12–24. $s = s_0 + v_0 t$

$R\left(\frac{4}{5}\right) = 0 + 20\left(\frac{3}{5}\right)t$

$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$-R\left(\frac{3}{5}\right) = 0 + 20\left(\frac{4}{5}\right)t + \frac{1}{2}(-9.81)t^2$

$t = 5.10 \text{ s}$

$R = 76.5 \text{ m}$ *Ans.*

F12–25. $x_B = x_A + (v_A)_x t_{AB}$

$12 \text{ ft} = 0 + (0.8660\, v_A)t_{AB}$

$v_A t_{AB} = 13.856$ (1)

$y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2} a_y t_{AB}^2$

$$(8 - 3)\text{ ft} = 0 + 0.5v_A t_{AB} + \tfrac{1}{2}(-32.2\text{ ft/s}^2)t_{AB}^2$$

Using Eq. (1),

$$5 = 0.5(13.856) - 16.1\, t_{AB}^2$$

$$t_{AB} = 0.3461\text{ s}$$

$$v_A = 40.0\text{ ft/s}$$ *Ans.*

F12–26. $y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2} a_y t_{AB}^2$

$$-150\text{ m} = 0 + (90\text{ m/s})t_{AB} + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

$$t_{AB} = 19.89\text{ s}$$

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 120\text{ m/s}(19.89\text{ s}) = 2386.37\text{ m}$$

$$= 2.39\text{ km}$$ *Ans.*

F12–27. $a_t = \dot{v} = \frac{dv}{dt} = \frac{d}{dt}(0.0625t^2) = (0.125t)\text{ m/s}^2\big|_{t=10\text{ s}}$

$$= 1.25\text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(0.0625t^2)^2}{40\text{ m}} = \left[97.656(10^{-6})t^4\right]\text{ m/s}^2\big|_{t=10\text{ s}}$$

$$= 0.9766\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(1.25\text{ m/s}^2)^2 + (0.9766\text{ m/s}^2)^2}$$

$$= 1.59\text{ m/s}^2$$ *Ans.*

F12–28. $v = 2s\,\big|_{s=10} = 20\text{ m/s}$

$$a_n = \frac{v^2}{\rho} = \frac{(20\text{ m/s})^2}{50\text{ m}} = 8\text{ m/s}^2$$

$$a_t = v\frac{dv}{ds} = 4s\big|_{s=10} = 40\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(40\text{ m/s}^2\right)^2 + \left(8\text{ m/s}^2\right)^2}$$

$$= 40.8\text{ m/s}^2$$ *Ans.*

F12–29. $v_C^2 = v_A^2 + 2a_t(s_C - s_A)$

$$(15\text{ m/s})^2 = (25\text{ m/s})^2 + 2a_t(300\text{ m} - 0)$$

$$a_t = -0.6667\text{ m/s}^2$$

$$v_B^2 = v_A^2 + 2a_t(s_B - s_A)$$

$$v_B^2 = (25\text{ m/s})^2 + 2(-0.6667\text{ m/s}^2)(250\text{ m} - 0)$$

$$v_B = 17.08\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(17.08\text{ m/s})^2}{300\text{ m}} = 0.9722\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.6667\text{ m/s}^2)^2 + (0.9722\text{ m/s}^2)^2}$$

$$= 1.18\text{ m/s}^2$$ *Ans.*

F12–30. $\tan\theta = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{24}x^2\right) = \frac{1}{12}x$

$$\theta = \tan^{-1}\left(\tfrac{1}{12}x\right)\Big|_{x=10\text{ ft}}$$

$$= \tan^{-1}\left(\tfrac{10}{12}\right) = 39.81° = 39.8°$$ *Ans.*

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{|d^2y/dx^2|} = \frac{\left[1 + \left(\frac{1}{12}x\right)^2\right]^{3/2}}{\left|\frac{1}{12}\right|}\Bigg|_{x=10\text{ ft}}$$

$$= 26.468\text{ ft}$$

$$a_n = \frac{v^2}{\rho} = \frac{(20\text{ ft/s})^2}{26.468\text{ ft}} = 15.11\text{ ft/s}^2$$

$$a = \sqrt{(a_t)^2 + (a_n)^2} = \sqrt{(6\text{ ft/s}^2)^2 + (15.11\text{ ft/s}^2)^2}$$

$$= 16.3\text{ ft/s}^2$$ *Ans.*

F12–31. $(a_B)_t = -0.001s = (-0.001)(300\text{ m})\left(\frac{\pi}{2}\text{ rad}\right)\text{ m/s}^2$

$$= -0.4712\text{ m/s}^2$$

$$v\,dv = a_t\,ds$$

$$\int_{25\text{ m/s}}^{v_B} v\,dv = \int_0^{150\pi\text{ m}} -0.001s\,ds$$

$$v_B = 20.07\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(20.07\text{ m/s})^2}{300\text{ m}} = 1.343\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.4712\text{ m/s}^2)^2 + (1.343\text{ m/s}^2)^2}$$

$$= 1.42\text{ m/s}^2$$ *Ans.*

F12–32. $a_t\,ds = v\,dv$

$$a_t = v\,\frac{dv}{ds} = (0.2s)(0.2) = (0.04s)\text{ m/s}^2$$

$$a_t = 0.04(50\text{ m}) = 2\text{ m/s}^2$$

$$v = 0.2(50\text{ m}) = 10\text{ m/s}$$

$$a_n = \frac{v^2}{\rho} = \frac{(10\text{ m/s})^2}{500\text{ m}} = 0.2\text{ m/s}^2$$
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(2\text{ m/s}^2)^2 + (0.2\text{ m/s}^2)^2}$$
$$= 2.01\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–33. $v_r = \dot{r} = 0$
$$v_\theta = r\dot{\theta} = (400\dot{\theta})\text{ ft/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$55\text{ ft/s} = \sqrt{0^2 + [(400\dot{\theta})\text{ ft/s}]^2}$$
$$\dot{\theta} = 0.1375\text{ rad/s} \qquad \textit{Ans.}$$

F12–34. $r = 0.1t^3\big|_{t=1.5\text{ s}} = 0.3375\text{ m}$
$$\dot{r} = 0.3t^2\big|_{t=1.5\text{ s}} = 0.675\text{ m/s}$$
$$\ddot{r} = 0.6t\big|_{t=1.5\text{ s}} = 0.900\text{ m/s}^2$$
$$\theta = 4t^{3/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad}$$
$$\dot{\theta} = 6t^{1/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad/s}$$
$$\ddot{\theta} = 3t^{-1/2}\big|_{t=1.5\text{ s}} = 2.449\text{ rad/s}^2$$
$$v_r = \dot{r} = 0.675\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3375\text{ m})(7.348\text{ rad/s}) = 2.480\text{ m/s}$$
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= (0.900\text{ m/s}^2) - (0.3375\text{ m})(7.348\text{ rad/s})^2$$
$$= -17.325\text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.3375\text{ m})(2.449\text{ rad/s}^2)$$
$$+ 2(0.675\text{ m/s})(7.348\text{ rad/s}) = 10.747\text{ m/s}^2$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$= \sqrt{(0.675\text{ m/s})^2 + (2.480\text{ m/s})^2}$$
$$= 2.57\text{ m/s} \qquad \textit{Ans.}$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-17.325\text{ m/s}^2)^2 + (10.747\text{ m/s}^2)^2}$$
$$= 20.4\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–35. $r = 2\theta$
$$\dot{r} = 2\dot{\theta}$$
$$\ddot{r} = 2\ddot{\theta}$$
At $\theta = \pi/4$ rad,
$$r = 2\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{2}\text{ ft}$$
$$\dot{r} = 2(3\text{ rad/s}) = 6\text{ ft/s}$$
$$\ddot{r} = 2(1\text{ rad/s}) = 2\text{ ft/s}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = 2\text{ ft/s}^2 - \left(\tfrac{\pi}{2}\text{ ft}\right)(3\text{ rad/s})^2$$
$$= -12.14\text{ ft/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= \left(\tfrac{\pi}{2}\text{ ft}\right)(1\text{ rad/s}^2) + 2(6\text{ ft/s})(3\text{ rad/s})$$
$$= 37.57\text{ ft/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-12.14\text{ ft/s}^2)^2 + (37.57\text{ ft/s}^2)^2}$$
$$= 39.5\text{ ft/s}^2 \qquad \textit{Ans.}$$

F12–36. $r = e^\theta$
$$\dot{r} = e^\theta\dot{\theta}$$
$$\ddot{r} = e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = (e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2) - e^\theta\dot{\theta}^2 = e^{\pi/4}(4)$$
$$= 8.77\text{ m/s}^2 \qquad \textit{Ans.}$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (e^\theta\ddot{\theta}) + (2(e^\theta\dot{\theta})\dot{\theta}) = e^\theta(\ddot{\theta} + 2\dot{\theta}^2)$$
$$= e^{\pi/4}(4 + 2(2)^2)$$
$$= 26.3\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–37. $r = [0.2(1 + \cos\theta)]\text{ m}\big|_{\theta=30^\circ} = 0.3732\text{ m}$
$$\dot{r} = \left[-0.2\,(\sin\theta)\dot{\theta}\right]\text{ m/s}\big|_{\theta=30^\circ}$$
$$= -0.2\sin 30^\circ(3\text{ rad/s})$$
$$= -0.3\text{ m/s}$$
$$v_r = \dot{r} = -0.3\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3732\text{ m})(3\text{ rad/s}) = 1.120\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(-0.3\text{ m/s})^2 + (1.120\text{ m/s})^2}$$
$$= 1.16\text{ m/s} \qquad \textit{Ans.}$$

F12–38. $30\text{ m} = r\sin\theta$
$$r = \left(\frac{30\text{ m}}{\sin\theta}\right) = (30\csc\theta)\text{ m}$$
$$r = (30\csc\theta)\big|_{\theta=45^\circ} = 42.426\text{ m}$$
$$\dot{r} = -30\csc\theta\,\text{ctn}\,\theta\,\dot{\theta}\big|_{\theta=45^\circ} = -\left(42.426\dot{\theta}\right)\text{ m/s}$$
$$v_r = \dot{r} = -(42.426\dot{\theta})\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (42.426\dot{\theta})\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$2 = \sqrt{(-42.426\dot{\theta})^2 + (42.426\dot{\theta})^2}$$
$$\dot{\theta} = 0.0333\text{ rad/s} \qquad \textit{Ans.}$$

F12–39. $l_T = 3s_D + s_A$
$$0 = 3v_D + v_A$$
$$0 = 3v_D + 3\text{ m/s}$$
$$v_D = -1\text{ m/s} = 1\text{ m/s}\uparrow \qquad \textit{Ans.}$$

F12–40. $s_B + 2s_A + 2h = l$

$v_B + 2v_A = 0$

$6 + 2v_A = 0 \quad v_A = -3 \text{ m/s} = 3 \text{ m/s} \uparrow$ *Ans.*

F12–41. $3s_A + s_B = l$

$3v_A + v_B = 0$

$3v_A + 1.5 = 0 \quad v_A = -0.5 \text{ m/s} = 0.5 \text{ m/s} \uparrow$ *Ans.*

F12–42. $l_T = 4\,s_A + s_F$

$0 = 4\,v_A + v_F$

$0 = 4\,v_A + 3 \text{ m/s}$

$v_A = -0.75 \text{ m/s} = 0.75 \text{ m/s} \uparrow$ *Ans.*

F12–43. $s_A + 2(s_A - a) + (s_A - s_P) = l$

$4s_A - s_P = l + 2a$

$4v_A - v_P = 0$

$4v_A - (-4) = 0$

$4v_A + 4 = 0 \qquad v_A = -1 \text{ m/s} = 1 \text{ m/s} \nearrow$ *Ans.*

F12–44. $s_C + s_B = l_{CED}$ (1)

$(s_A - s_C) + (s_B - s_C) + s_B = l_{ACDF}$

$s_A + 2s_B - 2s_C = l_{ACDF}$ (2)

Thus

$v_C + v_B = 0$

$v_A + 2v_B - 2v_C = 0$

Eliminating v_C,

$v_A + 4v_B = 0$

Thus,

$4 \text{ ft/s} + 4v_B = 0$

$v_B = -1 \text{ ft/s} = 1 \text{ ft/s} \uparrow$ *Ans.*

F12–45. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$100\mathbf{i} = 80\mathbf{j} + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = 100\mathbf{i} - 80\mathbf{j}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(100 \text{ km/h})^2 + (-80 \text{ km/h})^2}$

$= 128 \text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\frac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\frac{80 \text{ km/h}}{100 \text{ km/h}}\right) = 38.7° \measuredangle$ *Ans.*

F12–46. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(-400\mathbf{i} - 692.82\mathbf{j}) = (650\mathbf{i}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = [-1050\mathbf{i} - 692.82\mathbf{j}] \text{ km/h}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(1050 \text{ km/h})^2 + (692.82 \text{ km/h})^2}$

$= 1258 \text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\frac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\frac{692.82 \text{ km/h}}{1050 \text{ km/h}}\right) = 33.4° \measuredangle$ *Ans.*

F12–47. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(5\mathbf{i} + 8.660\mathbf{j}) = (12.99\mathbf{i} + 7.5\mathbf{j}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = [-7.990\mathbf{i} + 1.160\mathbf{j}] \text{ m/s}$

$v_{B/A} = \sqrt{(-7.990 \text{ m/s})^2 + (1.160 \text{ m/s})^2}$

$= 8.074 \text{ m/s}$

$d_{AB} = v_{B/A}t = (8.074 \text{ m/s})(4 \text{ s}) = 32.3 \text{ m}$ *Ans.*

F12–48. $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

$-20 \cos 45°\mathbf{i} + 20 \sin 45°\mathbf{j} = 65\mathbf{i} + \mathbf{v}_{A/B}$

$\mathbf{v}_{A/B} = -79.14\mathbf{i} + 14.14\mathbf{j}$

$\mathbf{v}_{A/B} = \sqrt{(-79.14)^2 + (14.14)^2}$

$= 80.4 \text{ km/h}$ *Ans.*

$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

$\frac{(20)^2}{0.1} \cos 45°\mathbf{i} + \frac{(20)^2}{0.1} \sin 45°\mathbf{j} = 1200\mathbf{i} + \mathbf{a}_{A/B}$

$\mathbf{a}_{A/B} = 1628\mathbf{i} + 2828\mathbf{j}$

$a_{A/B} = \sqrt{(1628)^2 + (2828)^2}$

$= 3.26(10^3) \text{ km/h}^2$ *Ans.*

Chapter 13

F13–1. $s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$6 \text{ m} = 0 + 0 + \frac{1}{2} a(3 \text{ s})^2$

$a = 1.333 \text{ m/s}^2$

$\Sigma F_y = ma_y; \quad N_A - 20(9.81) \text{ N} \cos 30° = 0$

$N_A = 169.91 \text{ N}$

$\Sigma F_x = ma_x; \quad T - 20(9.81) \text{ N} \sin 30°$

$- 0.3(169.91 \text{ N}) = (20 \text{ kg})(1.333 \text{ m/s}^2)$

$T = 176 \text{ N}$ *Ans.*

F13–2. $(F_f)_{max} = \mu_s N_A = 0.3(245.25 \text{ N}) = 73.575 \text{ N}.$ Since $F = 100 \text{ N} > (F_f)_{max}$ when $t = 0$, the crate will start to move immediately after **F** is applied.

$+\uparrow \Sigma F_y = ma_y; \quad N_A - 25(9.81) \text{ N} = 0$

$N_A = 245.25 \text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = ma_x;$

$10t^2 + 100 - 0.25(245.25 \text{ N}) = (25 \text{ kg})a$

$a = (0.4t^2 + 1.5475) \text{ m/s}^2$

$dv = a\,dt$

$\int_0^v dv = \int_0^{4 \text{ s}} (0.4t^2 + 1.5475)\,dt$

$v = 14.7 \text{ m/s} \rightarrow$ *Ans.*

F13–3. $\xrightarrow{+} \Sigma F_x = ma_x;$

$\left(\frac{4}{5}\right)500 \text{ N} - (500s)\text{N} = (10 \text{ kg})a$

$a = (40 - 50s) \text{ m/s}^2$

$v\,dv = a\,ds$

$\int_0^v v\,dv = \int_0^{0.5 \text{ m}} (40 - 50s)\,ds$

$\frac{v^2}{2}\Big|_0^v = \left(40s - 25s^2\right)\Big|_0^{0.5 \text{ m}}$

$v = 5.24 \text{ m/s}$ *Ans.*

F13–4. $\xrightarrow{+} \Sigma F_x = ma_x \quad 100(s + 1) \text{ N} = (2000 \text{ kg})a$

$a = (0.05(s + 1)) \text{ m/s}^2$

$v\,dv = a\,ds$

$\int_0^v v\,dv = \int_0^{10 \text{ m}} 0.05(s + 1)\,ds$

$v = 2.45 \text{ m/s}$

F13–5. $F_{sp} = k(l - l_0) = (200 \text{ N/m})(0.5 \text{ m} - 0.3 \text{ m})$

$= 40 \text{ N}$

$\theta = \tan^{-1}\left(\frac{0.3 \text{ m}}{0.4 \text{ m}}\right) = 36.86°$

$\xrightarrow{+} \Sigma F_x = ma_x;$

$100 \text{ N} - (40 \text{ N})\cos 36.86° = (25 \text{ kg})a$

$a = 2.72 \text{ m/s}^2$

F13–6. Blocks A and B:

$\xrightarrow{+} \Sigma F_x = ma_x;\ 6 = \frac{70}{32.2}a;\ a = 2.76 \text{ ft/s}^2$

Check if slipping occurs between A and B.

$\xrightarrow{+} \Sigma F_x = ma_x;\ 6 - F = \frac{20}{32.2}(2.76);$

$F = 4.29 \text{ lb} < 0.4(20) = 8 \text{ lb}$

$a_A = a_B = 2.76 \text{ ft/s}^2$ *Ans.*

F13–7. $\Sigma F_n = m\frac{v^2}{\rho};\ (0.3)m(9.81) = m\frac{v^2}{2}$

$v = 2.43 \text{ m/s}$ *Ans.*

F13–8. $+\downarrow \Sigma F_n = ma_n;\ m(32.2) = m\left(\frac{v^2}{250}\right)$

$v = 89.7 \text{ ft/s}$ *Ans.*

F13–9. $+\downarrow \Sigma F_n = ma_n;\ 150 + N_p = \frac{150}{32.2}\left(\frac{(120)^2}{400}\right)$

$N_p = 17.7 \text{ lb}$ *Ans.*

F13–10. $\xleftarrow{+} \Sigma F_n = ma_n;$

$N_c \sin 30° + 0.2\,N_c \cos 30° = m\frac{v^2}{500}$

$+\uparrow \Sigma F_b = 0;$

$N_c \cos 30° - 0.2N_c \sin 30° - m(32.2) = 0$

$v = 119 \text{ ft/s}$ *Ans.*

F13–11. $\Sigma F_t = ma_t;\quad 10(9.81) \text{ N} \cos 45° = (10 \text{ kg})a_t$

$a_t = 6.94 \text{ m/s}^2$ *Ans.*

$\Sigma F_n = ma_n;$

$T - 10(9.81) \text{ N} \sin 45° = (10 \text{ kg})\frac{(3 \text{ m/s})^2}{2 \text{ m}}$

$T = 114 \text{ N}$ *Ans.*

F13–12. $\Sigma F_n = ma_n;$

$F_n = (500 \text{ kg})\frac{(15 \text{ m/s})^2}{200 \text{ m}} = 562.5 \text{ N}$

$\Sigma F_t = ma_t;$

$F_t = (500 \text{ kg})(1.5 \text{ m/s}^2) = 750 \text{ N}$

$F = \sqrt{F_n^2 + F_t^2} = \sqrt{(562.5 \text{ N})^2 + (750 \text{ N})^2}$

$= 938 \text{ N}$ *Ans.*

F13–13. $a_r = \ddot{r} - r\dot{\theta}^2 = 0 - (1.5 \text{ m} + (8 \text{ m})\sin 45°)\dot{\theta}^2$

$= (-7.157\,\dot{\theta}^2) \text{ m/s}^2$

$\Sigma F_z = ma_z;$

$T\cos 45° - m(9.81) = m(0) \quad T = 13.87\,m$

$\Sigma F_r = ma_r;$

$-(13.87m)\sin 45° = m(-7.157\,\dot{\theta}^2)$

$\dot{\theta} = 1.17 \text{ rad/s}$ *Ans.*

F13–14. $\theta = \pi t^2\big|_{t=0.5 \text{ s}} = (\pi/4) \text{ rad}$

$\dot{\theta} = 2\pi t\big|_{t=0.5 \text{ s}} = \pi \text{ rad/s}$

$\ddot{\theta} = 2\pi \text{ rad/s}^2$

$r = 0.6 \sin\theta\big|_{\theta=\pi/4 \text{ rad}} = 0.4243 \text{ m}$

$\dot{r} = 0.6(\cos\theta)\dot{\theta}\big|_{\theta=\pi/4 \text{ rad}} = 1.3329 \text{ m/s}$

$\ddot{r} = 0.6\,[(\cos\theta)\ddot{\theta} - (\sin\theta)\dot{\theta}^2]\big|_{\theta=\pi/4 \text{ rad}} = -1.5216 \text{ m/s}^2$

$a_r = \ddot{r} - r\dot{\theta}^2 = -1.5216 \text{ m/s}^2 - (0.4243 \text{ m})(\pi \text{ rad/s})^2$

$= -5.7089 \text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.4243 \text{ m}(2\pi \text{ rad/s}^2)$

$+ 2(1.3329 \text{ m/s})(\pi \text{ rad/s})$

$= 11.0404 \text{ m/s}^2$

$\Sigma F_r = ma_r;$

$F\cos 45° - N\cos 45° - 0.2(9.81)\cos 45°$

$= 0.2(-5.7089)$

$\Sigma F_\theta = ma_\theta;$

$F\sin 45° + N\sin 45° - 0.2(9.81)\sin 45°$

$= 0.2(11.0404)$

$N = 2.37 \text{ N} \qquad F = 2.72 \text{ N}$ *Ans.*

F13–15. $r = 50e^{2\theta}\big|_{\theta=\pi/6\text{ rad}} = \left[50e^{2(\pi/6)}\right]\text{ m} = 142.48\text{ m}$

$\dot{r} = 50\left(2e^{2\theta}\,\dot{\theta}\right) = 100e^{2\theta}\,\dot{\theta}\big|_{\theta=\pi/6\text{ rad}}$

$= \left[100e^{2(\pi/6)}(0.05)\right] = 14.248\text{ m/s}$

$\ddot{r} = 100\left((2e^{2\theta}\dot{\theta})\dot{\theta} + e^{2\theta}(\ddot{\theta})\right)\big|_{\theta=\pi/6\text{ rad}}$

$= 100\left[2e^{2(\pi/6)}(0.05^2) + e^{2(\pi/6)}(0.01)\right]$

$= 4.274\text{ m/s}^2$

$a_r = \ddot{r} - r\dot{\theta}^2 = 4.274\text{ m/s}^2 - 142.48\text{ m}(0.05\text{ rad/s})^2$

$= 3.918\text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 142.48\text{ m}(0.01\text{ rad/s}^2)$

$+ 2(14.248\text{ m/s})(0.05\text{ rad/s})$

$= 2.850\text{ m/s}^2$

$\Sigma F_r = ma_r;$

$F_r = (2000\text{ kg})(3.918\text{ m/s}^2) = 7836.55\text{ N}$

$\Sigma F_\theta = ma_\theta;$

$F_\theta = (2000\text{ kg})(2.850\text{ m/s}^2) = 5699.31\text{ N}$

$F = \sqrt{F_r^2 + F_\theta^2}$

$= \sqrt{(7836.55\text{ N})^2 + (5699.31\text{ N})^2}$

$= 9689.87\text{ N} = 9.69\text{ kN}$

F13–16. $r = (0.6\cos 2\theta)\text{ m}\big|_{\theta=0^\circ} = [0.6\cos 2(0^\circ)]\text{ m} = 0.6\text{ m}$

$\dot{r} = (-1.2\sin 2\theta\dot{\theta})\text{ m/s}\big|_{\theta=0^\circ}$

$= \left[-1.2\sin 2(0^\circ)(-3)\right]\text{ m/s} = 0$

$\ddot{r} = -1.2\left(\sin 2\theta\ddot{\theta} + 2\cos 2\theta\dot{\theta}^2\right)\text{ m/s}^2\big|_{\theta=0^\circ}$

$= -21.6\text{ m/s}^2$

Thus,

$a_r = \ddot{r} - r\dot{\theta}^2 = -21.6\text{ m/s}^2 - 0.6\text{ m}(-3\text{ rad/s})^2$

$= -27\text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.6\text{ m}(0) + 2(0)(-3\text{ rad/s}) = 0$

$\Sigma F_\theta = ma_\theta; \quad F - 0.2(9.81)\text{ N} = 0.2\text{ kg}(0)$

$F = 1.96\text{ N}\uparrow$ *Ans.*

Chapter 14

F14–1. $T_1 + \Sigma U_{1-2} = T_2$

$0 + \left(\frac{4}{5}\right)(500\text{ N})(0.5\text{ m}) - \frac{1}{2}(500\text{ N/m})(0.5\text{ m})^2$

$= \frac{1}{2}(10\text{ kg})v^2$

$v = 5.24\text{ m/s}$ *Ans.*

F14–2. $\Sigma F_y = ma_y;\ N_A - 20(9.81)\text{ N}\cos 30^\circ = 0$

$N_A = 169.91\text{ N}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 300\text{ N}(10\text{ m}) - 0.3(169.91\text{ N})(10\text{ m})$

$- 20(9.81)\text{N}(10\text{ m})\sin 30^\circ$

$= \frac{1}{2}(20\text{ kg})v^2$

$v = 12.3\text{ m/s}$ *Ans.*

F14–3. $T_1 + \Sigma U_{1-2} = T_2$

$0 + 2\left[\int_0^{15\text{ m}}(600 + 2s^2)\text{ N}\,ds\right] - 100(9.81)\text{ N}(15\text{ m})$

$= \frac{1}{2}(100\text{ kg})v^2$

$v = 12.5\text{ m/s}$ *Ans.*

F14–4. $T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(1800\text{ kg})(125\text{ m/s})^2 - \left[\frac{(50\,000\text{ N} + 20\,000\text{ N})}{2}(400\text{ m})\right]$

$= \frac{1}{2}(1800\text{ kg})v^2$

$v = 8.33\text{ m/s}$ *Ans.*

F14–5. $T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(10\text{ kg})(5\text{ m/s})^2 + 100\text{ N}s' + [10(9.81)\text{ N}]\,s'\sin 30^\circ$

$-\frac{1}{2}(200\text{ N/m})(s')^2 = 0$

$s' = 2.09\text{ m}$

$s = 0.6\text{ m} + 2.09\text{ m} = 2.69\text{ m}$ *Ans.*

F14–6. $T_A + \Sigma U_{A-B} = T_B$

Consider difference in cord length $AC - BC$, which is distance F moves.

$0 + 10\text{ lb}\left(\sqrt{(3\text{ ft})^2 + (4\text{ ft})^2} - 3\text{ ft}\right)$

$= \frac{1}{2}\left(\frac{5}{32.2}\text{ slug}\right)v_B^2$

$v_B = 16.0\text{ ft/s}$ *Ans.*

F14–7. $\overset{+}{\rightarrow}\Sigma F_x = ma_x;$

$30\left(\frac{4}{5}\right) = 20a \quad a = 1.2\text{ m/s}^2 \rightarrow$

$v = v_0 + a_c t$

$v = 0 + 1.2(4) = 4.8\text{ m/s}$

$P = \mathbf{F}\cdot\mathbf{v} = F(\cos\theta)v$

$= 30\left(\frac{4}{5}\right)(4.8)$

$= 115\text{ W}$ *Ans.*

F14–8. $\overset{+}{\rightarrow}\Sigma F_x = ma_x;$

$10s = 20a \quad a = 0.5s\text{ m/s}^2 \rightarrow$

$v\,dv = a\,ds$

$\int_1^v v\,dv = \int_0^{5\text{ m}} 0.5\,s\,ds$

$v = 3.674\text{ m/s}$

$P = \mathbf{F}\cdot\mathbf{v} = [10(5)](3.674) = 184\text{ W}$ *Ans.*

F14–9. $(+\uparrow)\Sigma F_y = 0;$
$T_1 - 100 \text{ lb} = 0 \qquad T_1 = 100 \text{ lb}$
$(+\uparrow)\Sigma F_y = 0;$
$100 \text{ lb} + 100 \text{ lb} - T_2 = 0 \quad T_2 = 200 \text{ lb}$
$P_{\text{out}} = \mathbf{T}_B \cdot \mathbf{v}_B = (200 \text{ lb})(3 \text{ ft/s}) = 1.091 \text{ hp}$
$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{1.091 \text{ hp}}{0.8} = 1.36 \text{ hp}$ *Ans.*

F14–10. $\Sigma F_{y'} = ma_{y'};\ N - 20(9.81)\cos 30° = 20(0)$
$N = 169.91 \text{ N}$
$\Sigma F_{x'} = ma_{x'};$
$F - 20(9.81)\sin 30° - 0.2(169.91) = 0$
$F = 132.08 \text{ N}$
$P = \mathbf{F}\cdot\mathbf{v} = 132.08(5) = 660 \text{ W}$ *Ans.*

F14–11. $+\uparrow\Sigma F_y = ma_y;$
$T - 50(9.81) = 50(0) \quad T = 490.5 \text{ N}$
$P_{\text{out}} = \mathbf{T}\cdot\mathbf{v} = 490.5(1.5) = 735.75 \text{ W}$
Also, for a point on the other cable
$P_{\text{out}} = \left(\dfrac{490.5}{2}\right)(1.5)(2) = 735.75 \text{ W}$
$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{735.75}{0.8} = 920 \text{ W}$ *Ans.*

F14–12. $2s_A + s_P = l$
$2a_A + a_P = 0$
$2a_A + 6 = 0$
$a_A = -3 \text{ m/s}^2 = 3 \text{ m/s}^2 \uparrow$
$\Sigma F_y = ma_y; \quad T_A - 490.5 \text{ N} = (50 \text{ kg})(3 \text{ m/s}^2)$
$T_A = 640.5 \text{ N}$
$P_{\text{out}} = \mathbf{T}\cdot\mathbf{v} = (640.5 \text{ N}/2)(12) = 3843 \text{ W}$
$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{3843}{0.8} = 4803.75 \text{ W} = 4.80 \text{ kW}$ *Ans.*

F14–13. $T_A + V_A = T_B + V_B$
$0 + 2(9.81)(1.5) = \frac{1}{2}(2)(v_B)^2 + 0$
$v_B = 5.42 \text{ m/s}$ *Ans.*
$+\uparrow\Sigma F_n = ma_n;\ T - 2(9.81) = 2\left(\dfrac{(5.42)^2}{1.5}\right)$
$T = 58.9 \text{ N}$ *Ans.*

F14–14. $T_A + V_A = T_B + V_B$
$\frac{1}{2}m_A v_A^2 + mgh_A = \frac{1}{2}m_B v_B^2 + mgh_B$
$\left[\frac{1}{2}(2 \text{ kg})(1 \text{ m/s})^2\right] + [2\,(9.81) \text{ N}(4 \text{ m})]$
$= \left[\frac{1}{2}(2 \text{ kg})v_B^2\right] + [0]$
$v_B = 8.915 \text{ m/s} = 8.92 \text{ m/s}$ *Ans.*
$+\uparrow\Sigma F_n = ma_n; \quad N_B - 2(9.81) \text{ N}$
$= (2 \text{ kg})\left(\dfrac{(8.915 \text{ m/s})^2}{2 \text{ m}}\right)$
$N_B = 99.1 \text{ N}$ *Ans.*

F14–15. $T_1 + V_1 = T_2 + V_2$
$\frac{1}{2}(2)(4)^2 + \frac{1}{2}(30)(2-1)^2$
$= \frac{1}{2}(2)(v)^2 - 2(9.81)(1) + \frac{1}{2}(30)(\sqrt{5}-1)^2$
$v = 5.26 \text{ m/s}$ *Ans.*

F14–16. $T_A + V_A = T_B + V_B$
$0 + \frac{1}{2}(4)(2.5-0.5)^2 + 5(2.5)$
$= \frac{1}{2}\left(\frac{5}{32.2}\right)v_B^2 + \frac{1}{2}(4)(1-0.5)^2$
$v_B = 16.0 \text{ ft/s}$ *Ans.*

F14–17. $T_1 + V_1 = T_2 + V_2$
$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}ks_1^2$
$= \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}ks_2^2$
$[0] + [0] + [0] = [0] +$
$[-75 \text{ lb}(5 \text{ ft} + s)] + \Big[2\left(\frac{1}{2}(1000 \text{ lb/ft})s^2\right)$
$+ \frac{1}{2}(1500 \text{ lb/ft})(s - 0.25 \text{ ft})^2\Big]$
$s = s_A = s_C = 0.580 \text{ ft}$ *Ans.*
Also,
$s_B = 0.5803 \text{ ft} - 0.25 \text{ ft} = 0.330 \text{ ft}$ *Ans.*

F14–18. $T_A + V_A = T_B + V_B$
$\frac{1}{2}mv_A^2 + \left(\frac{1}{2}ks_A^2 + mgy_A\right)$
$= \frac{1}{2}mv_B^2 + \left(\frac{1}{2}ks_B^2 + mgy_B\right)$
$\frac{1}{2}(4 \text{ kg})(2 \text{ m/s})^2 + \frac{1}{2}(400 \text{ N/m})(0.1 \text{ m} - 0.2 \text{ m})^2 + 0$
$= \frac{1}{2}(4 \text{ kg})v_B^2 + \frac{1}{2}(400 \text{ N/m})(\sqrt{(0.4 \text{ m})^2 + (0.3 \text{ m})^2}$
$- 0.2 \text{ m})^2 + [4(9.81) \text{ N}](-(0.1 \text{ m} + 0.3 \text{ m}))$
$v_B = 1.962 \text{ m/s} = 1.96 \text{ m/s}$ *Ans.*

Chapter 15

F15–1. $(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma\displaystyle\int_{t1}^{t2} F_x\,dt = m(v_2)_x$
$(0.5 \text{ kg})(25 \text{ m/s})\cos 45° - \displaystyle\int F_x\,dt$
$= (0.5 \text{ kg})(10 \text{ m/s})\cos 30°$
$I_x = \displaystyle\int F_x\,dt = 4.509 \text{ N}\cdot\text{s}$
$(+\uparrow) \quad m(v_1)_y + \Sigma\displaystyle\int_{t1}^{t2} F_y\,dt = m(v_2)_y$
$-(0.5 \text{ kg})(25 \text{ m/s})\sin 45° + \displaystyle\int F_y\,dt$
$= (0.5 \text{ kg})(10 \text{ m/s})\sin 30°$
$I_y = \displaystyle\int F_y\,dt = 11.339 \text{ N}\cdot\text{s}$
$I = \displaystyle\int F\,dt = \sqrt{(4.509 \text{ N}\cdot\text{s})^2 + (11.339 \text{ N}\cdot\text{s})^2}$
$= 12.2 \text{ N}\cdot\text{s}$ *Ans.*

F15–2. $(+\uparrow)\quad m(v_1)_y + \Sigma\int_{t1}^{t2} F_y\,dt = m(v_2)_y$

$0 + N(4\text{ s}) + (100\text{ lb})(4\text{ s})\sin 30^\circ - (150\text{ lb})(4\text{ s}) = 0$

$N = 100\text{ lb}$

$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$0 + (100\text{ lb})(4\text{ s})\cos 30^\circ - 0.2(100\text{ lb})(4\text{ s}) = \left(\frac{150}{32.2}\text{ slug}\right)v$

$v = 57.2\text{ ft/s}$ *Ans.*

F15–3. Time to start motion,

$+\uparrow\Sigma F_y = 0;\quad N - 25(9.81)\text{ N} = 0\quad N = 245.25\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = 0;\quad 20t^2 - 0.3(245.25\text{ N}) = 0\quad t = 1.918\text{ s}$

$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$0 + \int_{1.918\text{ s}}^{4\text{ s}} 20t^2\,dt - (0.25(245.25\text{ N}))(4\text{ s} - 1.918\text{ s}) = (25\text{ kg})v$

$v = 10.1\text{ m/s}$ *Ans.*

F15–4. $(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$(1500\text{ kg})(0) + \left[\frac{1}{2}(6000\text{ N})(2\text{ s}) + (6000\text{ N})(6\text{ s} - 2\text{ s})\right] = (1500\text{ kg})\,v$

$v = 20\text{ m/s}$ *Ans.*

F15–5. SUV and trailer,

$m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$0 + (9000\text{ N})(20\text{ s}) = (1500\text{ kg} + 2500\text{ kg})v$

$v = 45.0\text{ m/s}$ *Ans.*

Trailer,

$m(v_1)_x + \Sigma\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$0 + T(20\text{ s}) = (1500\text{ kg})(45.0\text{ m/s})$

$T = 3375\text{ N} = 3.375\text{ kN}$ *Ans.*

F15–6. Block B:

$(+\downarrow)\ mv_1 + \int F\,dt = mv_2$

$0 + 8(5) - T(5) = \frac{8}{32.2}(1)$

$T = 7.95\text{ lb}$ *Ans.*

Block A:

$(\overset{+}{\rightarrow})\ mv_1 + \int F\,dt = mv_2$

$0 + 7.95(5) - \mu_k(10)(5) = \frac{10}{32.2}(1)$

$\mu_k = 0.789$ *Ans.*

F15–7. $(\overset{+}{\rightarrow})\quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$(20(10^3)\text{ kg})(3\text{ m/s}) + (15(10^3)\text{ kg})(-1.5\text{ m/s})$

$= (20(10^3)\text{ kg})(v_A)_2 + (15(10^3)\text{ kg})(2\text{ m/s})$

$(v_A)_2 = 0.375\text{ m/s} \rightarrow$ *Ans.*

$(\overset{+}{\rightarrow})\quad m(v_B)_1 + \Sigma\int_{t1}^{t2} F\,dt = m(v_B)_2$

$(15(10^3)\text{ kg})(-1.5\text{ m/s}) + F_{avg}(0.5\text{ s}) = (15(10^3)\text{ kg})(2\text{ m/s})$

$F_{avg} = 105(10^3)\text{ N} = 105\text{ kN}$ *Ans.*

F15–8. $(\overset{+}{\rightarrow})\quad m_p[(v_p)_1]_x + m_c[(v)_1]_x = (m_p + m_c)v_2$

$5\left[10\left(\frac{4}{5}\right)\right] + 0 = (5 + 20)v_2$

$v_2 = 1.6\text{ m/s}$ *Ans.*

F15–9. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m_A(v_A)_1^2 + (V_g)_1 = \frac{1}{2}m_A(v_A)_2^2 + (V_g)_2$

$\frac{1}{2}(5)(5)^2 + 5(9.81)(1.5) = \frac{1}{2}(5)(v_A)_2^2 + 0$

$(v_A)_2 = 7.378\text{ m/s}$

$(\overset{+}{\leftarrow})\quad m_A(v_A)_2 + m_B(v_B)_2 = (m_A + m_B)v$

$5(7.378) + 0 = (5 + 8)v$

$v = 2.84\text{ m/s}$ *Ans.*

F15–10. $(\overset{+}{\rightarrow})\quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$0 + 0 = 10(v_A)_2 + 15(v_B)_2 \qquad (1)$

$T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m_A(v_A)_1^2 + \frac{1}{2}m_B(v_B)_1^2 + (V_e)_1 = \frac{1}{2}m_A(v_A)_2^2 + \frac{1}{2}m_B(v_B)_2^2 + (V_e)_2$

$0 + 0 + \frac{1}{2}\left[5\left(10^3\right)\right]\left(0.2^2\right) = \frac{1}{2}(10)(v_A)_2^2 + \frac{1}{2}(15)(v_B)_2^2 + 0$

$5(v_A)_2^2 + 7.5(v_B)_2^2 = 100 \qquad (2)$

Solving Eqs. (1) and (2),

$(v_B)_2 = 2.31\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -3.464\text{ m/s} = 3.46\text{ m/s} \leftarrow$ *Ans.*

F15–11. $(\overset{+}{\leftarrow})\quad m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$

$0 + 10(15) = (15 + 10)v_2$

$v_2 = 6\text{ m/s}$

$T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}(m_A + m_B)v_2^2 + (V_e)_2 = \frac{1}{2}(m_A + m_B)v_3^2 + (V_e)_3$

$\frac{1}{2}(15 + 10)\left(6^2\right) + 0 = 0 + \frac{1}{2}\left[10\left(10^3\right)\right]s_{max}^2$

$s_{max} = 0.3\text{ m} = 300\text{ mm}$ *Ans.*

F15–12. $(\overset{+}{\rightarrow})\quad 0 + 0 = m_p(v_p)_x - m_c v_c$

$0 = (20\text{ kg})\,(v_p)_x - (250\text{ kg})v_c$

$(v_p)_x = 12.5\, v_c \qquad (1)$

$\mathbf{v}_p = \mathbf{v}_c + \mathbf{v}_{p/c}$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = -v_c\mathbf{i} + [(400\text{ m/s})\cos 30°\mathbf{i} + (400\text{ m/s})\sin 30°\mathbf{j}]$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = (346.41 - v_c)\mathbf{i} + 200\mathbf{j}$

$(v_p)_x = 346.41 - v_c$

$(v_p)_y = 200\text{ m/s}$

$(v_p)_x = 320.75\text{ m/s}\quad v_c = 25.66\text{ m/s}$

$v_p = \sqrt{(v_p)_x^2 + (v_p)_y^2}$

$= \sqrt{(320.75\text{ m/s})^2 + (200\text{ m/s})^2}$

$= 378\text{ m/s}$ *Ans.*

F15–13. $(\overset{+}{\rightarrow})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$= \dfrac{(9\text{ m/s}) - (1\text{ m/s})}{(8\text{ m/s}) - (-2\text{ m/s})} = 0.8$

F15–14. $(\overset{+}{\rightarrow})\quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$[15(10^3)\text{ kg}](5\text{ m/s}) + [25(10^3)](-7\text{ m/s})$

$= [15(10^3)\text{ kg}](v_A)_2 + [25(10^3)](v_B)_2$

$15(v_A)_2 + 25(v_B)_2 = -100 \qquad (1)$

Using the coefficient of restitution equation,

$(\overset{+}{\rightarrow})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$0.6 = \dfrac{(v_B)_2 - (v_A)_2}{5\text{ m/s} - (-7\text{ m/s})}$

$(v_B)_2 - (v_A)_2 = 7.2 \qquad (2)$

Solving,

$(v_B)_2 = 0.2\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -7\text{ m/s} = 7\text{ m/s} \leftarrow$ *Ans.*

F15–15. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m(v_A)_1^2 + mg(h_A)_1 = \frac{1}{2}m(v_A)_2^2 + mg(h_A)_2$

$\frac{1}{2}\left(\frac{30}{32.2}\text{ slug}\right)(5\text{ ft/s})^2 + (30\text{ lb})(10\text{ft})$

$= \frac{1}{2}\left(\frac{30}{32.2}\text{ slug}\right)(v_A)_2^2 + 0$

$(v_A)_2 = 25.87\text{ ft/s} \leftarrow$

$(\overset{+}{\leftarrow})\quad m_A(v_A)_2 + m_B(v_B)_2 = m_A(v_A)_3 + m_B(v_B)_3$

$\left(\frac{30}{32.2}\text{ slug}\right)(25.87\text{ ft/s}) + 0$

$= \left(\frac{30}{32.2}\text{ slug}\right)(v_A)_3 + \left(\frac{80}{32.2}\text{ slug}\right)(v_B)_3$

$30(v_A)_3 + 80(v_B)_3 = 775.95 \qquad (1)$

$(\overset{+}{\leftarrow})\quad e = \dfrac{(v_B)_3 - (v_A)_3}{(v_A)_2 - (v_B)_2}$

$0.6 = \dfrac{(v_B)_3 - (v_A)_3}{25.87\text{ ft/s} - 0}$

$(v_B)_3 - (v_A)_3 = 15.52 \qquad (2)$

Solving Eqs. (1) and (2), yields

$(v_B)_3 = 11.3\text{ ft/s} \leftarrow$

$(v_A)_3 = -4.23\text{ ft/s} = 4.23\text{ ft/s} \rightarrow$ *Ans.*

F15–16. $(+\uparrow)\quad m[(v_b)_1]_y = m[(v_b)_2]_y$

$[(v_b)_2]_y = [(v_b)_1]_y = (20\text{ m/s})\sin 30° = 10\text{ m/s} \uparrow$

$(\overset{+}{\rightarrow})\quad e = \dfrac{(v_w)_2 - [(v_b)_2]_x}{[(v_b)_1]_x - (v_w)_1}$

$0.75 = \dfrac{0 - [(v_b)_2]_x}{(20\text{ m/s})\cos 30° - 0}$

$[(v_b)_2]_x = -12.99\text{ m/s} = 12.99\text{ m/s} \leftarrow$

$(v_b)_2 = \sqrt{[(v_b)_2]_x^2 + [(v_b)_2]_y^2}$

$= \sqrt{(12.99\text{ m/s})^2 + (10\text{ m/s})^2}$

$= 16.4\text{ m/s}$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{[(v_b)_2]_y}{[(v_b)_2]_x}\right) = \tan^{-1}\left(\dfrac{10\text{ m/s}}{12.99\text{ m/s}}\right)$

$= 37.6°$ *Ans.*

F15–17. $\Sigma m(v_x)_1 = \Sigma m(v_x)_2$

$0 + 0 = 2\,(1) + 11\,(v_{Bx})_2$

$(v_{Bx})_2 = -0.1818\text{ m/s}$

$\Sigma m(v_y)_1 = \Sigma m(v_y)_2$

$2\,(3) + 0 = 0 + 11\,(v_{By})_2$

$(v_{By})_2 = 0.545\text{ m/s}$

$(v_B)_2 = \sqrt{(-0.1818)^2 + (0.545)^2}$

$= 0.575\text{ m/s}$ *Ans.*

F15–18. $+\nearrow\quad 1\,(3)\left(\frac{3}{5}\right) - 1\,(4)\left(\frac{4}{5}\right)$

$= 1\,(v_B)_{2x} + 1\,(v_A)_{2x}$

$+\nearrow\quad 0.5 = [(v_A)_{2x} - (v_B)_{2x}]/[(3)\left(\frac{3}{5}\right) - (-4)\left(\frac{4}{5}\right)]$

Solving,

$(v_A)_{2x} = 0.550\text{ m/s}, (v_B)_{2x} = -1.95\text{ m/s}$

Disc A,

$+\nwarrow\quad -1\,(4)\left(\frac{3}{5}\right) = 1\,(v_A)_{2y}$

$(v_A)_{2y} = -2.40\text{ m/s}$

Disc B,

$$-1(3)\left(\tfrac{4}{5}\right) = 1(v_B)_{2y}$$

$$(v_B)_{2y} = -2.40 \text{ m/s}$$

$$(v_A)_2 = \sqrt{(0.550)^2 + (2.40)^2} = 2.46 \text{ m/s} \qquad \textit{Ans.}$$

$$(v_B)_2 = \sqrt{(1.95)^2 + (2.40)^2} = 3.09 \text{ m/s} \qquad \textit{Ans.}$$

F15–19. $H_O = \Sigma mvd;$

$$H_O = \left[2(10)\left(\tfrac{4}{5}\right)\right](4) - \left[2(10)\left(\tfrac{3}{5}\right)\right](3)$$

$$= 28 \text{ kg}\cdot\text{m}^2/\text{s}$$

F15–20. $H_P = \Sigma mvd;$

$$H_P = [2(15)\sin 30°](2) - [2(15)\cos 30°](5)$$

$$= -99.9 \text{ kg}\cdot\text{m}^2/\text{s} = 99.9 \text{ kg}\cdot\text{m}^2/\text{s} \curvearrowright$$

F15–21. $(H_z)_1 + \Sigma \int M_z\,dt = (H_z)_2$

$$5(2)(1.5) + 5(1.5)(3) = 5v(1.5)$$

$$v = 5 \text{ m/s} \qquad \textit{Ans.}$$

F15–22. $(H_z)_1 + \Sigma \int M_z\,dt = (H_z)_2$

$$0 + \int_0^{4\text{ s}} (10t)\left(\tfrac{4}{5}\right)(1.5)dt = 5v(1.5)$$

$$v = 12.8 \text{ m/s} \qquad \textit{Ans.}$$

F15–23. $(H_z)_1 + \Sigma \int M_z\,dt = (H_z)_2$

$$0 + \int_0^{5\text{ s}} 0.9t^2\,dt = 2v(0.6)$$

$$v = 31.2 \text{ m/s} \qquad \textit{Ans.}$$

F15–24. $(H_z)_1 + \Sigma \int M_z\,dt = (H_z)_2$

$$0 + \int_0^{4\text{ s}} 8t dt + 2(10)(0.5)(4) = 2[10v(0.5)]$$

$$v = 10.4 \text{ m/s} \qquad \textit{Ans.}$$

Chapter 16

F16–1. $\theta = (20 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 40\pi \text{ rad}$

$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$(30 \text{ rad/s})^2 = 0^2 + 2\alpha_c[(40\pi \text{ rad}) - 0]$$

$$\alpha_c = 3.581 \text{ rad/s}^2 = 3.58 \text{ rad/s}^2 \qquad \textit{Ans.}$$

$$\omega = \omega_0 + \alpha_c t$$

$$30 \text{ rad/s} = 0 + (3.581 \text{ rad/s}^2)t$$

$$t = 8.38 \text{ s} \qquad \textit{Ans.}$$

F16–2. $\frac{d\omega}{d\theta} = 2(0.005\theta) = (0.01\theta)$

$$\alpha = \omega\frac{d\omega}{d\theta} = \left(0.005\,\theta^2\right)(0.01\theta) = 50\left(10^{-6}\right)\theta^3 \text{ rad/s}^2$$

When $\theta = 20 \text{ rev}(2\pi \text{ rad}/1 \text{ rev}) = 40\pi \text{ rad}$,

$$\alpha = \left[50\left(10^{-6}\right)(40\pi)^3\right] \text{ rad/s}^2$$

$$= 99.22 \text{ rad/s}^2 = 99.2 \text{ rad/s}^2 \qquad \textit{Ans.}$$

F16–3. $\omega = 4\theta^{1/2}$

$$150 \text{ rad/s} = 4\,\theta^{1/2}$$

$$\theta = 1406.25 \text{ rad}$$

$$dt = \frac{d\theta}{\omega}$$

$$\int_0^t dt = \int_{1 \text{ rad}}^{\theta} \frac{d\theta}{4\theta^{1/2}}$$

$$t\Big|_0^t = \tfrac{1}{2}\theta^{1/2}\Big|_{1 \text{ rad}}^{\theta}$$

$$t = \tfrac{1}{2}\theta^{1/2} - \tfrac{1}{2}$$

$$t = \tfrac{1}{2}(1406.25)^{1/2} - \tfrac{1}{2} = 18.25 \text{ s} \qquad \textit{Ans.}$$

F16–4. $\omega = \frac{d\theta}{dt} = (1.5t^2 + 15) \text{ rad/s}$

$$\alpha = \frac{d\omega}{dt} = (3t) \text{ rad/s}^2$$

$$\omega = [1.5(3^2) + 15] \text{ rad/s} = 28.5 \text{ rad/s}$$

$$\alpha = 3(3) \text{ rad/s}^2 = 9 \text{ rad/s}^2$$

$$v = \omega r = (28.5 \text{ rad/s})(0.75 \text{ ft}) = 21.4 \text{ ft/s} \qquad \textit{Ans.}$$

$$a = \alpha r = (9 \text{ rad/s}^2)(0.75 \text{ ft}) = 6.75 \text{ ft/s}^2 \qquad \textit{Ans.}$$

F16–5. $\omega\,d\omega = \alpha\,d\theta$

$$\int_{2 \text{ rad/s}}^{\omega} \omega\,d\omega = \int_0^{\theta} 0.5\theta\,d\theta$$

$$\frac{\omega^2}{2}\Big|_{2 \text{ rad/s}}^{\omega} = 0.25\theta^2\Big|_0^{\theta}$$

$$\omega = (0.5\,\theta^2 + 4)^{1/2} \text{ rad/s}$$

When $\theta = 2 \text{ rev} = 4\pi \text{ rad}$,

$$\omega = [0.5(4\pi)^2 + 4]^{1/2} \text{ rad/s} = 9.108 \text{ rad/s}$$

$$v_P = \omega r = (9.108 \text{ rad/s})(0.2 \text{ m}) = 1.82 \text{ m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha r = (0.5\theta \text{ rad/s}^2)(0.2 \text{ m})\Big|_{\theta = 4\pi \text{ rad}}$$

$$= 1.257 \text{ m/s}^2$$

$$(a_P)_n = \omega^2 r = (9.108 \text{ rad/s})^2(0.2 \text{ m}) = 16.59 \text{ m/s}^2$$

$$a_p = \sqrt{(a_P)_t^2 + (a_P)_n^2}$$

$$= \sqrt{(1.257 \text{ m/s}^2)^2 + (16.59 \text{ m/s}^2)^2}$$

$$= 16.6 \text{ m/s}^2 \qquad \textit{Ans.}$$

F16–6. $\alpha_B = \alpha_A\left(\frac{r_A}{r_B}\right)$

$= (4.5 \text{ rad/s}^2)\left(\frac{0.075 \text{ m}}{0.225 \text{ m}}\right) = 1.5 \text{ rad/s}^2$

$\omega_B = (\omega_B)_0 + \alpha_B t$

$\omega_B = 0 + (1.5 \text{ rad/s}^2)(3 \text{ s}) = 4.5 \text{ rad/s}$

$\theta_B = (\theta_B)_0 + (\omega_B)_0 t + \frac{1}{2}\alpha_B t^2$

$\theta_B = 0 + 0 + \frac{1}{2}(1.5 \text{ rad/s}^2)(3 \text{ s})^2$

$\theta_B = 6.75 \text{ rad}$

$v_C = \omega_B r_D = (4.5 \text{ rad/s})(0.125 \text{ m})$

$= 0.5625 \text{ m/s}$ *Ans.*

$s_C = \theta_B r_D = (6.75 \text{ rad})(0.125 \text{ m}) = 0.84375 \text{ m}$

$= 844 \text{ mm}$ *Ans.*

F16–7. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$-v_B\,\mathbf{j} = (3\mathbf{i}) \text{ m/s}$

$+ (\omega\mathbf{k}) \times (-1.5 \cos 30°\mathbf{i} + 1.5 \sin 30°\mathbf{j})$

$-v_B\,\mathbf{j} = [3 - \omega_{AB}(1.5 \sin 30°)]\mathbf{i} - \omega(1.5 \cos 30°)\mathbf{j}$

$0 = 3 - \omega(1.5 \sin 30°)$ (1)

$-v_B = 0 - \omega(1.5 \cos 30°)$ (2)

$\omega = 4 \text{ rad/s}$ $\quad v_B = 5.20 \text{ m/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\downarrow v_B\right] = \left[\underset{\rightarrow}{3}\right] + \left[\omega(1.5) \measuredangle 30°\right]$

This yields Eqs. (1) and (2).

F16–8. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = \mathbf{0} + (-10\mathbf{k}) \times (-0.6\mathbf{i} + 0.6\mathbf{j})$

$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 6\mathbf{i} + 6\mathbf{j}$

$(v_B)_x = 6 \text{ m/s}$ and $(v_B)_y = 6 \text{ m/s}$

$v_B = \sqrt{(v_B)_x^2 + (v_B)_y^2}$

$= \sqrt{(6 \text{ m/s})^2 + (6 \text{ m/s})^2}$

$= 8.49 \text{ m/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\underset{\rightarrow}{(v_B)_x}\right] + \left[(v_B)_y \uparrow\right] = \left[0\right] + \left[\measuredangle 45° \;\; 10\left(\frac{0.6}{\cos 45°}\right)\right]$

$\overset{+}{\rightarrow}\ (v_B)_x = 0 + 10(0.6/\cos 45°)\cos 45° = 6 \text{ m/s} \rightarrow$

$+\uparrow (v_B)_y = 0 + 10(0.6/\cos 45°)\sin 45° = 6 \text{ m/s}\uparrow$

F16–9. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$(4 \text{ ft/s})\mathbf{i} = (-2 \text{ ft/s})\mathbf{i} + (-\omega\mathbf{k}) \times (3 \text{ ft})\mathbf{j}$

$4\mathbf{i} = (-2 + 3\omega)\mathbf{i}$

$\omega = 2 \text{ rad/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\underset{\rightarrow}{4}\right] = \left[\underset{\leftarrow}{2}\right] + \left[\underset{\rightarrow}{\omega(3)}\right]$

$\overset{+}{\rightarrow}\ \ 4 = -2 + \omega(3); \qquad \omega = 2 \text{ rad/s}$

F16–10. Vector Analysis

$\mathbf{v}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_A$

$= (12 \text{ rad/s})\mathbf{k} \times (0.3 \text{ m})\mathbf{j}$

$= [-3.6\mathbf{i}] \text{ m/s}$

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$

$v_B\,\mathbf{j} = (-3.6 \text{ m/s})\mathbf{i}$

$+ (\omega_{AB}\,\mathbf{k}) \times (0.6 \cos 30°\mathbf{i} - 0.6 \sin 30°\mathbf{j}) \text{ m}$

$v_B\,\mathbf{j} = [\omega_{AB}(0.6 \sin 30°) - 3.6]\mathbf{i} + \omega_{AB}(0.6 \cos 30°)\mathbf{j}$

$0 = \omega_{AB}(0.6 \sin 30°) - 3.6$ (1)

$v_B = \omega_{AB}(0.6 \cos 30°)$ (2)

$\omega_{AB} = 12 \text{ rad/s}$ $\quad v_B = 6.24 \text{ m/s}\uparrow$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[v_B\uparrow\right] = \left[\overset{\leftarrow}{12(0.3)}\right] + \left[\measuredangle 30° \omega(0.6)\right]$

This yields Eqs. (1) and (2).

F16–11. Vector Analysis

$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$

$v_C\,\mathbf{j} = (-60\mathbf{i}) \text{ ft/s}$

$+ (-\omega_{BC}\mathbf{k}) \times (-2.5 \cos 30°\mathbf{i} + 2.5 \sin 30°\mathbf{j}) \text{ ft}$

$v_C\mathbf{j} = (-60)\mathbf{i} + 2.165\omega_{BC}\mathbf{j} + 1.25\omega_{BC}\,\mathbf{i}$

$0 = -60 + 1.25\omega_{BC}$ (1)

$v_C = 2.165\,\omega_{BC}$ (2)

$\omega_{BC} = 48 \text{ rad/s}$ *Ans.*

$v_C = 104 \text{ ft/s}$

Scalar Solution

$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$

$\left[v_C\uparrow\right] = \left[\underset{\leftarrow}{v_B}\right] + \left[\measuredangle 30°\ \omega\,(2.5)\right]$

This yields Eqs. (1) and (2).

F16–12. Vector Analysis

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$-v_B \cos 30^\circ\, \mathbf{i} + v_B \sin 30^\circ\, \mathbf{j} = (-3 \text{ m/s})\mathbf{j} + (-\omega\mathbf{k}) \times (-2 \sin 45^\circ\mathbf{i} - 2 \cos 45^\circ\mathbf{j}) \text{ m}$$

$$-0.8660 v_B\, \mathbf{i} + 0.5 v_B\, \mathbf{j} = -1.4142\omega\mathbf{i} + (1.4142\omega - 3)\mathbf{j}$$

$$-0.8660 v_B = -1.4142\omega \qquad (1)$$

$$0.5 v_B = 1.4142\omega - 3 \qquad (2)$$

$\omega = 5.02 \text{ rad/s} \quad v_B = 8.20 \text{ m/s}$ *Ans.*

Scalar Solution

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[\measuredangle 30^\circ\, v_B\right] = \left[\downarrow 3\right] + \left[\measuredangle 45^\circ\, \omega(2)\right]$$

This yields Eqs. (1) and (2).

F16–13. $\omega_{AB} = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2 \text{ rad/s}$ *Ans.*

$$\phi = \tan^{-1}\left(\tfrac{2}{1.5}\right) = 53.13^\circ$$

$$r_{C/IC} = \sqrt{(3)^2 + (2.5)^2 - 2(3)(2.5)\cos 53.13^\circ} = 2.5 \text{ m}$$

$v_C = \omega_{AB}\, r_{C/IC} = 2(2.5) = 5 \text{ m/s}$ *Ans.*

$\theta = 90^\circ - \phi = 90^\circ - 53.13^\circ = 36.9^\circ \measuredangle$ *Ans.*

F16–14. $v_B = \omega_{AB}\, r_{B/A} = 12(0.6) = 7.2 \text{ m/s} \downarrow$

$v_C = 0$ *Ans.*

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{7.2}{1.2} = 6 \text{ rad/s}$ *Ans.*

F16–15. $\omega = \dfrac{v_O}{r_{O/IC}} = \dfrac{6}{0.3} = 20 \text{ rad/s}$ *Ans.*

$$r_{A/IC} = \sqrt{0.3^2 + 0.6^2} = 0.6708 \text{ m}$$

$$\phi = \tan^{-1}\left(\tfrac{0.3}{0.6}\right) = 26.57^\circ$$

$v_A = \omega r_{A/IC} = 20(0.6708) = 13.4 \text{ m/s}$ *Ans.*

$\theta = 90^\circ - \phi = 90^\circ - 26.57^\circ = 63.4^\circ \measuredangle$ *Ans.*

F16–16. The location of *IC* can be determined using similar triangles.

$$\frac{0.5 - r_{C/IC}}{3} = \frac{r_{C/IC}}{1.5} \qquad r_{C/IC} = 0.1667 \text{ m}$$

$\omega = \dfrac{v_C}{r_{C/IC}} = \dfrac{1.5}{0.1667} = 9 \text{ rad/s}$ *Ans.*

Also, $r_{O/IC} = 0.3 - r_{C/IC} = 0.3 - 0.1667 = 0.1333$ m.

$v_O = \omega r_{O/IC} = 9(0.1333) = 1.20 \text{ m/s}$ *Ans.*

F16–17. $v_B = \omega r_{B/A} = 6(0.2) = 1.2 \text{ m/s}$

$$r_{B/IC} = 0.8 \tan 60^\circ = 1.3856 \text{ m}$$

$$r_{C/IC} = \tfrac{0.8}{\cos 60^\circ} = 1.6 \text{ m}$$

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{1.2}{1.3856} = 0.8660 \text{ rad/s}$$

$= 0.866 \text{ rad/s}$ *Ans.*

Then,

$v_C = \omega_{BC}\, r_{C/IC} = 0.8660(1.6) = 1.39 \text{ m/s}$ *Ans.*

F16–18. $v_B = \omega_{AB}\, r_{B/A} = 10(0.2) = 2 \text{ m/s}$

$$v_C = \omega_{CD}\, r_{C/D} = \omega_{CD}(0.2) \rightarrow$$

$$r_{B/IC} = \tfrac{0.4}{\cos 30^\circ} = 0.4619 \text{ m}$$

$$r_{C/IC} = 0.4 \tan 30^\circ = 0.2309 \text{ m}$$

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{2}{0.4619} = 4.330 \text{ rad/s}$$

$= 4.33 \text{ rad/s}$ *Ans.*

$$v_C = \omega_{BC}\, r_{C/IC}$$

$$\omega_{CD}(0.2) = 4.330(0.2309)$$

$\omega_{CD} = 5 \text{ rad/s}$ *Ans.*

F16–19. $\omega = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2 \text{ rad/s}$

Vector Analysis

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\, \mathbf{r}_{B/A}$$

$$a_B\mathbf{i} = -5\mathbf{j} + \left(\alpha\mathbf{k}\right) \times (3\mathbf{i} - 4\mathbf{j}) - 2^2(3\mathbf{i} - 4\mathbf{j})$$

$$a_B\mathbf{i} = \left(4\alpha - 12\right)\mathbf{i} + \left(3\alpha + 11\right)\mathbf{j}$$

$$a_B = 4\alpha - 12 \qquad (1)$$

$$0 = 3\alpha + 11 \qquad (2)$$

$\alpha = -3.67 \text{ rad/s}^2$ *Ans.*

$a_B = -26.7 \text{ m/s}^2$ *Ans.*

Scalar Solution

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\begin{bmatrix} a_B \\ \rightarrow \end{bmatrix} = \left[\downarrow 5\right] + \left[\alpha(5) \; {}^{5}\!\measuredangle^{3}_{4}\right] + \left[{}_{4}\measuredangle^{5}_{3}\; (2)^2(5)\right]$$

This yields Eqs. (1) and (2).

F16–20. Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 1.8\mathbf{i} + (-6\mathbf{k}) \times (0.3\mathbf{j}) - 12^2\,(0.3\mathbf{j})$$
$$= \{3.6\mathbf{i} - 43.2\mathbf{j}\}\text{m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$
$$\begin{bmatrix}(a_A)_x\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_A)_y\uparrow\end{bmatrix} = \begin{bmatrix}(6)(0.3)\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(6)(0.3)\\ \rightarrow\end{bmatrix} + \begin{bmatrix}\downarrow(12)^2(0.3)\end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad (a_A)_x = 1.8 + 1.8 = 3.6\text{ m/s}^2 \rightarrow$$
$$+\uparrow \qquad (a_A)_y = -43.2\text{ m/s}^2$$

F16–21. Using

$$v_O = \omega r; \qquad 6 = \omega(0.3)$$
$$\omega = 20\text{ rad/s}$$
$$a_O = \alpha r; \qquad 3 = \alpha(0.3)$$
$$\alpha = 10\text{ rad/s}^2 \qquad \textit{Ans.}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 3\mathbf{i} + (-10\mathbf{k}) \times (-0.6\mathbf{i}) - 20^2(-0.6\mathbf{i})$$
$$= \{243\mathbf{i} + 6\mathbf{j}\}\text{ m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$
$$\begin{bmatrix}(a_A)_x\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_A)_y\\ \uparrow\end{bmatrix} = \begin{bmatrix}3\\ \rightarrow\end{bmatrix} + \begin{bmatrix}10(0.6)\\ \uparrow\end{bmatrix} + \begin{bmatrix}(20)^2(0.6)\\ \rightarrow\end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad (a_A)_x = 3 + 240 = 243\text{ m/s}^2$$
$$+\uparrow \qquad (a_A)_y = 10(0.6) = 6\text{ m/s}^2 \uparrow$$

F16–22. $\dfrac{r_{A/IC}}{3} = \dfrac{0.5 - r_{A/IC}}{1.5}; \qquad r_{A/IC} = 0.3333\text{ m}$

$$\omega = \frac{v_A}{r_{A/IC}} = \frac{3}{0.3333} = 9\text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{A/C} - \omega^2\,\mathbf{r}_{A/C}$$
$$1.5\mathbf{i} - (a_A)_n\,\mathbf{j} = -0.75\mathbf{i} + (a_C)_n\,\mathbf{j}$$
$$+ (-\alpha\mathbf{k}) \times 0.5\mathbf{j} - 9^2\,(0.5\mathbf{j})$$
$$1.5\mathbf{i} - (a_A)_n\,\mathbf{j} = (0.5\alpha - 0.75)\mathbf{i} + \left[(a_C)_n - 40.5\right]\mathbf{j}$$
$$1.5 = 0.5\alpha - 0.75$$
$$\alpha = 4.5\text{ rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \mathbf{a}_{A/C}$$
$$\begin{bmatrix}1.5\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_A)_n\\ \downarrow\end{bmatrix} = \begin{bmatrix}0.75\\ \leftarrow\end{bmatrix} + \begin{bmatrix}(a_C)_n\\ \uparrow\end{bmatrix} + \begin{bmatrix}\alpha(0.5)\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(9)^2(0.5)\\ \downarrow\end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad 1.5 = -0.75 + \alpha(0.5)$$
$$\alpha = 4.5\text{ rad/s}^2$$

F16–23. $v_B = \omega r_{B/A} = 12(0.3) = 3.6\text{ m/s}$

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{3.6}{1.2} = 3\text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$
$$= (-6\mathbf{k}) \times (0.3\mathbf{i}) - 12^2(0.3\mathbf{i})$$
$$= \{-43.2\mathbf{i} - 1.8\mathbf{j}\}\text{ m/s}^2$$
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$
$$a_C\,\mathbf{i} = (-43.2\mathbf{i} - 1.8\mathbf{j})$$
$$+ (\alpha_{BC}\,\mathbf{k}) \times (1.2\mathbf{i}) - 3^2(1.2\mathbf{i})$$
$$a_C\,\mathbf{i} = -54\mathbf{i} + (1.2\alpha_{BC} - 1.8)\mathbf{j}$$
$$a_C = -54\text{ m/s}^2 = 54\text{ m/s}^2 \leftarrow \qquad \textit{Ans.}$$
$$0 = 1.2\alpha_{BC} - 1.8 \quad \alpha_{BC} = 1.5\text{ rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$$
$$\begin{bmatrix}a_C\\ \leftarrow\end{bmatrix} = \begin{bmatrix}6(0.3)\\ \downarrow\end{bmatrix} + \begin{bmatrix}(12)^2(0.3)\\ \leftarrow\end{bmatrix} + \begin{bmatrix}\alpha_{BC}(1.2)\\ \uparrow\end{bmatrix} + \begin{bmatrix}(3)^2(1.2)\\ \leftarrow\end{bmatrix}$$
$$\overset{+}{\leftarrow} \qquad a_C = 43.2 + 10.8 = 54\text{ m/s}^2 \leftarrow$$
$$+\uparrow \qquad 0 = -6(0.3) + 1.2\alpha_{BC}$$
$$\alpha_{BC} = 1.5\text{ rad/s}^2$$

F16–24. $v_B = \omega\, r_{B/A} = 6(0.2) = 1.2\text{ m/s} \rightarrow$

$$r_{B/IC} = 0.8\tan 60^\circ = 1.3856\text{ m}$$
$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{1.2}{1.3856} = 0.8660\text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$
$$= (-3\mathbf{k}) \times (0.2\mathbf{j}) - 6^2\,(0.2\mathbf{j})$$
$$= [0.6\mathbf{i} - 7.2\mathbf{j}]\text{ m/s}$$
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B}$$
$$a_C\cos 30^\circ\mathbf{i} + a_C\sin 30^\circ\mathbf{j}$$
$$= (0.6\mathbf{i} - 7.2\mathbf{j}) + (\alpha_{BC}\,\mathbf{k} \times 0.8\mathbf{i}) - 0.8660^2(0.8\mathbf{i})$$

$0.8660a_C\,\mathbf{i} + 0.5a_C\,\mathbf{j} = (0.8\alpha_{BC} - 7.2)\mathbf{j}$

$0.8660a_C = 0$ (1)

$0.5a_C = 0.8\alpha_{BC} - 7.2$ (2)

$a_C = 0 \qquad \alpha_{BC} = 9\ \text{rad/s}^2$ *Ans.*

Scalar Analysis

$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$

$$\begin{bmatrix} a_C \\ \measuredangle 30^\circ \end{bmatrix} = \begin{bmatrix} 3(0.2) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6)^2(0.2) \\ \downarrow \end{bmatrix} + \begin{bmatrix} \alpha_{BC}(0.8) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.8660)^2(0.8) \\ \leftarrow \end{bmatrix}$$

This yields Eqs. (1) and (2).

Chapter 17

F17–1. $\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;\ 100\left(\frac{4}{5}\right) = 100a$

$a = 0.8\ \text{m/s}^2 \rightarrow$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100\left(\frac{3}{5}\right) - 100(9.81) = 0$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$N_A(0.6) + 100\left(\frac{3}{5}\right)(0.7)$
$\quad - N_B(0.4) - 100\left(\frac{4}{5}\right)(0.7) = 0$ (2)

$N_A = 430.4\ \text{N} = 430\ \text{N}$ *Ans.*

$N_B = 610.6\ \text{N} = 611\ \text{N}$ *Ans.*

F17–2. $\Sigma F_{x'} = m(a_G)_{x'};\quad 80(9.81)\sin 15^\circ = 80a$

$a = 2.54\ \text{m/s}^2$ *Ans.*

$\Sigma F_{y'} = m(a_G)_{y'};$

$N_A + N_B - 80(9.81)\cos 15^\circ = 0$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$N_A(0.5) - N_B(0.5) = 0$ (2)

$N_A = N_B = 379\ \text{N}$ *Ans.*

F17–3. $\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A;\quad 10\left(\frac{3}{5}\right)(7) = \frac{20}{32.2}a(3.5)$

$a = 19.3\ \text{ft/s}^2$ *Ans.*

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;\quad A_x + 10\left(\frac{3}{5}\right) = \frac{20}{32.2}(19.32)$

$A_x = 6\ \text{lb}$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;\quad A_y - 20 + 10\left(\frac{4}{5}\right) = 0$

$A_y = 12\ \text{lb}$ *Ans.*

F17–4. $F_A = \mu_s N_A = 0.2N_A \quad F_B = \mu_s N_B = 0.2N_B$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;$

$0.2N_A + 0.2N_B = 100a$ (1)

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100(9.81) = 0$ (2)

$\circlearrowleft + \Sigma M_G = 0;$

$0.2N_A(0.75) + N_A(0.9) + 0.2N_B(0.75)$
$\quad - N_B(0.6) = 0$ (3)

Solving Eqs. (1), (2), and (3),

$N_A = 294.3\ \text{N} = 294\ \text{N}$

$N_B = 686.7\ \text{N} = 687\ \text{N}$

$a = 1.96\ \text{m/s}^2$ *Ans.*

Since N_A is positive, the table will indeed slide before it tips.

F17–5. $(a_G)_t = \alpha r = \alpha(1.5\ \text{m})$

$(a_G)_n = \omega^2 r = (5\ \text{rad/s})^2(1.5\ \text{m}) = 37.5\ \text{m/s}^2$

$\Sigma F_t = m(a_G)_t;\qquad 100\ \text{N} = 50\ \text{kg}[\alpha(1.5\ \text{m})]$

$\alpha = 1.33\ \text{rad/s}^2$ *Ans.*

$\Sigma F_n = m(a_G)_n;\qquad T_{AB} + T_{CD} - 50(9.81)\ \text{N}$
$\quad = 50\ \text{kg}(37.5\ \text{m/s}^2)$

$T_{AB} + T_{CD} = 2365.5$

$\circlearrowleft + \Sigma M_G = 0;\quad T_{CD}(1\ \text{m}) - T_{AB}(1\ \text{m}) = 0$

$T_{AB} = T_{CD} = 1182.75\ \text{N} = 1.18\ \text{kN}$ *Ans.*

F17–6. $\circlearrowleft + \Sigma M_C = 0;$

$\mathbf{a}_G = \mathbf{a}_D = \mathbf{a}_B$

$D_y(0.6) - 450 = 0 \quad D_y = 750\ \text{N}$ *Ans.*

$(a_G)_n = \omega^2 r = 6^2(0.6) = 21.6\ \text{m/s}^2$

$(a_G)_t = \alpha r = \alpha(0.6)$

$+\uparrow \Sigma F_t = m(a_G)_t;$

$750 - 50(9.81) = 50[\alpha(0.6)]$

$\alpha = 8.65\ \text{rad/s}^2$ *Ans.*

$\overset{+}{\rightarrow} \Sigma F_n = m(a_G)_n;$

$F_{AB} + D_x = 50(21.6)$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$D_x(0.4) + 750(0.1) - F_{AB}(0.4) = 0$ (2)

$D_x = 446.25\ \text{N} = 446\ \text{N}$ *Ans.*

$F_{AB} = 633.75\ \text{N} = 634\ \text{N}$ *Ans.*

F17–7. $I_O = mk_O^2 = 100(0.5^2) = 25\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha; \quad -100(0.6) = -25\alpha$

$\alpha = 2.4\text{ rad/s}^2$

$\omega = \omega_0 + \alpha_c t$

$\omega = 0 + 2.4(3) = 7.2\text{ rad/s}$ *Ans.*

F17–8. $I_O = \frac{1}{2}mr^2 = \frac{1}{2}(50)(0.3^2) = 2.25\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha;$

$-9t = -2.25\alpha \quad \alpha = (4t)\text{ rad/s}^2$

$d\omega = \alpha\, dt$

$$\int_0^\omega d\omega = \int_0^t 4t\, dt$$

$\omega = (2t^2)\text{ rad/s}$

$\omega = 2(4^2) = 32\text{ rad/s}$ *Ans.*

F17–9. $(a_G)_t = \alpha r_G = \alpha(0.15)$

$(a_G)_n = \omega^2 r_G = 6^2(0.15) = 5.4\text{ m/s}^2$

$I_O = I_G + md^2 = \frac{1}{12}(30)(0.9^2) + 30(0.15^2)$

$= 2.7\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha; \quad 60 - 30(9.81)(0.15) = 2.7\alpha$

$\alpha = 5.872\text{ rad/s}^2 = 5.87\text{ rad/s}^2$ *Ans.*

$\overset{+}{\leftarrow}\Sigma F_n = m(a_G)_n; \quad O_n = 30(5.4) = 162\text{ N}$ *Ans.*

$+\uparrow\Sigma F_t = m(a_G)_t;$

$O_t - 30(9.81) = 30[5.872(0.15)]$

$O_t = 320.725\text{ N} = 321\text{ N}$ *Ans.*

F17–10. $(a_G)_t = \alpha r_G = \alpha(0.3)$

$(a_G)_n = \omega^2 r_G = 10^2(0.3) = 30\text{ m/s}^2$

$I_O = I_G + md^2 = \frac{1}{2}(30)(0.3^2) + 30(0.3^2)$

$= 4.05\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha;$

$50\left(\frac{3}{5}\right)(0.3) + 50\left(\frac{4}{5}\right)(0.3) = 4.05\alpha$

$\alpha = 5.185\text{ rad/s}^2 = 5.19\text{ rad/s}^2$ *Ans.*

$+\uparrow\Sigma F_n = m(a_G)_n;$

$O_n + 50\left(\frac{3}{5}\right) - 30(9.81) = 30(30)$

$O_n = 1164.3\text{ N} = 1.16\text{kN}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_t = m(a_G)_t;$

$O_t + 50\left(\frac{4}{5}\right) = 30[5.185(0.3)]$

$O_t = 6.67\text{ N}$ *Ans.*

F17–11. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(15\text{ kg})(0.9\text{ m})^2 = 1.0125\text{ kg}\cdot\text{m}^2$

$(a_G)_n = \omega^2 r_G = 0$

$(a_G)_t = \alpha(0.15\text{ m})$

$I_O = I_G + md_{OG}^2$

$= 1.0125\text{ kg}\cdot\text{m}^2 + 15\text{ kg}(0.15\text{ m})^2$

$= 1.35\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha;$

$[15(9.81)\text{ N}](0.15\text{ m}) = (1.35\text{ kg}\cdot\text{m}^2)\alpha$

$\alpha = 16.35\text{ rad/s}^2$ *Ans.*

$+\downarrow\Sigma F_t = m(a_G)_t; \quad -O_t + 15(9.81)\text{N}$

$= (15\text{ kg})[16.35\text{ rad/s}^2(0.15\text{ m})]$

$O_t = 110.36\text{ N} = 110\text{ N}$ *Ans.*

$\overset{+}{\rightarrow}\Sigma F_n = m(a_G)_n; \quad O_n = 0$ *Ans.*

F17–12. $(a_G)_t = \alpha r_G = \alpha(0.45)$

$(a_G)_n = \omega^2 r_G = 6^2(0.45) = 16.2\text{ m/s}^2$

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}(30)(0.9^2) = 8.1\text{ kg}\cdot\text{m}^2$

$\zeta+\Sigma M_O = I_O\alpha;$

$300\left(\frac{4}{5}\right)(0.6) - 30(9.81)(0.45) = 8.1\alpha$

$\alpha = 1.428\text{ rad/s}^2 = 1.43\text{ rad/s}^2$ *Ans.*

$\overset{+}{\leftarrow}\Sigma F_n = m(a_G)_n; \quad O_n + 300\left(\frac{3}{5}\right) = 30(16.2)$

$O_n = 306\text{ N}$ *Ans.*

$+\uparrow\Sigma F_t = m(a_G)_t; \quad O_t + 300\left(\frac{4}{5}\right) - 30(9.81)$

$= 30[1.428(0.45)]$

$O_t = 73.58\text{ N} = 73.6\text{ N}$ *Ans.*

F17–13. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(60)(3^2) = 45\text{ kg}\cdot\text{m}^2$

$+\uparrow\Sigma F_y = m(a_G)_y;$

$80 - 20 = 60a_G \quad a_G = 1\text{ m/s}^2\uparrow$

$\zeta+\Sigma M_G = I_G\alpha; \quad 80(1) + 20(0.75) = 45\alpha$

$\alpha = 2.11\text{ rad/s}^2$ *Ans.*

F17–14. $\zeta+\Sigma M_A = (\mathcal{M}_k)_A;$

$-200(0.3) = -100a_G(0.3) - 4.5\alpha$

$30a_G + 4.5\alpha = 60 \quad (1)$

$a_G = \alpha r = \alpha(0.3) \quad (2)$

$\alpha = 4.44\text{ rad/s}^2 \quad a_G = 1.33\text{ m/s}^2\rightarrow$ *Ans.*

F17–15. $+\uparrow\Sigma F_y = m(a_G)_y;$

$N - 20(9.81) = 0 \quad N = 196.2\text{ N}$

$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x; \quad 0.5(196.2) = 20a_O$

$a_O = 4.905\text{ m/s}^2\rightarrow$ *Ans.*

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$0.5(196.2)(0.4) - 100 = -1.8\alpha$

$\alpha = 33.8 \text{ rad/s}^2$ *Ans.*

F17–16. Sphere $I_G = \frac{2}{5}(20)(0.15)^2 = 0.18 \text{ kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_{IC} = (\mathcal{M}_k)_{IC};$

$20(9.81)\sin 30°(0.15) = 0.18\alpha + (20a_G)(0.15)$

$0.18\alpha + 3a_G = 14.715$

$a_G = \alpha r = \alpha(0.15)$

$\alpha = 23.36 \text{ rad/s}^2 = 23.4 \text{ rad/s}^2$ *Ans.*

$a_G = 3.504 \text{ m/s}^2 = 3.50 \text{ m/s}^2$ *Ans.*

F17–17. $+\uparrow \Sigma F_y = m(a_G)_y;$

$N - 200(9.81) = 0 \quad N = 1962 \text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;$

$T - 0.2(1962) = 200a_G \qquad (1)$

$\circlearrowleft + \Sigma M_A = (\mathcal{M}_k)_A; \quad 450 - 0.2(1962)(1)$

$= 18\alpha + 200a_G(0.4) \qquad (2)$

$(a_A)_t = 0 \quad a_A = (a_A)_n$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$a_G\mathbf{i} = -a_A\mathbf{j} + \alpha\mathbf{k} \times (-0.4\mathbf{j}) - \omega^2(-0.4\mathbf{j})$

$a_G\mathbf{i} = 0.4\alpha\mathbf{i} + (0.4\omega^2 - a_A)\mathbf{j}$

$a_G = 0.4\alpha \qquad (3)$

Solving Eqs. (1), (2), and (3),

$\alpha = 1.15 \text{ rad/s}^2 \quad a_G = 0.461 \text{ m/s}^2$

$T = 485 \text{ N}$ *Ans.*

F17–18. $\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x; \quad 0 = 12(a_G)_x \quad (a_G)_x = 0$

$\circlearrowleft + \Sigma M_A = (\mathcal{M}_k)_A$

$-12(9.81)(0.3) = 12(a_G)_y(0.3) - \frac{1}{12}(12)(0.6)^2\alpha$

$0.36\alpha - 3.6(a_G)_y = 35.316 \qquad (1)$

$\omega = 0$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$(a_G)_y\,\mathbf{j} = a_A\mathbf{i} + (-\alpha\mathbf{k}) \times (0.3\mathbf{i}) - \mathbf{0}$

$(a_G)_y\,\mathbf{j} = (a_A)\mathbf{i} - 0.3\,\mathbf{j}$

$a_A = 0$ *Ans.*

$(a_G)_y = -0.3\alpha \qquad (2)$

Solving Eqs. (1) and (2)

$\alpha = 24.5 \text{ rad/s}^2$

$(a_G)_y = -7.36 \text{ m/s}^2 = 7.36 \text{ m/s}^2\downarrow$ *Ans.*

Chapter 18

F18–1. $I_O = mk_O^2 = 80(0.4^2) = 12.8 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(12.8)\omega^2 = 6.4\omega^2$

$s = \theta r = 20(2\pi)(0.6) = 24\pi \text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 50(24\pi) = 6.4\omega^2$

$\omega = 24.3 \text{ rad/s}$ *Ans.*

F18–2. $T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$

$= \frac{1}{2}\left(\frac{50}{32.2}\text{ slug}\right)(2.5\omega_2)^2$

$+ \frac{1}{2}\left[\frac{1}{12}\left(\frac{50}{32.2}\text{ slug}\right)(5 \text{ ft})^2\right]\omega_2^2$

$T_2 = 6.4700\omega_2^2$

Or,

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{50}{32.2}\text{ slug}\right)(5 \text{ ft})^2$

$= 12.9400 \text{ slug}\cdot\text{ft}^2$

So that

$T_2 = \frac{1}{2}I_O\omega_2^2 = \frac{1}{2}(12.9400 \text{ slug}\cdot\text{ft}^2)\omega_2^2$

$= 6.4700\omega_2^2$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + [-Wy_G + M\theta] = T_2$

$0 + \left[-(50 \text{ lb})(2.5 \text{ ft}) + (100 \text{ lb}\cdot\text{ft})\left(\frac{\pi}{2}\right)\right]$

$= 6.4700{\omega_2}^2$

$\omega_2 = 2.23 \text{ rad/s}$ *Ans.*

F18–3. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(2.5)$

$I_G = \frac{1}{12}ml^2 = \frac{1}{12}(50)(5^2) = 104.17 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$

$= \frac{1}{2}(50)[\omega_2(2.5)]^2 + \frac{1}{2}(104.17)\omega_2^2 = 208.33\omega_2^2$

$U_P = Ps_P = 600(3) = 1800 \text{ J}$

$U_W = -Wh = -50(9.81)(2.5 - 2) = -245.25 \text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 1800 + (-245.25) = 208.33\omega_2^2$

$\omega_2 = 2.732 \text{ rad/s} = 2.73 \text{ rad/s}$ *Ans.*

F18–4. $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(50\text{ kg})(0.4\omega)^2 + \frac{1}{2}\left[50\text{ kg}(0.3\text{ m})^2\right]\omega^2$

$= 6.25\omega^2\text{ J}$

Or,

$T = \frac{1}{2} I_{IC}\omega^2$

$= \frac{1}{2}\left[50\text{ kg}(0.3\text{ m})^2 + 50\text{ kg}(0.4\text{ m})^2\right]\omega^2$

$= 6.25\omega^2\text{ J}$

$s_G = \theta r = 10(2\pi\text{ rad})(0.4\text{ m}) = 8\pi\text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + P\cos 30^\circ\, s_G = T_2$

$0 + (50\text{ N})\cos 30^\circ(8\pi\text{ m}) = 6.25\omega^2\text{ J}$

$\omega = 13.2\text{ rad/s}$ *Ans.*

F18–5. $I_G = \frac{1}{12} m l^2 = \frac{1}{12}(30)(3^2) = 22.5\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(30)[\omega(0.5)]^2 + \frac{1}{2}(22.5)\omega^2 = 15\omega^2$

Or,

$I_O = I_G + md^2 = \frac{1}{12}(30)(3^2) + 30(0.5^2)$

$= 30\text{ kg}\cdot\text{m}^2$

$T_2 = \frac{1}{2} I_O\omega^2 = \frac{1}{2}(30)\omega^2 = 15\omega^2$

$s_1 = \theta r_1 = 8\pi(0.5) = 4\pi\text{ m}$

$s_2 = \theta r_2 = 8\pi(1.5) = 12\pi\text{ m}$

$U_{P_1} = P_1 s_1 = 30(4\pi) = 120\pi\text{ J}$

$U_{P_2} = P_2 s_2 = 20(12\pi) = 240\pi\text{ J}$

$U_M = M\theta = 20[4(2\pi)] = 160\pi\text{ J}$

$U_W =$ (0 bar returns to same position)

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 120\pi + 240\pi + 160\pi = 15\omega^2$

$\omega = 10.44\text{ rad/s} = 10.4\text{ rad/s}$ *Ans.*

F18–6. $v_G = \omega r = \omega(0.4)$

$I_G = m k_G^2 = 20(0.3^2) = 1.8\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G\omega^2$

$= \frac{1}{2}(20)[\omega(0.4)]^2 + \frac{1}{2}(1.8)\omega^2$

$= 2.5\omega^2$

$U_M = M\theta = M\left(\frac{s_O}{r}\right) = 50\left(\frac{20}{0.4}\right) = 2500\text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 2500 = 2.5\omega^2$

$\omega = 31.62\text{ rad/s} = 31.6\text{ rad/s}$ *Ans.*

F18–7. $v_G = \omega r = \omega(0.3)$

$I_G = \frac{1}{2} m r^2 = \frac{1}{2}(30)(0.3^2) = 1.35\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_G)_2^2 + \frac{1}{2} I_G\omega_2^2$

$= \frac{1}{2}(30)[\omega_2(0.3)]^2 + \frac{1}{2}(1.35)\omega_2^2 = 2.025\omega_2^2$

$(V_g)_1 = W y_1 = 0$

$(V_g)_2 = -W y_2 = -30(9.81)(0.3) = -88.29\text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 2.025\omega_2^2 + (-88.29)$

$\omega_2 = 6.603\text{ rad/s} = 6.60\text{ rad/s}$ *Ans.*

F18–8. $v_O = \omega r_{O/IC} = \omega(0.2)$

$I_O = m k_O^2 = 50(0.3^2) = 4.5\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_O)_2^2 + \frac{1}{2} I_O\omega_2^2$

$= \frac{1}{2}(50)[\omega_2(0.2)]^2 + \frac{1}{2}(4.5)\omega_2^2$

$= 3.25\omega_2^2$

$(V_g)_1 = W y_1 = 0$

$(V_g)_2 = -W y_2 = -50(9.81)(6\sin 30^\circ)$

$= -1471.5\text{J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 3.25\omega_2^2 + (-1471.5)$

$\omega_2 = 21.28\text{ rad/s} = 21.3\text{ rad/s}$ *Ans.*

F18–9. $v_G = \omega r_G = \omega(1.5)$

$I_G = \frac{1}{12}(60)(3^2) = 45\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_G)_2^2 + \frac{1}{2} I_G\omega_2^2$

$= \frac{1}{2}(60)[\omega_2(1.5)]^2 + \frac{1}{2}(45)\omega_2^2$

$= 90\omega_2^2$

Or,

$T_2 = \frac{1}{2} I_O\omega_2^2 = \frac{1}{2}\left[45 + 60(1.5^2)\right]\omega_2^2 = 90\omega_2^2$

$(V_g)_1 = W y_1 = 0$

$(V_g)_2 = -W y_2 = -60(9.81)(1.5\sin 45^\circ)$

$= -624.30\text{ J}$

$(V_e)_1 = \frac{1}{2} k s_1^2 = 0$

$(V_e)_2 = \frac{1}{2} k s_2^2 = \frac{1}{2}(150)(3\sin 45^\circ)^2 = 337.5\text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 90\omega_2^2 + [-624.30 + 337.5]$

$\omega_2 = 1.785\text{ rad/s} = 1.79\text{ rad/s}$ *Ans.*

F18–10. $v_G = \omega r_G = \omega(0.75)$

$I_G = \frac{1}{12}(30)(1.5^2) = 5.625\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$
$= \frac{1}{2}(30)[\omega(0.75)]^2 + \frac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$

Or,

$T_2 = \frac{1}{2}I_O\omega_2^2 = \frac{1}{2}\left[5.625 + 30\left(0.75^2\right)\right]\omega_2^2$
$= 11.25\omega_2^2$

$\left(V_g\right)_1 = Wy_1 = 0$

$\left(V_g\right)_2 = -Wy_2 = -30(9.81)(0.75)$
$= -220.725 \text{ J}$

$(V_e)_1 = \frac{1}{2}ks_1^2 = 0$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(80)\left(\sqrt{2^2 + 1.5^2} - 0.5\right)^2 = 160 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 11.25\omega_2^2 + (-220.725 + 160)$

$\omega_2 = 2.323 \text{ rad/s} = 2.32 \text{ rad/s}$ *Ans.*

F18–11. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(0.75)$

$I_G = \frac{1}{12}(30)\left(1.5^2\right) = 5.625 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$
$= \frac{1}{2}(30)[\omega_2(0.75)]^2 + \frac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$

$\left(V_g\right)_1 = Wy_1 = 30(9.81)(0.75 \sin 45°) = 156.08 \text{ J}$

$\left(V_g\right)_2 = -Wy_2 = 0$

$\left(V_e\right)_1 = \frac{1}{2}ks_1^2 = 0$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(300)(1.5 - 1.5\cos 45°)^2$
$= 28.95 \text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + (156.08 + 0) = 11.25\omega_2^2 + (0 + 28.95)$

$\omega_2 = 3.362 \text{ rad/s} = 3.36 \text{ rad/s}$ *Ans.*

F18–12. $(V_g)_1 = -Wy_1 = -[20(9.81) \text{ N}](1 \text{ m}) = -196.2 \text{ J}$

$(V_g)_2 = 0$

$(V_e)_1 = \frac{1}{2}ks_1^2$
$= \frac{1}{2}(100 \text{ N/m})\left(\sqrt{(3 \text{ m})^2 + (2 \text{ m})^2} - 0.5 \text{ m}\right)^2$
$= 482.22 \text{ J}$

$(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(100 \text{ N/m})(1 \text{ m} - 0.5 \text{ m})^2$
$= 12.5 \text{ J}$

$T_1 = 0$

$T_2 = \frac{1}{2}I_A\omega^2 = \frac{1}{2}\left[\frac{1}{3}(20 \text{ kg})(2 \text{ m})^2\right]\omega^2$
$= 13.3333\omega^2$

$T_1 + V_1 = T_2 + V_2$

$0 + [-196.2 \text{ J} + 482.22 \text{ J}]$
$= 13.3333\omega_2^2 + [0 + 12.5 \text{ J}]$

$\omega_2 = 4.53 \text{ rad/s}$ *Ans.*

Chapter 19

F19–1. $\circlearrowleft + I_O\omega_1 + \Sigma\int_{t_1}^{t_2} M_O\,dt = I_O\omega_2$

$0 + \int_0^{4\text{ s}} 3t^2\,dt = \left[60\left(0.3\right)^2\right]\omega_2$

$\omega_2 = 11.85 \text{ rad/s} = 11.9 \text{ rad/s}$ *Ans.*

F19–2. $\circlearrowleft + (H_A)_1 + \Sigma\int_{t_1}^{t_2} M_A dt = (H_A)_2$

$0 + 300(6) = 300\left(0.4^2\right)\omega_2 + 300[\omega(0.6)](0.6)$

$\omega_2 = 11.54 \text{ rad/s} = 11.5 \text{ rad/s}$ *Ans.*

$\overset{+}{\rightarrow}\quad m(v_1)_x + \Sigma\int_{t_1}^{t_2} F_x dt = m(v_2)_x$

$0 + F_f(6) = 300[11.54(0.6)]$

$F_f = 346 \text{ N}$ *Ans.*

F19–3. $v_A = \omega_A r_{A/IC} = \omega_A(0.15)$

$\circlearrowleft + \Sigma M_O = 0;\quad 9 - A_t(0.45) = 0\quad A_t = 20 \text{ N}$

$\circlearrowleft + (H_C)_1 + \Sigma\int_{t_1}^{t_2} M_C\,dt = (H_C)_2$

$0 + [20(5)](0.15)$
$= 10[\omega_A(0.15)](0.15)$
$+ \left[10\left(0.1^2\right)\right]\omega_A$

$\omega_A = 46.2 \text{ rad/s}$ *Ans.*

F19–4. $I_A = mk_A^2 = 10\left(0.08^2\right) = 0.064 \text{ kg}\cdot\text{m}^2$

$I_B = mk_B^2 = 50\left(0.15^2\right) = 1.125 \text{ kg}\cdot\text{m}^2$

$\omega_A = \left(\frac{r_B}{r_A}\right)\omega_B = \left(\frac{0.2}{0.1}\right)\omega_B = 2\omega_B$

$\circlearrowleft + I_A(\omega_A)_1 + \Sigma\int_{t_1}^{t_2} M_A\,dt = I_A(\omega_A)_2$

$0 + 10(5) - \int_0^{5\text{ s}} F(0.1)dt = 0.064[2(\omega_B)_2]$

$\int_0^{5\text{ s}} F dt = 500 - 1.28(\omega_B)_2 \quad (1)$

$\circlearrowleft + I_B(\omega_B)_1 + \Sigma\int_{t_1}^{t_2} M_B\,dt = I_B(\omega_B)_2$

$0 + \int_0^{5\text{ s}} F(0.2)dt = 1.125(\omega_B)_2$

$\int_0^{5\text{ s}} F dt = 5.625(\omega_B)_2 \quad (2)$

Equating Eqs. (1) and (2),

$500 - 1.28(\omega_B)_2 = 5.625(\omega_B)_2$

$(\omega_B)_2 = 72.41 \text{ rad/s} = 72.4 \text{ rad/s}$ *Ans.*

F19–5. $(\xrightarrow{+})\quad m\left[(v_O)_x\right]_1 + \Sigma\int F_x\,dt = m\left[(v_O)_x\right]_2$

$$0 + (150\text{ N})(3\text{ s}) + F_A(3\text{ s}) = (50\text{ kg})(0.3\omega_2)$$

$$\circlearrowleft + I_G\omega_1 + \Sigma\int M_G\,dt = I_G\omega_2$$

$$0 + (150\text{ N})(0.2\text{ m})(3\text{ s}) - F_A(0.3\text{ m})(3\text{ s}) = [(50\text{ kg})(0.175\text{ m})^2]\,\omega_2$$

$$\omega_2 = 37.3\text{ rad/s}$$ *Ans.*

$$F_A = 36.53\text{ N}$$

Also,

$$I_{IC}\omega_1 + \Sigma\int M_{IC}\,dt = I_{IC}\omega_2$$

$$0 + [(150\text{ N})(0.2 + 0.3)\text{ m}](3\text{ s}) = [(50\text{ kg})(0.175\text{ m})^2 + (50\text{ kg})(0.3\text{ m})^2]\omega_2$$

$$\omega_2 = 37.3\text{ rad/s}$$ *Ans.*

F19–6. $(+\uparrow)\; m\left[(v_G)_1\right]_y + \Sigma\int F_y\,dt = m\left[(v_G)_2\right]_y$

$$0 + N_A(3\text{ s}) - (150\text{ lb})(3\text{ s}) = 0$$

$$N_A = 150\text{ lb}$$

$$\circlearrowleft + (H_{IC})_1 + \Sigma\int M_{IC}\,dt = (H_{IC})_2$$

$$0 + (25\text{ lb}\cdot\text{ft})(3\text{ s}) - [0.15(150\text{ lb})(3\text{ s})](0.5\text{ ft}) = \left[\tfrac{150}{32.2}\text{ slug}(1.25\text{ ft})^2\right]\omega_2 + \left(\tfrac{150}{32.2}\text{ slug}\right)\left[\omega_2(1\text{ ft})\right](1\text{ ft})$$

$$\omega_2 = 3.46\text{ rad/s}$$ *Ans.*

Preliminary Problems Dynamics Solutions

Chapter 12

P12–1. a) $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(2t^3) = 6t^2\Big|_{t=2\text{ s}} = 24 \text{ m/s}$

b) $a\,ds = v\,dv, \quad v = 5s, \quad dv = 5\,ds$

$a\,ds = (5s)\,5\,ds$

$a = 25s\Big|_{s=1\text{ m}} = 25 \text{ m/s}^2$

c) $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(4t + 5) = 4 \text{ m/s}^2$

d) $v = v_0 + a_c t$

$v = 0 + 2(2) = 4 \text{ m/s}$

e) $v^2 = v_0^2 + 2a_c(s - s_0)$

$v^2 = (3)^2 + 2(2)(4 - 0)$

$v = 5 \text{ m/s}$

f) $a\,ds = v\,dv$

$\displaystyle\int_{s_1}^{s_2} s\,ds = \int_0^v v\,dv$

$s^2\Big|_4^5 = v^2\Big|_0^v$

$25 - 16 = v^2$

$v = 3 \text{ m/s}$

g) $s = s_0 + v_0 t + \dfrac{1}{2} a_c t^2$

$s = 2 + 2(3) + \dfrac{1}{2}(4)(3)^2 = 26 \text{ m}$

h) $dv = a\,dt$

$\displaystyle\int_0^v dv = \int_0^1 (8t^2)\,dt$

$v = 2.67t^3\Big|_0^1 = 2.67 \text{ m/s}$

i) $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(3t^2 + 2) = 6t\Big|_{t=2\text{ s}} = 12 \text{ m/s}$

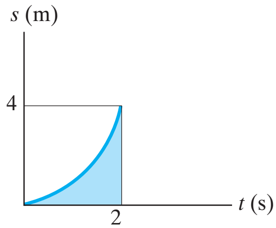
j) $v_{avg} = \dfrac{\Delta s}{\Delta t} = \dfrac{6\text{ m} - (-1\text{ m})}{10\text{ s} - 0} = 0.7 \text{ m/s} \rightarrow$

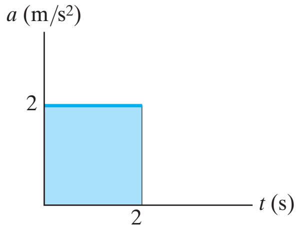
$(v_{sp})_{avg} = \dfrac{s_T}{\Delta t} = \dfrac{7\text{ m} + 14\text{ m}}{10\text{ s} - 0} = 2.1 \text{ m/s}$

P12–2. a) $v = 2t$

$s = t^2$

$a = 2$

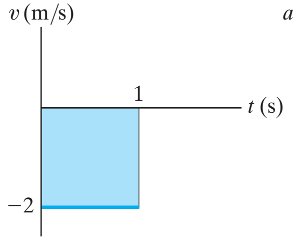


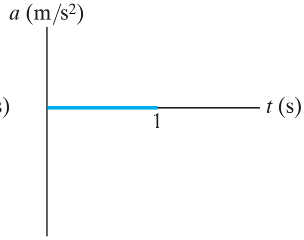


b) $s = -2t + 2$

$v = -2$

$a = 0$

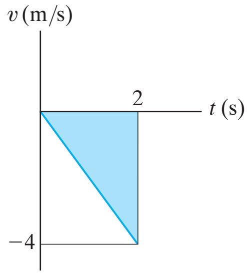


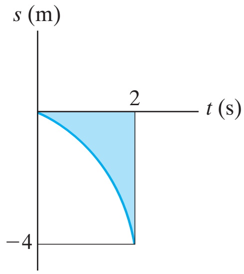


c) $a = -2$

$v = -2t$

$s = -t^2$





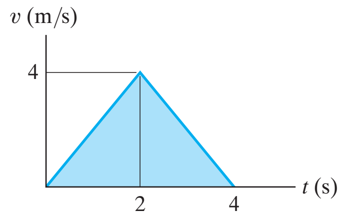
d)

$\Delta s = \displaystyle\int_0^3 v\,dt = \text{Area} = \frac{1}{2}(2)(2) + 2(3 - 2) = 4 \text{ m}$

$s - 0 = 4 \text{ m}, \qquad s = 4 \text{ m}$

$a = \dfrac{dv}{dt} = \text{slope at } t = 3 \text{ s}, a = 0$

e) For $a = 2$,
$v = 2t$
When $t = 2$ s, $v = 4$ m/s.
For $a = -2$,
$$\int_4^v dv = \int_2^t -2\,dt$$
$$v - 4 = -2t + 4$$
$$v = -2t + 8$$



f) $$\int_1^v v\,dv = \int_0^2 a\,ds = \text{Area}$$
$$\frac{1}{2}v^2 - \frac{1}{2}(1)^2 = \frac{1}{2}(2)(4)$$
$$v = 3 \text{ m/s}$$

g) $v\,dv = a\,ds$ At $s = 1$ m, $v = 2$ m/s.
$$a = v\frac{dv}{ds} = v(\text{slope}) = 2(-2) = -4 \text{ m/s}$$

P12–3. a) $y = 4x^2$
$$\dot{y} = 8x\dot{x}$$
$$\ddot{y} = (8\dot{x})\dot{x} + 8x(\ddot{x})$$

b) $y = 3e^x$
$$\dot{y} = 3e^x\dot{x}$$
$$\ddot{y} = (3e^x\dot{x})\dot{x} + 3e^x(\ddot{x})$$

c) $y = 6 \sin x$
$$\dot{y} = (6\cos x)\dot{x}$$
$$\ddot{y} = [(-6\sin x)\dot{x}]\,\dot{x} + (6\cos x)(\ddot{x})$$

P12–4. $y_A, t_{AB}, (v_B)_y$
$$20 = 0 + 40t_{AB}$$
$$0 = y_A + 0 + \frac{1}{2}(-9.81)(t_{AB})^2$$
$$(v_B)_y^2 = 0^2 + 2(-9.81)(0 - y_A)$$

P12–5. $x_B, t_{AB}, (v_B)_y$
$$x_B = 0 + (10\cos 30°)(t_{AB})$$
$$0 = 8 + (10\sin 30°)t_{AB} + \frac{1}{2}(-9.81)t_{AB}^2$$
$$(v_B)_y^2 = 0^2 + 2(-9.81)(0 - 8)$$

P12–6. $x_B, y_B, (v_B)_y$
$$x_B = 0 + (60\cos 20°)(5)$$
$$y_B = 0 + (60\sin 20°)(5) + \frac{1}{2}(-9.81)(5)^2$$
$$(v_B)_y = 60\sin 20° + (-9.81)(5)$$

P12–7. a) $a_t = \dot{v} = 3 \text{ m/s}^2$
$$a_n = \frac{v^2}{\rho} = \frac{(2)^2}{1} = 4 \text{ m/s}^2$$
$$a = \sqrt{(3)^2 + (4)^2} = 5 \text{ m/s}^2$$

b) $a_t = \dot{v} = 4 \text{ m/s}^2$
$$v^2 = v_0^2 + 2a_c(s - s_0)$$
$$v^2 = 0 + 2(4)(2 - 0)$$
$$v = 4 \text{ m/s}$$
$$a_n = \frac{v^2}{\rho} = \frac{(4)^2}{2} = 8 \text{ m/s}^2$$

c) $a_t = 0$
$$\rho = \left.\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right|_{x=0} = \frac{1 + 0}{4} = \frac{1}{4}$$
$$a_n = \frac{v^2}{\rho} = \frac{(2)^2}{\frac{1}{4}} = 16 \text{ m/s}^2$$
$$a = \sqrt{(0)^2 + (16)^2} = 16 \text{ m/s}^2$$

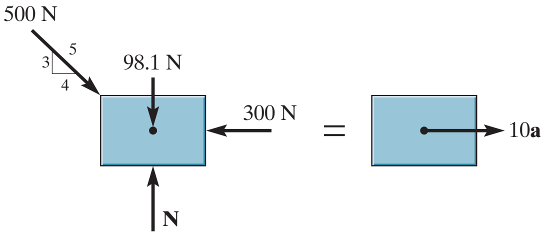
d) $a_t\,ds = v\,dv$
$$a_t\,ds = (4s + 1)(4\,ds)$$
$$a_t = (16s + 4)\big|_{s=0} = 4 \text{ m/s}^2$$
$$a_n = \frac{v^2}{\rho} = \frac{(4(0) + 1)^2}{2} = 0.5 \text{ m/s}^2$$

e) $a_t\,ds = v\,dv$
$$\int_0^s 2s\,ds = \int_1^v v\,dv$$
$$s^2 = \frac{1}{2}(v^2 - 1)$$
$$v = \sqrt{2s^2 + 1}\bigg|_{s = 2\text{ m}} = 3 \text{ m/s}$$
$$a_t = \dot{v} = 2(2) = 4 \text{ m/s}^2$$
$$a_n = \frac{v^2}{\rho} = \frac{(3)^2}{3} = 3 \text{ m/s}^2$$
$$a = \sqrt{(4)^2 + (3)^2} = 5 \text{ m/s}^2$$

f) $a_t = \dot{v} = 8t\Big|_{t=1} = 8 \text{ m/s}^2$
$$a_n = \frac{v^2}{\rho} = \frac{(4(1)^2 + 2)^2}{6} = 6 \text{ m/s}^2$$
$$a = \sqrt{(8)^2 + (6)^2} = 10 \text{ m/s}^2$$

Chapter 13

P13–1. a)

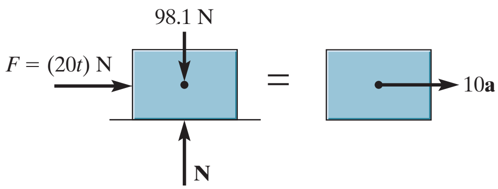


$$\xrightarrow{+} \Sigma F_x = ma_x; \quad \left(\frac{4}{5}\right)(500 \text{ N}) - 300 \text{ N} = 10a$$

$$a = 10 \text{ m/s}^2$$

$$\xrightarrow{+} v = v_0 + a_c t; \quad v = 0 + 10(2) = 20 \text{ m/s}$$

b)



$$\xrightarrow{+} \Sigma F_x = ma_x; \quad 20t = 10a$$

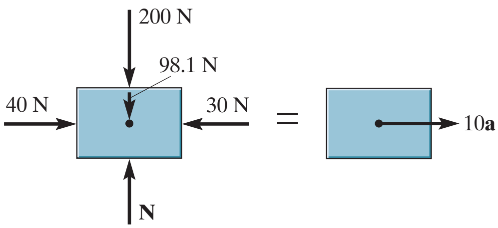
$$a = 2t$$

$$dv = a\,dt; \quad \int_0^v dv = \int_0^2 2t\,dt$$

$$v = 4 \text{ m/s}$$

P13–2.

a)



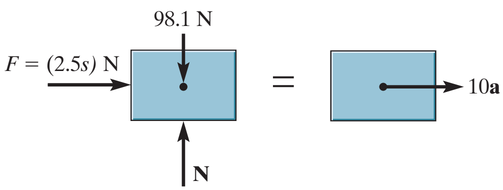
$$\xrightarrow{+} \Sigma F_x = ma_x; \quad 40 \text{ N} - 30 \text{ N} = 10a$$

$$a = 1 \text{ m/s}^2$$

$$\xrightarrow{+} v^2 = v_0^2 + 2a_c(s - s_0); \quad v^2 = (3)^2 + 2(1)(8 - 0)$$

$$v = 5 \text{ m/s}$$

b)



$$\xrightarrow{+} \Sigma F_x = ma_x; \quad 2.5s = 10a$$

$$a = 2.5s$$

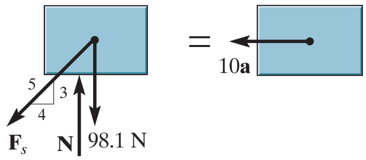
$$v\,dv = a\,ds$$

$$\int_3^v v\,dv = \int_0^8 2.5s\,ds$$

$$v^2 - (3)^2 = 2.5(8 - 0)^2$$

$$v = 13 \text{ m/s}$$

P13–3.

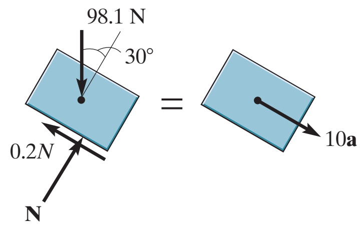


$$F_s = kx = (10 \text{ N/m})(5 \text{ m} - 1 \text{ m}) = 40 \text{ N}$$

$$\xleftarrow{+} \Sigma F_x = ma_x; \quad \frac{4}{5}(40 \text{ N}) = 10a$$

$$a = 3.2 \text{ m/s}^2$$

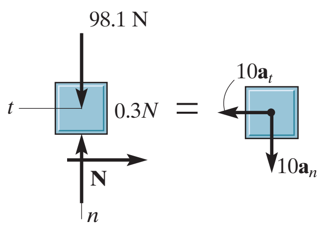
P13–4.



$$\searrow^{+} \Sigma F_x = ma_x; \quad 98.1 \sin 30° - 0.2N = 10a$$

$$^{+}\nearrow \Sigma F_y = ma_y; \quad N - 98.1 \cos 30° = 0$$

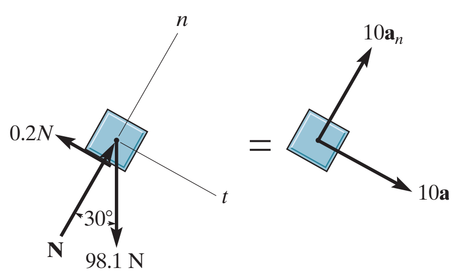
P13–5. a)



$$\xleftarrow{+} \Sigma F_t = ma_t; \quad -0.3N = 10a_t$$

$$+\downarrow \Sigma F_n = ma_n; \quad 98.1 - N = 10\left(\frac{(6)^2}{10}\right)$$

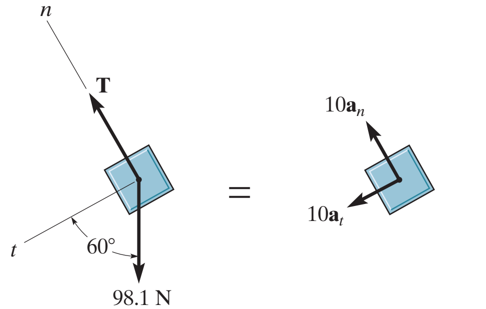
b)



$\overset{+}{\searrow} \Sigma F_t = ma_t;\ 98.1 \sin 30° - 0.2N = 10a_t$

$\overset{+}{\nearrow} \Sigma F_n = ma_n;\quad N - 98.1 \cos 30° = 10\left(\frac{(4)^2}{5}\right)$

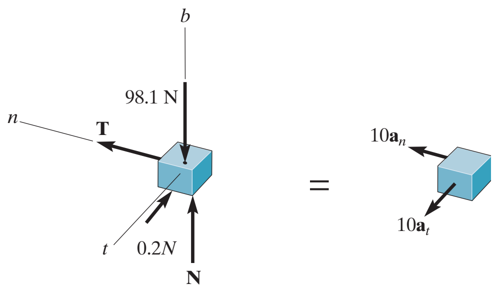
c)



$\overset{+}{\swarrow} \Sigma F_t = ma_t;\quad 98.1 \cos 60° = 10a_t$

$\overset{+}{\nwarrow} \Sigma F_n = ma_n;\quad T - 98.1 \sin 60° = 10\left(\frac{8^2}{6}\right)$

P13–6. a)

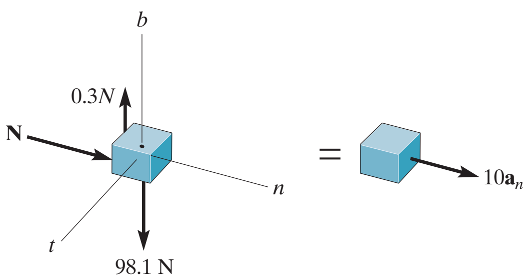


$\Sigma F_b = 0;\quad N - 98.1 = 0$

$\Sigma F_t = ma_t;\quad -0.2N = 10a_t$

$\Sigma F_n = ma_n;\quad T = 10\frac{(8)^2}{4}$

b)



$\Sigma F_b = 0;\quad 0.3N - 98.1 = 0$

$\Sigma F_t = ma_t;\quad 0 = 0$

$\Sigma F_n = ma_n;\quad N = 10\frac{v^2}{2}$

Chapter 14

P14–1. a) $U = \frac{3}{5}(500\text{ N})(2\text{ m}) = 600\text{ J}$

b) $U = 0$

c) $U = \int_0^2 6s^2\, ds = 2(2)^3 = 16\text{ J}$

d) $U = 100\text{ N}\left(\frac{3}{5}(2\text{ m})\right) = \frac{3}{5}(100\text{ N})(2\text{ m}) = 120\text{ J}$

e) $U = \frac{4}{5}(\text{Area}) = \frac{4}{5}\left[\frac{1}{2}(1)(20) + (1)(20)\right] = 24\text{ J}$

f) $U = \frac{1}{2}(10\text{ N/m})((3\text{ m})^2 - (1\text{ m}^2)) = 40\text{ J}$

g) $U = -\left(\frac{4}{5}\right)(100\text{ N})(2\text{ m}) = -160\text{ J}$

P14–2. a) $T = \frac{1}{2}(10\text{ kg})(2\text{ m/s})^2 = 20\text{ J}$

b) $T = \frac{1}{2}(10\text{ kg})(6\text{ m/s})^2 = 180\text{ J}$

P14–3. a) $V = (100\text{ N})(2\text{ m}) = 200\text{ J}$

b) $V = (100\text{ N})(3\text{ m}) = 300\text{ J}$

c) $V = 0$

P14–4. a) $V = \frac{1}{2}(10\text{ N/m})(5\text{ m} - 4\text{ m})^2 = 5\text{ J}$

b) $V = \frac{1}{2}(10\text{ N/m})(10\text{ m} - 4\text{ m})^2 = 180\text{ J}$

c) $V = \frac{1}{2}(10\text{ N/m})(5\text{ m} - 4\text{ m})^2 = 5\text{ J}$

Chapter 15

P15–1. a) $I = (100\text{ N})(2\text{ s}) = 200\text{ N}\cdot\text{s} \swarrow$

b) $I = (200\text{ N})(2\text{ s}) = 400\text{ N}\cdot\text{s}\downarrow$

c) $I = \int_0^2 6t\, dt = 3(2)^2 = 12\text{ N}\cdot\text{s} \searrow$

d) $I = \text{Area} = \frac{1}{2}(1)(20) + (2)(20) = 50\text{ N}\cdot\text{s} \nearrow$

e) $I = (80\text{ N})(2\text{ s}) = 160\text{ N}\cdot\text{s} \rightarrow$

f) $I = (60\text{ N})(2\text{ s}) = 120\text{ N}\cdot\text{s} \nearrow$

P15–2. a) $L = (10\text{ kg})(10\text{ m/s}) = 100\text{ kg}\cdot\text{m/s}\searrow$

b) $L = (10\text{ kg})(2\text{ m/s}) = 20\text{ kg}\cdot\text{m/s} \swarrow$

c) $L = (10\text{ kg})(3\text{ m/s}) = 30\text{ kg}\cdot\text{m/s} \rightarrow$

Chapter 16

P16–1. a) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 18\text{ m/s} + 2\omega \quad (60^\circ)$$

Also,

$$-v_B\mathbf{j} = -18\mathbf{j} + (-\omega\mathbf{k}) \times (-2\cos 60^\circ\mathbf{i}) - 2\sin 60^\circ\mathbf{j})$$

b) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$(v_B)_x + (v_B)_y = 4(0.5)\text{ m/s} + 4(0.5)\text{ m/s} \quad (30^\circ)$$

Also,

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 2\mathbf{i} + (-4\mathbf{k}) \times (-0.5\cos 30^\circ\mathbf{i} + \mathbf{0.5}\sin 30^\circ\mathbf{j})$$

c) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 6\text{ m/s} + \omega(5) \quad (45^\circ,\ 30^\circ)$$

Also,

$$v_B\cos 45^\circ\mathbf{i} + v_B\sin 45^\circ\mathbf{j} = 6\mathbf{i} + (\omega\mathbf{k}) \times (4\mathbf{i} - 3\mathbf{j})$$

d) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 6\text{ m/s} + \omega(3) \quad (30^\circ)$$

Also,

$$v_B\mathbf{i} = 6\cos 30^\circ\mathbf{i} + 6\sin 30^\circ\mathbf{j} + (\omega\mathbf{k}) \times (3\mathbf{i})$$

e) $v_A = 12\text{ m/s} = \omega(0.5\text{ m}) \quad \omega = 24\text{ rad/s}$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$$

$$(v_B)_x + (v_B)_y = 12\text{ m/s} + (24)(0.5)$$

Also,

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 12\mathbf{j} + (24\mathbf{k}) \times (0.5\mathbf{j})$$

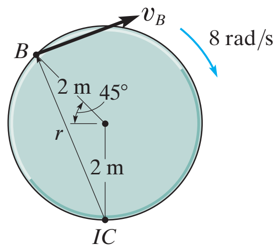
f) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 6\text{ m/s} + \omega(5) \quad (3,\ 4,\ 5)$$

Also,

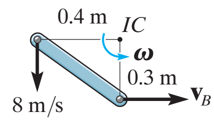
$$v_B\mathbf{i} = 6\mathbf{i} + (\omega\mathbf{k}) \times (4\mathbf{i} + 3\mathbf{j})$$

P16–2. a)

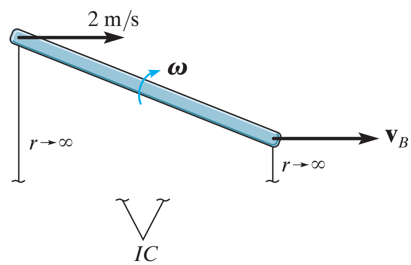


$$r = \sqrt{(2\cos 45^\circ)^2 + (2 + 2\sin 45^\circ)^2}$$

b)

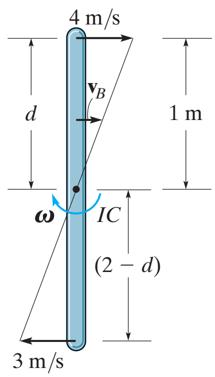


c)

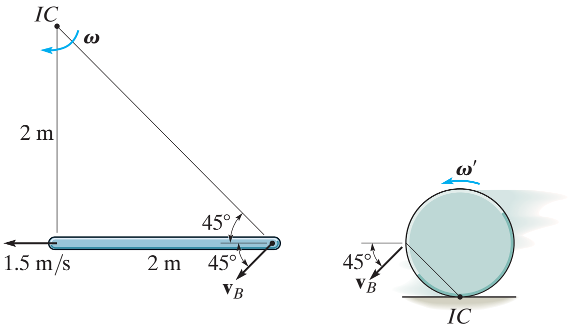


$$v_B = 2\text{ m/s},\ \omega = 0$$

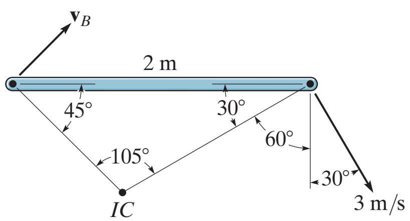
d)



e)



f)



P16–3. a)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

$$\underset{\downarrow}{a_B} = \underset{\leftarrow}{2\ \text{m/s}^2} + \underset{\uparrow}{\frac{(3)^2}{3}} + \underset{\searrow 45^\circ}{2\alpha} + \underset{45^\circ \swarrow}{(2.12)^2(2)}$$

Also,

$$-a_B\mathbf{j} = -2\mathbf{i} + 3\mathbf{j} + (-\alpha\mathbf{k}) \times (2\sin 45^\circ\mathbf{i} + 2\cos 45^\circ\mathbf{j}) - (2.12)^2(2\sin 45^\circ\mathbf{i} + 2\cos 45^\circ\mathbf{j})$$

b)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

$$\underset{\rightarrow}{(a_B)_x} + \underset{\uparrow}{(a_B)_y} = \underset{\rightarrow}{(2)(2)\ \text{m/s}^2} + \underset{\nearrow 45^\circ}{\alpha(2)} + \underset{\searrow 45^\circ}{(4)^2(2)}$$

Also,

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = 4\mathbf{i} + (-\alpha\mathbf{k}) \times (-2\cos 45^\circ\mathbf{i} + 2\sin 45^\circ\mathbf{j}) - (4)^2(-2\cos 45^\circ\mathbf{i} + 2\sin 45^\circ\mathbf{j})$$

c)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

$$\underset{\rightarrow}{(a_B)_x} + \underset{\downarrow}{(6)^2(1)} = \underset{\rightarrow}{2(2)} + \underset{\downarrow}{(3)^2(2)} + \underset{\downarrow}{\alpha(4)}$$

Also,

$$(a_B)_x\mathbf{i} - 36\mathbf{j} = 4\mathbf{i} - 18\mathbf{j} + (-\alpha\mathbf{k}) \times (4\mathbf{i})$$

d)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

$$\underset{\rightarrow}{a_B} = \underset{60^\circ \swarrow}{6} + \underset{\uparrow}{\alpha(2)} + \underset{\rightarrow}{(3)^2(2)}$$

Also,

$$a_B\mathbf{i} = -6\cos 60^\circ\mathbf{i} - 6\sin 60^\circ\mathbf{j} + (-\alpha\mathbf{k}) \times (-2\mathbf{i}) - (3)^2(-2\mathbf{i})$$

e)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

$$\underset{\leftarrow}{a_B} = \underset{\downarrow}{8(0.5)} + \underset{\rightarrow}{(4)^2(0.5)} + \underset{\nwarrow 30^\circ}{\alpha(2)} + \underset{\nearrow 30^\circ}{(1.15)^2(2)}$$

Also,

$$-a_B\mathbf{i} = -4\mathbf{j} + 8\mathbf{i} + (-\alpha\mathbf{k}) \times (-2\cos 30^\circ\mathbf{i} - 2\sin 30^\circ\mathbf{j}) - (1.15)^2(-2\cos 30^\circ\mathbf{i} - 2\sin 30^\circ\mathbf{j})$$

f)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\ (\text{pin})}$$

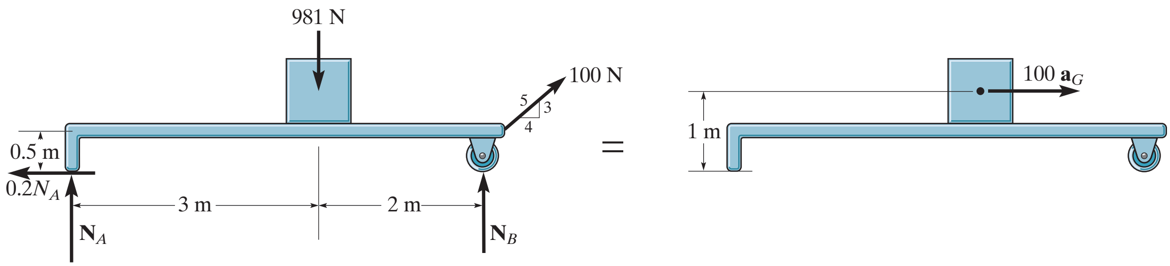
$$\underset{\rightarrow}{(a_B)_x} + \underset{\uparrow}{(a_B)_y} = \underset{\downarrow}{2(0.5)} + \underset{\rightarrow}{2(0.5)} + \underset{\downarrow}{(4)^2(0.5)}$$

Also,

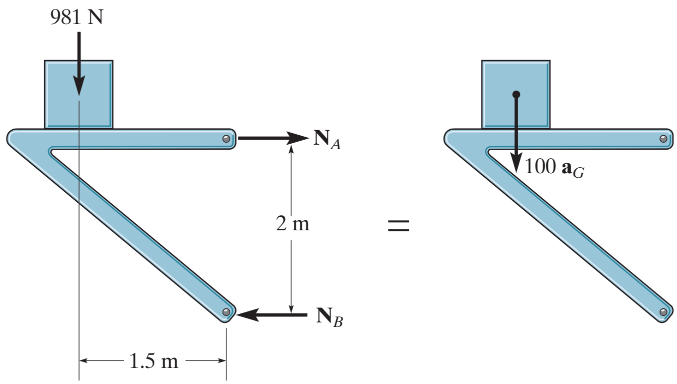
$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -1\mathbf{j} + (-2\mathbf{k}) \times (0.5\mathbf{j}) - (4)^2(0.5\mathbf{j})$$

Chapter 17

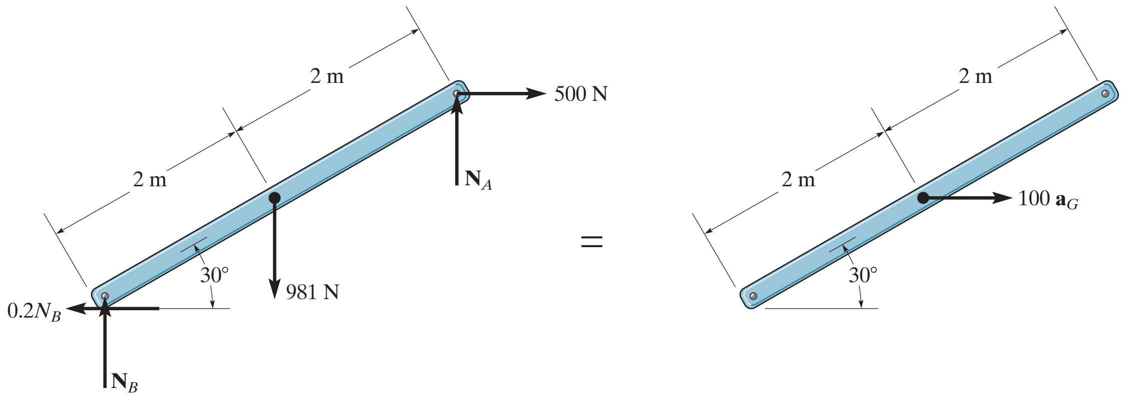
P17–1. a)



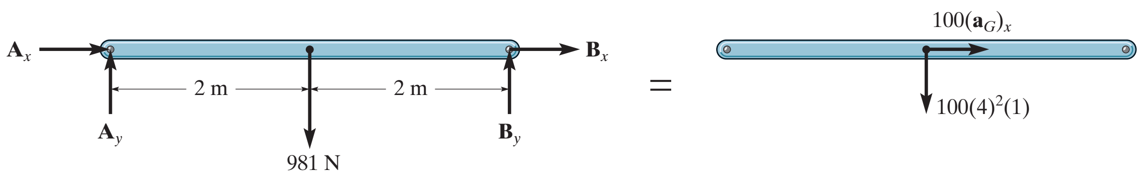
b)



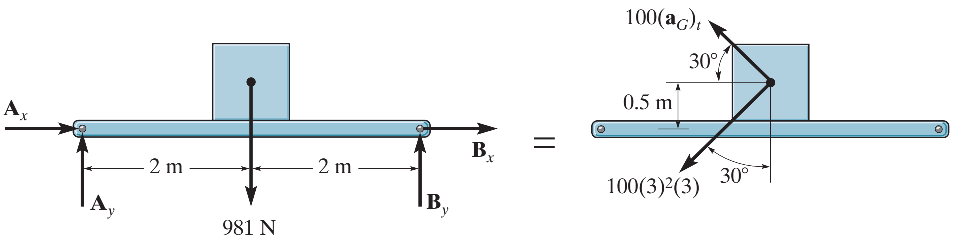
c)



d)

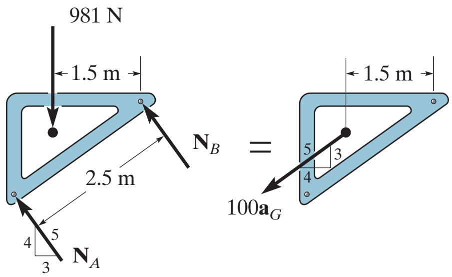


e)



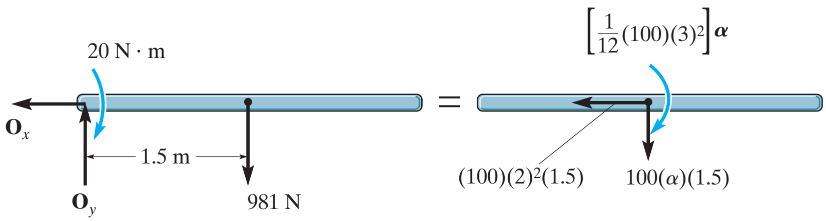
A_x
2 m
2 m
B_x
A_y
B_y
981 N
=
$100(a_G)_t$
30°
0.5 m
$100(3)^2(3)$
30°

f)



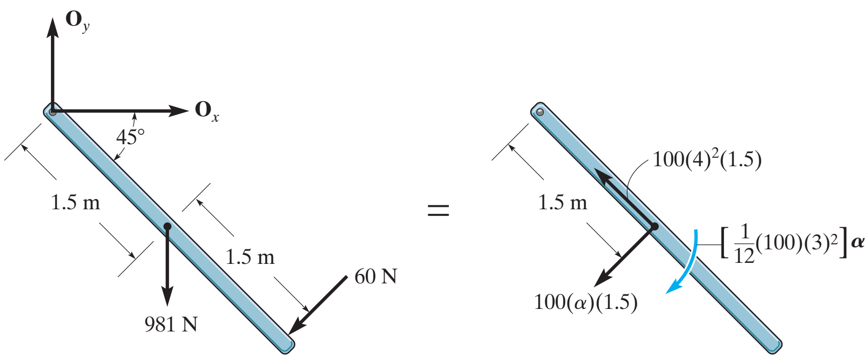
981 N
1.5 m
N_B
2.5 m
5
4
3
N_A
=
1.5 m
5
3
4
$100a_G$

P17–2. a)



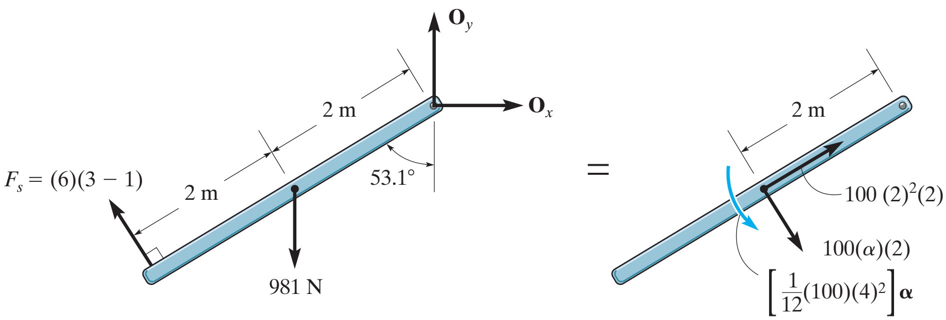
20 N · m
O_x
1.5 m
O_y
981 N
=
$\left[\frac{1}{12}(100)(3)^2\right]\alpha$
$(100)(2)^2(1.5)$
$100(\alpha)(1.5)$

b)



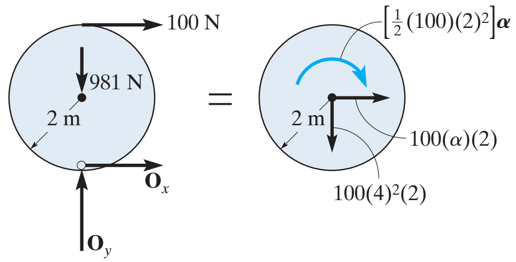
O_y
O_x
45°
1.5 m
1.5 m
60 N
981 N
=
1.5 m
$100(4)^2(1.5)$
$\left[\frac{1}{12}(100)(3)^2\right]\alpha$
$100(\alpha)(1.5)$

c)

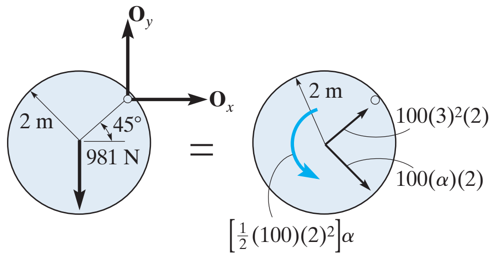


O_y
O_x
2 m
2 m
53.1°
$F_s = (6)(3 - 1)$
981 N
=
2 m
$100\ (2)^2(2)$
$100(\alpha)(2)$
$\left[\frac{1}{12}(100)(4)^2\right]\alpha$

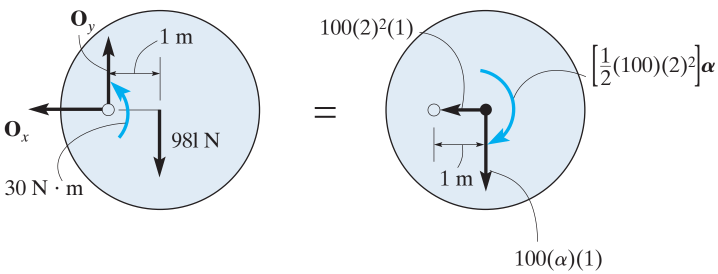
d)



e)



f)



Chapter 18

P18–1. a) $T = \frac{1}{2}\left[\frac{100(2)^2}{2}\right](3)^2 = 900 \text{ J}$

b) $T = \frac{1}{2}(100)[2(1)]^2 + \frac{1}{2}\left[\frac{1}{12}(100)(6)^2\right](2)^2$

$= 800 \text{ J}$

Also,

$$T = \frac{1}{2}\left[\frac{1}{12}(100)(6)^2 + 100(1)^2\right](2)^2 = 800 \text{ J}$$

c) $T = \frac{1}{2}(100)[2(2)]^2 + \frac{1}{2}\left[\frac{1}{2}(100)(2)^2\right](2)^2$

$= 1200 \text{ J}$

Also,

$$T = \frac{1}{2}\left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](2)^2$$

$= 1200 \text{ J}$

d) $T = \frac{1}{2}(100)[2(1.5)]^2 + \frac{1}{2}\left[\frac{1}{12}(100)(3)^2\right](2)^2$

$= 600 \text{ J}$

Also,

$$T = \frac{1}{2}\left[\frac{1}{12}(100)(3)^2 + 100(1.5)^2\right](2)^2$$

$= 600 \text{ J}$

e) $T = \frac{1}{2}(100)[4(2)]^2 + \frac{1}{2}\left[\frac{1}{2}(100)(2)^2\right](4)^2$

$= 4800 \text{ J}$

Also,

$$T = \frac{1}{2}\left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](4)^2$$

$= 4800 \text{ J}$

f) $T = \frac{1}{2}(100)[(4)(2)]^2 = 3200 \text{ J}$

Chapter 19

P19–1. a) $H_G = \left[\frac{1}{2}(100)(2)^2\right](3) = 600 \text{ kg} \cdot \text{m}^2/\text{s} \circlearrowright$

$$H_O = \left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](3)$$

$= 1800 \text{ kg} \cdot \text{m}^2/\text{s} \circlearrowright$

b) $H_G = \left[\frac{1}{12}(100)(3)^2\right](4) = 300 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

$H_O = \left[\frac{1}{12}(100)(3)^2 + (100)(1.5)^2\right](4)$

$= 1200 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

c) $H_G = \left[\frac{1}{2}(100)(2)^2\right](4) = 800 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

$H_O = \left[\frac{1}{2}(100)(2)^2 + (100)(2)^2\right](4)$

$= 2400 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

d) $H_G = \left[\frac{1}{12}(100)(4)^2\right]3 = 400 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowleft$

$H_O = \left[\frac{1}{12}(100)(4)^2 + (100)(1)^2\right]3$

$= 700 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowleft$

P19–2. a) $\int M_O\,dt = \left(\frac{4}{5}\right)(500)(2)(3) = 2400 \text{ N}\cdot\text{s}\cdot\text{m} \circlearrowright$

b) $\int M_O\,dt = \left[2(20) + \frac{1}{2}(3 - 2)(20)\right]4$

$= 200 \text{ N}\cdot\text{s}\cdot\text{m} \circlearrowright$

c) $\int M_O\,dt = \frac{3}{5}\int_0^3 4(2t + 2)dt = 36 \text{ N}\cdot\text{s}\cdot\text{m} \circlearrowright$

d) $\int M_O\,dt = \int_0^3 (30t^2)dt = 270 \circlearrowright$

Review Problem Solutions

Chapter 12

R12–1. $s = t^3 - 9t^2 + 15t$

$v = \dfrac{ds}{dt} = 3t^2 - 18t + 15$

$a = \dfrac{dv}{dt} = 6t - 18$

a_{max} occurs at $t = 10$ s.

$a_{max} = 6(10) - 18 = 42\ \text{ft/s}^2$ *Ans.*

v_{max} occurs when $t = 10$ s

$v_{max} = 3(10)^2 - 18(10) + 15 = 135\ \text{ft/s}$ *Ans.*

R12–2. $(\overset{+}{\rightarrow})\quad s = s_0 + v_0 t + \dfrac{1}{2}a_c t^2$

$s = 0 + 12(10) + \dfrac{1}{2}(-2)(10)^2$

$s = 20.0$ ft *Ans.*

R12–3. $v = \dfrac{ds}{dt} = 1800(1 - e^{-0.3t})$

$\displaystyle\int_0^x ds = \int_0^t 1800(1 - e^{-0.3t})\,dt$

$s = 1800\left(t + \dfrac{1}{0.3}e^{-0.3t}\right) - 6000$

Thus, in $t = 3$ s

$s = 1800\left(3 + \dfrac{1}{0.3}e^{-0.3(3)}\right) - 6000$

$s = 1839.4\ \text{mm} = 1.84\ \text{m}$ *Ans.*

R12–4. $0 \le t \le 5\quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{20}{5} = 4\ \text{m/s}^2$ *Ans.*

$5 \le t \le 20\quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{20 - 20}{20 - 5} = 0\ \text{m/s}^2$ *Ans.*

$20 \le t \le 30\quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{0 - 20}{30 - 20} = -2\ \text{m/s}^2$ *Ans.*

At $t_1 = 5$ s, $t_2 = 20$ s, and $t_3 = 30$ s,

$s_1 = A_1 = \dfrac{1}{2}(5)(20) = 50$ m *Ans.*

$s_2 = A_1 + A_2 = 50 + 20(20 - 5) = 350$ m *Ans.*

$s_3 = A_1 + A_2 + A_3 = 350$

$+ \dfrac{1}{2}(30 - 20)(20) = 450$ m *Ans.*

R12–5. $v_A = 20\mathbf{i}$

$v_B = 21.21\mathbf{i} + 21.21\mathbf{j}$

$v_C = 40\mathbf{i}$

$\mathbf{a}_{AB} = \dfrac{\Delta v}{\Delta t} = \dfrac{21.21\mathbf{i} + 21.21\mathbf{j} - 20\mathbf{i}}{3}$

$\mathbf{a}_{AB} = \{0.404\mathbf{i} + 7.07\mathbf{j}\}\ \text{m/s}^2$ *Ans.*

$\mathbf{a}_{AC} = \dfrac{\Delta v}{\Delta t} = \dfrac{40\mathbf{i} - 20\mathbf{i}}{8}$

$\mathbf{a}_{AC} = \{2.50\mathbf{i}\}\ \text{m/s}^2$ *Ans.*

R12–6. $(\overset{+}{\rightarrow})\quad s = s_0 + v_0 t$

$126 = 0 + (v_0)_x(3.6)$

$(v_0)_x = 35\ \text{ft/s}$

$(+\uparrow)\quad s = s_0 + v_0 t + \dfrac{1}{2}a_c t^2$

$O = 0 + (v_0)_y(3.6) + \dfrac{1}{2}(-32.2)(3.6)^2$

$(v_0)_y = 57.96\ \text{ft/s}$

$v_0 = \sqrt{(35)^2 + (57.96)^2} = 67.7\ \text{ft/s}$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{57.96}{35}\right) = 58.9°$ *Ans.*

R12–7. $v\,dv = a_t\,ds$

$\displaystyle\int_4^v v\,dv = \int_0^{10} 0.05s\,ds$

$0.5v^2 - 8 = \dfrac{0.05}{2}(10)^2$

$v = 4.583 = 4.58\ \text{m/s}$ *Ans.*

$a_n = \dfrac{v^2}{\rho} = \dfrac{(4.583)^2}{50} = 0.420\ \text{m/s}^2$

$a_t = 0.05(10) = 0.5\ \text{m/s}^2$

$a = \sqrt{(0.420)^2 + (0.5)^2} = 0.653\ \text{m/s}^2$ *Ans.*

R12–8. $dv = a\,dt$

$\displaystyle\int_0^v dv = \int_0^t 0.5e^t\,dt$

$v = 0.5(e^t - 1)$

When $t = 2$ s, $v = 0.5(e^2 - 1) = 3.195\ \text{m/s}$

$= 3.19\ \text{m/s}$ *Ans.*

When $t = 2$ s $a_t = 0.5e^2 = 3.695\ \text{m/s}^2$

$$a_n = \frac{v^2}{\rho} = \frac{3.195^2}{5} = 2.041\ \text{m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{3.695^2 + 2.041^2}$$

$$= 4.22\ \text{m/s}^2 \qquad \textit{Ans.}$$

R12–9. $r = 2$ m $\qquad \theta = 5t^2$

$\dot{r} = 0 \qquad \dot{\theta} = 10t$

$\ddot{r} = 0 \qquad \ddot{\theta} = 10$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$

$$= \left[0 - 2(10t)^2\right]\mathbf{u}_r + [2(10) + 0]\mathbf{u}_\theta$$

$$= \{-200t^2\mathbf{u}_r + 20\mathbf{u}_\theta\}\ \text{m/s}^2$$

When $\theta = 30° = 30\left(\dfrac{\pi}{180}\right) = 0.524$ rad

$$0.524 = 5t^2$$

$$t = 0.324\ \text{s}$$

$$\mathbf{a} = \left[-200(0.324)^2\right]\mathbf{u}_r + 20\mathbf{u}_\theta$$

$$= \{-20.9\mathbf{u}_r + 20\mathbf{u}_\theta\}\ \text{m/s}^2$$

$$a = \sqrt{(-20.9)^2 + (20)^2} = 29.0\ \text{m/s}^2 \qquad \textit{Ans.}$$

R12–10. $4s_B + s_A = l$

$4v_B = -v_A$

$4a_B = -a_A$

$4a_B = -0.2$

$a_B = -0.05\ \text{m/s}^2$

$(+\downarrow) \qquad v_B = (v_B)_0 + a_B t$

$-8 = 0 - (0.05)(t)$

$t = 160$ s $\qquad$ *Ans.*

R12–11. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$[500 \leftarrow] = [600 \ \overset{75°}{\angle\theta}\ ] + v_{B/A}$

$(\overset{+}{\leftarrow}) \qquad 500 = -600\cos 75° + (v_{B/A})_x$

$(v_{B/A})_x = 655.29 \leftarrow$

$(+\uparrow) \qquad 0 = -600\sin 75° + (v_{B/A})_y$

$(v_{B/A})_y = 579.56 \uparrow$

$(v_{B/A}) = \sqrt{(655.29)^2 + (579.56)^2}$

$v_{B/A} = 875$ km/h $\qquad$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{579.56}{655.29}\right) = 41.5°$ $\qquad$ *Ans.*

Chapter 13

R13–1. $20\ \text{km/h} = \dfrac{20(10)^3}{3600} = 5.556\ \text{m/s}$

$(\overset{+}{\leftarrow}) \qquad v^2 = v_0^2 + 2a_c(s - s_0)$

$$a = -0.3429\ \text{m/s}^2 = 0.3429\ \text{m/s}^2 \rightarrow$$

$\overset{+}{\rightarrow}\Sigma F_x = ma_x; \qquad F = 250(0.3429) = 85.7$ N $\qquad$ *Ans.*

R13–2. $\nwarrow + \Sigma F_y = ma_y; \qquad N_C - 50(9.81)\cos 30° = 0$

$N_C = 424.79$

$\nearrow + \Sigma F_x = ma_x; \qquad 3T - 0.3(424.79) - 50(9.81)\sin 30° = 50a_C \qquad$ **(1)**

Kinematics, $2s_C + (s_C - s_p) = l$

Taking two time derivatives, yields

$$3a_C = a_p$$

Thus, $a_C = \dfrac{6}{3} = 2$

Substituting into Eq. (1) and solving,

$T = 158$ N $\qquad$ *Ans.*

R13–3. Suppose the two blocks move together.

Then

$$50\ \text{lb} = \frac{50 + 20}{32.2}a$$

$$a = 23\ \text{m/s}^2$$

Then the friction force on block B is

$$F_B = \frac{50}{32.2}(23) = 35.7\ \text{lb}$$

The maximum friction force between blocks A and B is

$$F_{\max} = 0.4(20) = 8\ \text{lb} < 35.7\ \text{1b}$$

The blocks have different accelerations.

Block A:

$\overset{+}{\rightarrow}\Sigma F_x = ma_x; \qquad 20(0.3) = \dfrac{20}{32.2}a_A$

$a_A = 70.8\ \text{ft/s}^2$ $\qquad$ *Ans.*

Block B:

$\overset{+}{\rightarrow}\Sigma F_x = ma_x; \qquad 20(0.3) = \dfrac{50}{32.2}a_B$

$a_B = 3.86\ \text{ft/s}^2$ $\qquad$ *Ans.*

R13–4. ***Kinematics:*** Since the motion of the crate is known, its acceleration **a** will be determined first.

$$a = v\frac{dv}{ds} = (0.05s^{3/2})\left[(0.05)\left(\frac{3}{2}\right)s^{1/2}\right]$$

$$= 0.00375s^2\ \text{m/s}^2$$

When $s = 10$ m,

$$a = 0.00375(10^2) = 0.375\ \text{m/s}^2 \rightarrow$$

Free-Body Diagram: The kinetic friction $F_f = \mu_k N = 0.2N$ must act to the left to oppose the motion of the crate which is to the right.

Equations of Motion: Here, $a_y = 0$. Thus,

$$+\uparrow \Sigma F_y = ma_y; \quad N - 20(9.81) = 20(0)$$
$$N = 196.2 \text{ N}$$

Using the results of **N** and **a**,

$$\xrightarrow{+} \Sigma F_x = ma_x; \quad T - 0.2(196.2) = 20(0.375)$$
$$T = 46.7 \text{ N} \qquad \textit{Ans.}$$

R13–5. $+\nwarrow \Sigma F_n = ma_n; \quad T - 30(9.81)\cos\theta = 30\left(\dfrac{v^2}{4}\right)$

$$+\nearrow \Sigma F_t = ma_t; \quad -30(9.81)\sin\theta = 30a_t$$
$$a_t = -9.81\sin\theta$$

$a_t\,ds = v\,dv$ Since $ds = 4\,d\theta$, then

$$-9.81\int_0^\theta \sin\theta\,(4\,d\theta) = \int_4^v v dv$$
$$9.81(4)\cos\theta\Big|_0^\theta = \frac{1}{2}(v)^2 - \frac{1}{2}(4)^2$$
$$39.24(\cos\theta - 1) + 8 = \frac{1}{2}v^2$$

At $\theta = 20°$

$v = 3.357$ m/s

$a_t = -3.36 \text{ m/s}^2 = 3.36 \text{ m/s}^2 \swarrow$ *Ans.*

$T = 361$ N *Ans.*

R13–6. $\Sigma F_z = ma_z; \quad N_z - mg = 0 \quad N_z = mg$

$$\Sigma F_x = ma_n; \quad 0.3(mg) = m\left(\frac{v^2}{r}\right)$$
$$v = \sqrt{0.3gr} = \sqrt{0.3(32.2)(3)} = 5.38 \text{ ft/s} \qquad \textit{Ans.}$$

R13–7. $v = \dfrac{1}{8}x^2$

$$\frac{dy}{dx} = \tan\theta = \frac{1}{4}x\Big|_{x=-6} = -1.5 \quad \theta = -56.31°$$
$$\frac{d^2y}{dx^2} = \frac{1}{4}$$
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1 + (-1.5)^2\right]^{\frac{3}{2}}}{\left|\frac{1}{4}\right|} = 23.436 \text{ ft}$$
$$+\nearrow \Sigma F_n = ma_n; \quad N - 10\cos 56.31° = \left(\frac{10}{32.2}\right)\left(\frac{(5)^2}{23.436}\right)$$
$$N = 5.8783 = 5.88 \text{ lb} \qquad \textit{Ans.}$$

$$+\searrow \Sigma F_t = ma_t; \quad -0.2(5.8783) + 10\sin 56.31° = \left(\frac{10}{32.2}\right)a_t$$
$$a_t = 23.0 \text{ ft/s}^2 \qquad \textit{Ans.}$$

R13–8. $r = 0.5$ m

$\dot{r} = 3$ m/s $\qquad \dot{\theta} = 6$ rad/s

$\ddot{r} = 1 \text{ m/s}^2 \qquad \ddot{\theta} = 2$ rad/s

$$a_r = \ddot{r} - r\dot{\theta}^2 = 1 - 0.5(6)^2 = -17$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.5(2) + 2(3)(6) = 37$$
$$\Sigma F_r = ma_r; \quad F_r = 4(-17) = -68 \text{ N}$$
$$\Sigma F_\theta = ma_\theta; \quad N_\theta = 4(37) = 148 \text{ N}$$
$$\Sigma F_z = ma_z; \quad N_z = 4(9.81) = 0$$
$$N_z = 39.24 \text{ N}$$
$$F_r = -68 \text{ N} \qquad \textit{Ans.}$$
$$N = \sqrt{(148)^2 + (39.24)^2} = 153 \text{ N} \qquad \textit{Ans.}$$

Chapter 14

R14–1. $+\nwarrow \Sigma F_y = 0; \quad N_C - 150\cos 30° = 0$

$$N_C = 129.9 \text{ lb}$$
$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 150\sin 30°(30) - (0.3)129.9(30) = \frac{1}{2}\left(\frac{150}{32.2}\right)v_2^2$$
$$v_2 = 21.5 \text{ ft/s} \qquad \textit{Ans.}$$

R14–2. $r_{AB} = r_B - r_A = -4\mathbf{i} + 8\mathbf{j} - 9\mathbf{k}$

$$T_1 + \Sigma\int F ds = T_2$$
$$0 + 2(10 - 1) + \int_4^0 10dx + \int_0^8 6y\,dy + \int_{10}^1 2z\,dz = \frac{1}{2}\left(\frac{2}{32.2}\right)v^2n$$
$$v_B = 47.8 \text{ ft/s} \qquad \textit{Ans.}$$

R14–3. $T_1 + V_1 = T_2 + V_2$

$$0 + 1.5(10) = \frac{1}{2}\left(\frac{1.5}{32.2}\right)v_B^2$$
$$v_B = 25.4 \text{ ft/s} \qquad \textit{Ans.}$$

R14–4. The work done by F depends upon the difference in the cord length $AC–BC$.

$$T_A + \Sigma U_{A-B} = T_B$$

$$0 + F\left[\sqrt{(0.3)^2 + (0.3)^2} - \sqrt{(0.3)^2 + (0.3 - 0.15)^2}\right] - 0.5(9.81)(0.15)$$

$$-\frac{1}{2}(100)(0.15)^2 = \frac{1}{2}(0.5)(2.5)^2$$

$$F(0.0889) = 3.423$$

$$F = 38.5 \text{ N} \qquad \textit{Ans.}$$

R14–5. $(+\uparrow) \quad v^2 = v_0^2 + 2a_c(s - s_0)$

$$(12)^2 = 0 + 2a_c(10 - 0)$$

$$a_c = 7.20 \text{ ft/s}^2$$

$$+\uparrow\Sigma F_y = ma_y; \quad 2T - 50 = \frac{50}{32.2}(7.20)$$

$$T = 30.6 \text{ lb}$$

$$s_C + (s_C - s_M) = l$$

$$v_M = 2v_C$$

$$v_M = 2(12) = 24 \text{ ft/s}$$

$$P_0 = \mathbf{T}\cdot\mathbf{v} = 30.6(24) = 734.2 \text{ lb}\cdot\text{ft/s}$$

$$P_i = \frac{734.2}{0.74} = 992.1 \text{ lb}\cdot\text{ft/s} = 1.80 \text{ hp} \qquad \textit{Ans.}$$

R14–6. $+\uparrow\Sigma F_y = m\,a_y; \quad 2(30) - 50 = \frac{50}{32.2}a_B$

$$a_B = 6.44 \text{ m/s}^2$$

$$(+\uparrow) \quad v^2 = v_0^2 + 2a_c(s - s_0)$$

$$v_B^2 = 0 + 2(6.44)(10 - 0)$$

$$v_B = 11.349 \text{ ft/s}$$

$$2s_B + s_M = l$$

$$2v_B = -v_M$$

$$v_M = -2(11.349) = 22.698 \text{ ft/s}$$

$$P_o = \mathbf{F}\cdot\mathbf{v} = 30(22.698) = 680.94 \text{ ft}\cdot\text{lb/s}$$

$$P_i = \frac{680.94}{0.76} = 895.97 \text{ ft}\cdot\text{lb/s}$$

$$P_i = 1.63 \text{ hp} \qquad \textit{Ans.}$$

R14–7. $T_A + V_A = T_B + V_B$

$$0 + (0.25)(9.81)(0.6) + \frac{1}{2}(150)(0.6 - 0.1)^2$$

$$= \frac{1}{2}(0.25)(v_B)^2 + \frac{1}{2}(150)(0.4 - 0.1)^2$$

$$v_B = 10.4 \text{ m/s} \qquad \textit{Ans.}$$

R14–8. $\frac{6}{z} = \frac{\sqrt{15^2 + 2^2}}{15}$

$$z = 5.95 \text{ ft}$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}\left(\frac{10}{32.2}\right)v_2^2 + \frac{1}{2}\left(\frac{30}{32.2}\right)v_2^2 + 10(5.95) - 30(5.95)$$

$$v_2 = 13.8 \text{ ft/s} \qquad \textit{Ans.}$$

Chapter 15

R15–1. $(+\uparrow) \quad m(v_1)_y + \Sigma\int F_y\,dt = m(v_2)_y$

$$0 + N_p(t) - 58.86(t) = 0$$

$$N_p = 58.86 \text{ N}$$

$$(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma\int F_x\,dt = m(v_2)_x$$

$$6(3) - 0.2(58.86)(t) = 6(1)$$

$$t = 1.02 \text{ s} \qquad \textit{Ans.}$$

R15–2. $+\nwarrow\Sigma F_x = 0; \quad N_B - 50(9.81)\cos 30^\circ = 0$

$$N_B = 424.79 \text{ N}$$

$$(+\nearrow) \quad m(v_x)_1 + \Sigma\int F_x\,dt = m(v_x)_2$$

$$50(2) + \int_0^2 \left(300 + 120\sqrt{t}\right)dt - 0.4(424.79)(2) - 50(9.81)\sin 30^\circ(2) = 50v_2$$

$$v_2 = 1.92 \text{ m/s} \qquad \textit{Ans.}$$

R15–3. The crate starts moving when

$$F = F_r = 0.6(196.2) = 117.72 \text{ N}$$

From the graph since

$$F = \frac{200}{5}t. \quad 0 \le t \le 5 \text{ s}$$

The time needed for the crate to start moving is

$$t = \frac{5}{200}(117.72) = 2.943 \text{ s}$$

Hence, the impulse due to F is equal to the area under the curve from $2.943 \text{ s} \le t \le 10 \text{ s}$

$$\overset{+}{\rightarrow} \quad m(v_x)_1 + \Sigma\int F_x\,dt = m(v_x)_2$$

$$0 + \int_{2.943}^{5} \frac{200}{5}t\,dt + \int_5^{10} 200\,dt - (0.5)196.2(10 - 2.943) = 20v_2$$

$$40\left(\frac{1}{2}t^2\right)\Big|_{2.943}^{5} + 200(10 - 5) - 692.292 = 20v_2$$

$$634.483 = 20v_2$$

$$v_2 = 31.7 \text{ m/s}$$ *Ans.*

R15–4. $(v_A)_1 = \left[20(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 5.556 \text{ m/s}$

$(v_B)_1 = \left[5(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 1.389 \text{ m/s},$

and $(v_C)_1 = \left[25(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 6.944 \text{ m/s}$

For the first case,

$(\overset{+}{\rightarrow})\quad m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$

$10000(5.556) + 5000(1.389) = (10000 + 5000)v_{AB}$

$v_{AB} = 4.167 \text{ m/s} \rightarrow$

Using the result of v_{AB} and considering the second case,

$(\overset{+}{\rightarrow})\quad (m_A + m_B)v_{AB} + m_C(v_C)_1 = (m_A + m_B + m_C)v_{ABC}$

$(10000 + 5000)(4.167) + [-20000(6.944)] = (10000 + 5000 + 20000)v_{ABC}$

$v_{ABC} = -2.183 \text{ m/s} = 2.18 \text{ m/s} \leftarrow$ *Ans.*

R15–5. $(\overset{+}{\rightarrow})\quad m_P(v_P)_1 + m_B(v_B)_1 = m_P(v_p)_2 + m_B(v_B)_2$

$0.2(900) + 15(0) = 0.2(300) + 15(v_B)_2$

$(v_B)_2 = 8 \text{ m/s} \rightarrow$ *Ans.*

$(+\uparrow)\quad m(v_1)_y + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_2)_y$

$15(0) + N(t) - 15(9.81)(t) = 15(0)$

$N = 147.15 \text{ N}$

$(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_2)_x$

$15(8) + [-0.2(147.15)(t)] = 15(0)$

$t = 4.077 \text{ s} = 4.08 \text{ s}$ *Ans.*

R15–6. $(\overset{+}{\rightarrow})\quad \Sigma mv_1 = \Sigma mv_2$

$3(2) + 0 = 3(v_A)_2 + 2(v_B)_2$

$(\overset{+}{\rightarrow})\quad e = \dfrac{(v_B)_2 - (v_A)_2)}{(v_A)_1 - (v_B)_1}$

$1 = \dfrac{(v_B)_2 - (v_A)_2}{2 - 0}$

Solving

$(v_A)_2 = 0.400 \text{ m/s} \rightarrow$ *Ans.*

$(v_B)_2 = 2.40 \text{ m/s} \rightarrow$ *Ans.*

Block A:

$T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(3)(0.400)^2 - 3(9.81)(0.3)d_A = 0$

$d_A = 0.0272 \text{ m}$

Block B:

$T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(2)(2.40)^2 - 2(9.81)(0.3)d_B = 0$

$d_B = 0.9786 \text{ m}$

$d = d_B - d_A = 0.951 \text{ m}$ *Ans.*

R15–7. $(v_A)_{x_1} = -2\cos 40° = -1.532 \text{ m/s}$

$(v_A)_{y_1} = -2\sin 40° = -1.285 \text{ m/s}$

$(\overset{+}{\rightarrow})\quad m_A(v_A)_{x_1} + m_B(v_B)_{x_1} = m_A(v_A)_{x_2} + m_B(v_B)_{x_2}$

$-2(1.532) + 0 = 0.2(v_A)_{x_2} + 0.2(v_B)_{x_2}$ **(1)**

$(\overset{+}{\rightarrow})\quad e = \dfrac{(v_{ref})_2}{(v_{ref})_1}$

$0.75 = \dfrac{(v_A)_{x_2} - (v_B)_{x_1}}{1.532}$ **(2)**

Solving Eqs. (1) and (2)

$(v_A)_{x_2} = -0.1915 \text{ m/s}$

$(v_B)_{x_2} = -1.3405 \text{ m/s}$

For A:

$(+\downarrow)\quad m_A(v_A)_{y_1} = m_A(v_A)_{y_2}$

$(v_A)_{y_2} = 1.285 \text{ m/s}$

For B:

$(+\uparrow)\quad m_B(v_B)_{y_1} = m_B(v_B)_{y_2}$

$(v_B)_{y_2} = 0$

Hence $(v_B)_2 = (v_B)_{x_2} = 1.34 \text{ m/s} \leftarrow$ *Ans.*

$(v_A)_2 = \sqrt{(-0.1915)^2 + (1.285)^2} = 1.30 \text{ m/s}$ *Ans.*

$(\theta_A)_2 = \tan^{-1}\left(\dfrac{0.1915}{1.285}\right) = 8.47°$ *Ans.*

R15–8. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$(10)(2)(0.75) + 60(2)\left(\frac{3}{5}\right)(0.75) + \int_0^2 (8t^2 + 5)\,dt = 10v(0.75)$

$69 + \left[\frac{8}{3}t^3 + 5t\right]_0^2 = 7.5v$

$v = 13.4 \text{ m/s}$ *Ans.*

Chapter 16

R16–1. $(\omega_A)_O = 60 \text{ rad/s}$

$\alpha_A = -1 \text{ rad/s}^2$

$\omega_A = (\omega_A)_O + \alpha_A t$

$\omega_A = 60 + (-1)(3) = 57 \text{ rad/s}$

$v_A = r\omega_A = (1)(57) = 57 \text{ ft/s} = v_B$

$\omega_B = \dfrac{v_B}{r} = 57/2 = 28.5 \text{ rad/s}$

$v_W = r_C\omega_C = (0.5)(28.5) = 14.2 \text{ ft/s}$ *Ans.*

$\alpha_A = 1$

$a_{A_t} = l(1) = 1 \text{ ft/s}^2$

$\alpha_B = \dfrac{1}{2} = 0.5 \text{ rad/s}^2$

$a_W = r\alpha_B = (0.5)(0.5) = 0.25 \text{ ft/s}^2$ *Ans.*

R16–2. $\alpha_a = 0.6\theta_A$

$\theta_C = \dfrac{0.5}{0.075} = 6.667 \text{ rad}$

$\theta_A(0.05) = (6.667)(0.15)$

$\theta_A = 20 \text{ rad}$

$\alpha d\theta = \omega d\omega$

$\displaystyle\int_0^{20} 0.6\theta_A d\theta_A = \int_3^{\omega_A} \omega_A d\omega_A$

$0.3\theta_A^2\Big|_0^{20} = \dfrac{1}{2}\omega_A^2\Big|_3^{\omega_A}$

$120 = \dfrac{1}{2}\omega_A^2 - 4.5$

$\omega_A = 15.780 \text{ rad/s}$

$15.780(0.05) = \omega_C(0.15)$

$\omega_C = 5.260 \text{ rad/s}$

$v_B = 5.260(0.075) = 0.394 \text{ m/s}$ *Ans.*

R16–3. A point on the drum which is in contact with the board has a tangential acceleration of

$a_t = 0.5 \text{ m/s}^2$

$a^2 = a_t^2 + a_n^2$

$(3)^2 = (0.5)^2 + a_n^2$

$a_n = 2.96 \text{ m/s}^2$

$a_n = \omega^2 r, \qquad \omega = \sqrt{\dfrac{2.96}{0.25}} = 3.44 \text{ rad/s}$

$v_B = \omega r = 3.44(0.25) = 0.860 \text{ m/s}$ *Ans.*

R16–4. $\mathbf{v}_B = \omega_{AB} \times \mathbf{r}_{B/A}$

$\mathbf{v}_C = \mathbf{v}_B + \omega \times \mathbf{r}_{C/B}$

$v_C\mathbf{i} = (6\mathbf{k}) \times (0.2\cos 45°\mathbf{i} + 0.2\sin 45°\mathbf{j}) + (\omega\mathbf{k}) \times (0.5\cos 30°\mathbf{i} - 0.5\sin 30°\mathbf{j})$

$v_C = -0.8485 + \omega(0.25)$

$0 = 0.8485 + 0.433\,\omega$

Solving

$\omega = 1.96 \text{ rad/s} \circlearrowright$

$v_C = 1.34 \text{ m/s}$ *Ans.*

R16–5. $\omega = \dfrac{2}{0.08} = 25 \text{ rad/s}$

$\alpha = \dfrac{4}{0.08} = 50 \text{ rad/s}^2$

$\mathbf{a}_C = \mathbf{a}_A + (\mathbf{a}_{C/A})_n + (\mathbf{a}_{C/A})_t$

$\mathbf{a}_C = 4\mathbf{j} + (25)^2(0.08)\mathbf{i} + 50(0.08)\mathbf{j}$

$\xrightarrow{+}\quad a_C\cos\theta = 0 + 50$

$+\uparrow\quad a_C\sin\theta = 4 + 0 + 4$

Solving, $a_C = 50.6 \text{ m/s}^2$ *Ans.*

$\theta = 9.09°$ ∠θ *Ans.*

The cylinder moves up with an acceleration

$a_B = (a_C)_t = 50.6\sin 9.09° = 8.00 \text{ m/s}^2\uparrow$ *Ans.*

R16–6. $\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$

$2.057 + (a_C)_t = 1.8 + 1.2 + \alpha_{CB}(0.5)$

$\rightarrow \quad \downarrow \quad \downarrow \quad \leftarrow \quad$ θ 30°

$(\xrightarrow{+})\quad 2.057 = -1.2 + \alpha_{CB}(0.5)\cos 30°$

$(+\downarrow)\quad (a_C)_t = 1.8 + \alpha_{CB}(0.5)\sin 30°$

$\alpha_{CB} = 7.52 \text{ rad/s}^2$ *Ans.*

$(a_C)_t = 3.68 \text{ m/s}^2$

$a_C = \sqrt{(3.68)^2 + (2.057)^2} = 4.22 \text{ m/s}^2$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{3.68}{2.057}\right) = 60.8°$ *Ans.*

Also,

$\mathbf{a}_C = \mathbf{a}_B + \alpha_{CB} \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B}$

$-(a_C)_t\mathbf{j} + \dfrac{(0.6)^2}{0.175}\mathbf{i} = -(2)^2(0.3)\mathbf{i} - 6(0.3)\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (-0.5\cos 60°\mathbf{i} - 0.5\sin 60°\mathbf{j}) - \mathbf{0}$

$2.057 = -1.20 + \alpha_{CB}(0.433)$

$-(a_C)_t = -1.8 - \alpha_{CB}(0.250)$

$\alpha_{CB} = 7.52 \text{ rad/s}^2$ *Ans.*

$a_t = 3.68 \text{ m/s}^2$

$a_C = \sqrt{(3.68)^2 + (2.057)^2} = 4.22 \text{ m/s}^2$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{3.68}{2.057}\right) = 60.8°$ θ *Ans.*

R16–7. $a_C = 0.5(8) = 4\text{ m/s}^2$

$\mathbf{a}_B = \mathbf{a}_C + \mathbf{a}_{B/C}$

$$\mathbf{a}_B = \begin{bmatrix} 4 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (3)^2(0.5) \\ \measuredangle 30^\circ \end{bmatrix} + \begin{bmatrix} (0.5)(8) \\ 30^\circ \end{bmatrix}$$

$(\overset{+}{\rightarrow})\quad (a_B)_x = -4 + 4.5\cos 30^\circ + 4\sin 30^\circ$

$= 1.897\text{ m/s}^2$

$(+\uparrow)\quad (a_B)_y = 0 + 4.5\sin 30^\circ - 4\cos 30^\circ$

$= -1.214\text{ m/s}^2$

$a_B = \sqrt{(1.897)^2 + (-1.214)^2}$

$= 2.25\text{ m/s}^2$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{1.214}{1.897}\right) = 32.6^\circ$ *Ans.*

Also,

$\mathbf{a}_B = \mathbf{a}_C + \alpha \times \mathbf{r}_{B/C} - \omega^2\mathbf{r}_{B/C}$

$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -4\mathbf{i} + (8\mathbf{k}) \times (-0.5\cos 30^\circ\mathbf{i}$

$-0.5\sin 30^\circ\mathbf{j}) - (3)^2(-0.5\cos 30^\circ\mathbf{i} - 0.5\sin 30^\circ\mathbf{j})$

$(\overset{+}{\rightarrow})\ (a_B)_x = -4 + 8(0.5\sin 30^\circ) + (3)^2(0.5\cos 30^\circ)$

$= 1.897\text{ m/s}^2$

$(+\uparrow)\ (a_B)_y = 0 - 8(0.5\cos 30^\circ) + (3)^2(0.5\sin 30^\circ)$

$= -1.214\text{ m/s}^2$

$\theta = \tan^{-1}\left(\dfrac{1.214}{1.897}\right) = 32.6^\circ$ *Ans.*

$a_B = \sqrt{(1.897)^2 + (-1.214)^2} = 2.25\text{ m/s}^2$ *Ans.*

R16–8. $v_B = 3(7) = 21\text{ in./s} \leftarrow$

$\mathbf{v}_C = \mathbf{v}_B + \omega \times \mathbf{r}_{C/B}$

$-v_C\left(\dfrac{4}{5}\right)\mathbf{i} - v_C\left(\dfrac{3}{5}\right)\mathbf{j} = -21\mathbf{i} + \omega\mathbf{k} \times (-5\mathbf{i} - 12\mathbf{j})$

$(\overset{+}{\rightarrow})\quad -0.8v_C = -21 + 12\omega$

$(+\uparrow)\quad -0.6v_C = -5\omega$

Solving:

$\omega = 1.125\text{ rad/s}$

$v_C = 9.375\text{ in./s} = 9.38\text{ in./s}$ (3–4–5 slope) *Ans.*

$(a_B)_n = (3)^2(7) = 63\text{ in./s}^2\downarrow$

$(a_B)_t = (2)(7) = 14\text{ in./s}^2 \leftarrow$

$\mathbf{a}_C = \mathbf{a}_B + \alpha \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B}$

$-a_C\left(\dfrac{4}{5}\right)\mathbf{i} - a_C\left(\dfrac{3}{5}\right)\mathbf{j} = -14\mathbf{i} - 63\mathbf{j} + (\alpha\mathbf{k})$

$\times (-5\mathbf{i} - 12\mathbf{j}) - (1.125)^2(-5\mathbf{i} - 12\mathbf{j})$

$(\overset{+}{\rightarrow})\quad -0.8a_C = -14 + 12\alpha + 6.328$

$(+\uparrow)\quad -0.6a_C = -63 - 5\alpha + 15.1875$

$a_C = 54.7\text{ in./s}^2$ (3–4–5 slope) *Ans.*

$\alpha = -3.00\text{ rad/s}^2$

Chapter 17

R17–1. $\overset{+}{\leftarrow}\Sigma F_x = ma_x;\quad 50\cos 60^\circ = 200a_G$ **(1)**

$+\uparrow\Sigma F_y = ma_y;\quad N_A + N_B - 200(9.81)$

$-50\sin 60^\circ = 0$ **(2)**

$\circlearrowleft +\Sigma M_G = 0;\quad -N_A(0.3) + N_B(0.2) +$

$50\cos 60^\circ(0.3)$

$-50\sin 60^\circ(0.6) = 0$ **(3)**

Solving,

$a_G = 0.125\text{ m/s}^2$

$N_A = 765.2\text{ N}$

$N_B = 1240\text{ N}$

At each wheel

$N'_A = \dfrac{N_A}{2} = 383\text{ N}$ *Ans.*

$N'_B = \dfrac{N_B}{2} = 620\text{ N}$ *Ans.*

R17–2. ***Curvilinear Translation***:

$(a_G)_t = 8(3) = 24\text{ ft/s}^2$

$(a_G)_n = (5)^2(3) = 75\text{ ft/s}^2$

$\bar{x} = \dfrac{\Sigma\tilde{x}m}{\Sigma m} = \dfrac{1(3) + 2(3)}{6} = 1.5\text{ ft}$

$+\downarrow\Sigma F_y = m(a_G)_y;\quad E_y + 6 = \dfrac{6}{32.2}(24)\cos 30^\circ$

$+\dfrac{6}{32.2}(75)\sin 30^\circ$

$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x;\quad E_x = \dfrac{6}{32.2}(75)\cos 30^\circ$

$-\dfrac{6}{32.2}(24)\sin 30^\circ$

$\circlearrowleft +\Sigma M_G = 0;\quad M_E - E_y(1.5) = 0$

$E_x = 9.87\text{ lb}$ *Ans.*

$E_y = 4.86\text{ lb}$ *Ans.*

$M_E = 7.29\text{ lb}\cdot\text{ft}$ *Ans.*

R17–3. (a) Rear wheel drive

Equations of motion:

$\overset{+}{\rightarrow}\Sigma F_x = m(a_G)_x;\quad 0.3N_B = 1.5(10)^3a_G$ **(1)**

$\circlearrowright +\Sigma M_A = \Sigma(M_k)_A;\quad 1.5(10)^3(9.81)(1.3)$

$$-N_B(2.9) = -1.5(10)^3 a_G(0.4) \qquad \textbf{(2)}$$

Solving Eqs. (1) and (2) yields:

$$N_B = 6881 \text{ N} = 6.88 \text{ kN}$$

$$a_G = 1.38 \text{ m/s}^2 \qquad \textit{Ans.}$$

R17–4. $\xrightarrow{+} \Sigma F_x = m(a_G)_x;\ 40 \sin 60° + N_C - \left(\frac{5}{13}\right)T = 0$

$+\uparrow \Sigma F_y = m(a_G)_y;\ -40 \cos 60° + 0.3N_C - 20(9.81) + \frac{12}{13}T = 0$

$\circlearrowleft + \Sigma M_A = I_A \alpha;\quad 40(0.120) - 0.3N_C(0.120) = \left[\frac{1}{2}(20)(0.120)^2\right]\alpha$

Solving,

$T = 218 \text{ N}$ *Ans.*

$N_C = 49.28 \text{ N}$

$\alpha = 21.0 \text{ rad/s}^2$ *Ans.*

R17–5. $(a_G)_t = 4\alpha$

$\xleftarrow{+} \Sigma F_t = m(a_G)_x;\quad F + 20 - 5 = \frac{30}{32.2}(4\alpha)$

$\circlearrowright + \Sigma M_O = I_O\alpha;\quad 20(3) + F(6) = \frac{1}{3}\left(\frac{30}{32.2}\right)(8)^2\alpha$

Solving,

$\alpha = 12.1 \text{ rad/s}^2$ *Ans.*

$F = 30.0 \text{ lb}$ *Ans.*

R17–6. $I_O = \frac{2}{5}\left(\frac{30}{32.2}\right)(1)^2 + \left(\frac{30}{32.2}\right)(3)^2 + \frac{1}{3}\left(\frac{10}{32.2}\right)(2)^2 = 9.17 \text{ slug} \cdot \text{ft}^2$

$\bar{x} = \frac{30(3) + 10(1)}{30 + 10} = 2.5 \text{ ft}$

$\xrightarrow{+} \Sigma F_n = ma_n;\quad O_x = 0$

$+\downarrow \Sigma F_t = ma_t;\quad 40 - O_y = \frac{40}{32.2}a_G$

$\circlearrowleft + \Sigma M_O = I_O\alpha;\quad 40(2.5) = 9.17\alpha$

Kinematics

$a_G = 2.5\alpha$

Solving,

$\alpha = 10.90 \text{ rad/s}^2$

$a_G = 27.3 \text{ ft/s}^2$

$O_x = 0$

$O_y = 6.14 \text{ lb}$

Thus:

$F_o = 6.14 \text{ lb} \rightarrow$ *Ans.*

R17–7. $+\uparrow \Sigma F_y = m(a_G)_y;\quad N_B - 20(9.81) = 0$

$N_B = 196.2 \text{ N}$

$F_B = 0.1(196.2) = 19.62 \text{ N}$

$\circlearrowleft + \Sigma M_{IC} = \Sigma(M_k)_{IC};\quad 30 - 19.62(0.6) = 20(0.2\alpha)(0.2) + [20(0.25)^2]\alpha$

$\alpha = 8.89 \text{ rad/s}^2$ *Ans.*

R17–8. $\xleftarrow{+} \Sigma F_x = m(a_G)_x;\quad 0.3N_A = \frac{20}{32.2}a_G$

$+\uparrow \Sigma F_y = m(a_G)_y;\quad N_A - 20 = 0$

$\circlearrowright + \Sigma M_G = I_G\alpha;\quad 0.3N_A(0.5) = \left[\frac{2}{5}\left(\frac{20}{32.2}\right)(0.5)^2\right]\alpha$

Solving,

$N_A = 20 \text{ lb}$

$a_G = 9.66 \text{ ft/s}^2$

$\alpha = 48.3 \text{ rad/s}^2$

$(\circlearrowleft +)\quad \omega = \omega_0 + \alpha_c t$

$0 = \omega_1 - 48.3t$

$\omega_1 = 48.3t$

$(\xrightarrow{+})\quad v = v_0 + a_c t$

$0 = 20 - 9.66\left(\frac{\omega}{48.3}\right)$

$\omega = 100 \text{ rad/s}$ *Ans.*

Chapter 18

R18–1. $T_1 + \Sigma U_{1-2} = T_2$

$0 + (50)(9.81)(1.25) = \frac{1}{2}\left[(50)(1.75)^2\right]\omega_2^2$

$\omega_2 = 2.83 \text{ rad/s}$ *Ans.*

R18–2. ***Kinetic Energy and Work***: The mass moment inertia of the flywheel about its mass center is $I_O = mk_O^2 = 50(0.2^2) = 2 \text{ kg} \cdot \text{m}^2$. Thus,

$$T = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(2)\omega^2 = \omega^2$$

Since the wheel is initially at rest, $T_1 = 0$. $\mathbf{W}$, $\mathbf{O}_x$, and $\mathbf{O}_y$ do no work while $\mathbf{M}$ does positive work. When the wheel rotates

$\theta = (5 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 10\pi$, the work done by M is

$$U_M = \int M d\theta = \int_0^{10\pi} (9\theta^{1/2} + 1)d\theta$$
$$= (6\theta^{3/2} + \theta)\Big|_0^{10\pi}$$
$$= 1087.93 \text{ J}$$

Principle of Work and Energy:

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 1087.93 = \omega^2$$
$$\omega = 33.0 \text{ rad/s} \qquad \textit{Ans.}$$

R18–3. Before braking:

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 15(9.81)(3) = \frac{1}{2}(15)v_B^2 + \frac{1}{2}\left[50(0.23)^2\right]\left(\frac{v_B}{0.15}\right)^2$$
$$v_B = 2.58 \text{ m/s} \qquad \textit{Ans.}$$
$$\frac{s_B}{0.15} = \frac{s_C}{0.25}$$

Set $s_B = 3$ m, then $s_C = 5$ m.

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 - F(5) + 15(9.81)(6) = 0$$
$$F = 176.6 \text{ N}$$
$$N = \frac{176.6}{0.5} = 353.2 \text{ N}$$

Brake arm:

$$\circlearrowleft + \Sigma M_A = 0; \qquad -353.2(0.5) + P(1.25) = 0$$
$$P = 141 \text{ N} \qquad \textit{Ans.}$$

R18–4.
$$\frac{s_G}{0.3} = \frac{s_A}{(0.5 - 0.3)}$$
$$s_A = 0.6667 s_G$$
$$+\nwarrow \Sigma F_y = 0; \qquad N_A - 60(9.81)\cos 30^\circ = 0$$
$$N_A = 509.7 \text{ N}$$
$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 60(9.81)\sin 30^\circ(s_G) - 0.2(509.7)(0.6667 s_G)$$
$$= \frac{1}{2}\left[60(0.3)^2\right](6)^2$$
$$+ \frac{1}{2}(60)\left[(0.3)(6)\right]^2$$
$$s_G = 0.859 \text{ m} \qquad \textit{Ans.}$$

R18–5. ***Conservation of Energy***: Originally, both gears are rotating with an angular velocity of $\omega_1 = \dfrac{2}{0.05} = 40$ rad/s. After the rack has traveled $s = 600$ mm, both gears rotate with an angular velocity of $\omega_2 = \dfrac{v_2}{0.05}$, where v_2 is the speed of the rack at that moment.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}(6)(2)^2 + 2\left\{\frac{1}{2}\left[4(0.03)^2\right](40)^2\right\} + 0$$
$$= \left\{\frac{1}{2}\left[4(0.03)^2\right]\left(\frac{v_2}{0.05}\right)^2\right\} - 6(9.81)(0.6)$$
$$v_2 = 3.46 \text{ m/s} \qquad \textit{Ans.}$$

R18–6. Datum through A.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}\left[\frac{1}{3}\left(\frac{50}{32.2}\right)(6)^2\right](2)^2 + \frac{1}{2}(6)(4 - 2)^2$$
$$= \frac{1}{2}\left[\frac{1}{3}\left(\frac{50}{32.2}\right)(6)^2\right]\omega^2 + \frac{1}{2}(6)(7 - 2)^2 - 50(1.5)$$
$$\omega = 2.30 \text{ rad/s} \qquad \textit{Ans.}$$

R18–7.
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 4(1.5 \sin 45^\circ) + 1(3 \sin 45^\circ)$$
$$= \frac{1}{2}\left[\frac{1}{3}\left(\frac{4}{32.2}\right)(3)^2\right]\left(\frac{v_C}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{32.2}\right)(v_C)^2 + 0$$
$$v_C = 13.3 \text{ ft/s} \qquad \textit{Ans.}$$

R18–8. Datum at lowest point.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}\left[\frac{1}{2}(40)(0.3)^2\right]\left(\frac{4}{0.3}\right)^2 + \frac{1}{2}(40)(4)^2$$
$$+ 40(9.81)d \sin 30^\circ = 0 + \frac{1}{2}(200)d^2$$
$$100d^2 - 196.2d - 480 = 0$$

Solving for the positive root,

$$d = 3.38 \text{ m} \qquad \textit{Ans.}$$

Chapter 19

R19–1. $I_O = mk_O^2 = \dfrac{150}{32.2}(1.25)^2 = 7.279 \text{ slug}\cdot\text{ft}^2$

$$I_O\omega_1 + \Sigma\int_{t_1}^{t_2} M_O dt = I_O\omega_2$$
$$0 - \int_0^{3\text{ s}} 10t^2(1)\,dt = 7.279\omega_2$$
$$\frac{10t^3}{3}\Big|_0^{3\text{ s}} = 7.279\omega_2$$
$$\omega_2 = 12.4 \text{ rad/s} \qquad \textit{Ans.}$$

R19–3. $+\swarrow m(v_G)_1 + \Sigma \int F\,dt = m(v_G)_2$

$$0 + 9(9.81)(\sin 30°)(3) - \int_0^3 F\,dt = 9(v_G)_2 \quad \textbf{(1)}$$

$$\circlearrowleft + (H_G)_1 + \Sigma \int M_G dt = (H_G)_2$$

$$0 + \left(\int_0^3 F\,dt\right)(0.3) = \left[9(0.225)^2\right]\omega_2 \quad \textbf{(2)}$$

Since $(v_G)_2 = 0.3\omega_2$,

Eliminating $\int_0^3 F\,dt$ from Eqs. (1) and (2) and solving for $(v_G)_2$ yields.

$$(v_G)_2 = 9.42 \text{ m/s} \qquad \textit{Ans.}$$

Also,

$$\circlearrowleft + (H_A)_1 + \Sigma \int M_A dt = (H_A)_2$$

$$0 + 9(9.81)\sin 30°(3)(0.3) = \left[9(0.225)^2 + 9(0.3)^2\right]\omega$$

$\omega = 31.39$ rad/s

$v = 0.3(31.39) = 9.42$ m/s *Ans.*

R19–4. $\overset{+}{\leftarrow}$ $m(v_x)_1 + \Sigma \int_{t_1}^{t_2} F_x dt = m(v_x)_2$

$0 + 200(3) = 100(v_O)_2$

$(v_O)_2 = 6$ m/s *Ans.*

and

$$I_z\omega_1 + \Sigma \int_{t_1}^{t_2} M_z dt = I_z\omega_2$$

$0 - [200(0.4)(3)] = -9\omega_2$

$\omega_2 = 26.7$ rad/s *Ans.*

R19–5. $(+\uparrow)$ $mv_1 + \Sigma \int F dt = mv_2$

$$0 + T(3) - 30(3) + 40(3) = \frac{30}{32.2}v_o$$

$(\circlearrowleft+)$ $(H_O)_1 + \Sigma \int M_O dt = (H_O)_2$

$$-T(0.5)3 + 40(1)3 = \left[\frac{30}{32.2}(0.65)^2\right]\omega$$

Kinematics,

$$v_o = 0.5\omega$$

Solving,

$T = 23.5$ lb

$\omega = 215$ rad/s *Ans.*

$v_O = 108$ ft/s

Also,

$(\circlearrowleft+)$ $(H_{IC})_1 + \Sigma \int M_{IC} dt = (H_{IC})_2$

$0 - 30(0.5)(3) + 40(1.5)(3)$

$$= \left[\frac{30}{32.2}(0.65)^2 + \frac{30}{32.2}(0.5)^2\right]\omega$$

$\omega = 215$ rad/s *Ans.*

R19–6. $\circlearrowright +$ $(H_A)_1 + \Sigma \int M_A dt = (H_A)_2$

$$\left[\frac{30}{32.2}(0.8)^2\right](6) - \int T\,dt(1.2) = \left[\frac{30}{32.2}(0.8)^2\right]\omega_A$$

$\circlearrowright +$ $(H_B)_1 + \Sigma \int M_B dt = (H_B)_2$

$$0 + \int T\,dt(0.4) = \left[\frac{15}{32.2}(0.6)^2\right]\omega_B$$

Kinematics:

$$1.2\omega_A = 0.4\omega_B$$

$$\omega_B = 3\omega_A$$

Thus,

$\omega_A = 1.70$ rad/s *Ans.*

$\omega_B = 5.10$ rad/s *Ans.*

R19–7. $H_1 = H_2$

$$\left(\frac{1}{2}mr^2\right)\omega_1 = \left[\frac{1}{2}mr^2 + mr^2\right]\omega_2$$

$\omega_2 = \dfrac{1}{3}\omega_1$ *Ans.*

R19–8. $H_1 = H_2$

$$(0.940)(0.5) + (4)\left[\frac{1}{12}(20)\left((0.75)^2 + (0.2)^2\right) + (20)(0.375 + 0.2)^2\right](0.5)$$

$$= (0.940)(\omega) + 4\left[\frac{1}{12}(20)(0.2)^2 + (20)(0.2)^2\right]\omega$$

$\omega = 3.56$ rad/s *Ans.*

Answers to Selected Problems

Chapter 12

12–1. $s = 80.7$ m

12–2. $s = 20$ ft

12–3. $a = -24\ \text{m/s}^2$, $\Delta s = -880$ m, $s_T = 912$ m

12–5. $s_T = 8$ m, $v_{avg} = 2.67$ m/s

12–6. $s|_{t=6\text{ s}} = -27.0$ ft, $s_{tot} = 69.0$ ft

12–7. $v_{avg} = 0$, $(v_{sp})_{avg} = 3$ m/s, $a\Big|_{t=6\text{ s}} = 2\ \text{m/s}^2$

12–9. $v = 32$ m/s, $s = 67$ m, $d = 66$ m

12–10. $v = 1.29$ m/s

12–11. $v_{avg} = 0.222$ m/s, $(v_{sp})_{avg} = 2.22$ m/s

12–13. Normal: $d = 517$ ft, drunk: $d = 616$ ft

12–14. $v = 165$ ft/s, $a = 48\ \text{ft/s}^2$, $s_T = 450$ ft, $v_{avg} = 25.0$ ft/s, $(v_{sp})_{avg} = 45.0$ ft/s

12–15. $v = \left(2kt + \dfrac{1}{v_0^2}\right)^{-1/2}$, $s = \dfrac{1}{k}\left[\left(2kt + \dfrac{1}{v_0^2}\right)^{1/2} - \dfrac{1}{v_0}\right]$

12–17. $d = 16.9$ ft

12–18. $t = 5.62$ s

12–19. $s = 28.4$ km

12–21. $s = 123$ ft, $a = 2.99\ \text{ft/s}^2$

12–22. $h = 314$ m, $v = 72.5$ m/s

12–23. $v = (20e^{-2t})$ m/s, $a = (-40e^{-2t})\ \text{m/s}^2$, $s = 10(1 - e^{-2t})$ m

12–25. (a) $v = 45.5$ m/s, (b) $v_{max} = 100$ m/s

12–26. (a) $s = -30.5$ m,
(b) $s_{Tot} = 56.0$ m,
(c) $v = 10$ m/s

12–27. $t = 0.549\left(\dfrac{v_f}{g}\right)$

12–29. $h = 20.4$ m, $t = 2$ s

12–30. $s = 54.0$ m

12–31. $s = \dfrac{v_0}{k}(1 - e^{-kt})$, $a = -kv_0e^{-kt}$

12–33. $v = 11.2$ km/s

12–34. $v = -R\sqrt{\dfrac{2g_0(y_0 - y)}{(R + y)(R + y_0)}}$, $v_{imp} = 3.02$ km/s

12–35. $t' = 27.3$ s.
When $t = 27.3$ s, $v = 13.7$ ft/s.

12–37. $\Delta s = 1.11$ km

12–38. $a|_{t=0} = -4\ \text{m/s}^2$, $a|_{t=2\text{ s}} = 0$,
$a|_{t=4\text{ s}} = 4\ \text{m/s}^2$, $v|_{t=0} = 3$ m/s,
$v|_{t=2\text{ s}} = -1$ m/s, $v|_{t=4\text{ s}} = 3$ m/s

12–39. $s = 2\sin\left(\dfrac{\pi}{5}t\right) + 4$, $v = \dfrac{2\pi}{5}\cos\left(\dfrac{\pi}{5}t\right)$,
$a = -\dfrac{2\pi^2}{25}\sin\left(\dfrac{\pi}{5}t\right)$

12–41. $t = 7.48$ s. When $t = 2.14$ s,
$v = v_{max} = 10.7$ ft/s, $h = 11.4$ ft.

12–42. $s = 600$ m. For $0 \le t < 40$ s, $a = 0$.
For $40\text{ s} < t \le 80$ s, $a = -0.250\ \text{m/s}^2$.

12–43. $t' = 35$ s
For $0 \le t < 10$ s, $s = \{300t\}$ ft, $v = 300$ ft/s
For $10\text{ s} < t < 20$ s,
$s = \left\{\dfrac{1}{6}t^3 - 15t^2 + 550t - 1167\right\}$ ft
$v = \left\{\dfrac{1}{2}t^2 - 30t + 550\right\}$ ft/s
For $20\text{ s} < t \le 35$ s,
$s = \{-5t^2 + 350t + 167\}$ ft
$v = (-10t + 350)$ ft/s

12–45. When $t = 0.1$ s, $s = 0.5$ m and a changes from $100\ \text{m/s}^2$ to $-100\ \text{m/s}^2$. When $t = 0.2$ s, $s = 1$ m.

12–46. $v\Big|_{s=75\text{ ft}} = 27.4$ ft/s, $v\Big|_{s=125\text{ ft}} = 37.4$ ft/s

12–47. For $0 \le t < 30$ s, $v = \left\{\dfrac{1}{5}t^2\right\}$ m/s, $s = \left\{\dfrac{1}{15}t^3\right\}$ m
For $30 \le t \le 60$ s, $v = \{24t - 540\}$ m/s,
$s = \{12t^2 - 540t + 7200\}$ m

12–49. $v_{max} = 100$ m/s, $t' = 40$ s

12–50. For $0 \le s < 300$ ft, $v = \{4.90\,s^{1/2}\}$ m/s.
For $300\text{ ft} < s \le 450$ ft,
$v = \{(-0.04s^2 + 48s - 3600)^{1/2}\}$ m/s.
$s = 200$ ft when $t = 5.77$ s.

12–51. For $0 \le t < 60$ s, $s = \left\{\dfrac{1}{20}t^2\right\}$ m, $a = 0.1\ \text{m/s}^2$.
For $60\text{ s} < t < 120$ s, $s = \{6t - 180\}$ m, $a = 0$.
For $120\text{ s} < t \le 180$ s, $s = \left\{\dfrac{1}{30}t^2 - 2t + 300\right\}$ m,
$a = 0.0667\ \text{m/s}^2$.

12–53. At $t = 8$ s, $a = 0$ and $s = 30$ m.
At $t = 12$ s, $a = -1\ \text{m/s}^2$ and $s = 48$ m.

12–54. For $0 \le t < 5$ s, $s = \{0.2t^3\}$ m,
$a = \{1.2t\}\ \text{m/s}^2$
For $5\text{ s} < t \le 15$ s, $s = \left\{\dfrac{1}{4}(90t - 3t^2 - 275)\right\}$ m
$a = -1.5\ \text{m/s}^2$,
At $t = 15$ s, $s = 100$ m, $v_{avg} = 6.67$ m/s

12–55. $t' = 33.3$ s, $s|_{t=5\text{ s}} = 550$ ft, $s|_{t=15\text{ s}} = 1500$ ft,
$s|_{t=20\text{ s}} = 1800$ ft, $s|_{t=33.3\text{ s}} = 2067$ ft

12–57. For $0 \le s < 100$ ft, $v = \left\{\sqrt{\dfrac{1}{50}(800s - s^2)}\right\}$ ft/s
For $100\text{ ft} < s \le 150$ ft,
$v = \left\{\dfrac{1}{5}\sqrt{-3s^2 + 900s - 25\,000}\right\}$ ft/s

12–58. For $0 \le t < 15$ s, $v = \left\{\frac{1}{2}t^2\right\}$ m/s, $s = \left\{\frac{1}{6}t^3\right\}$ m.
For 15 s $< t \le 40$ s,
$v = \{20t - 187.5 \text{ m/s}\}$,
$s = \{10t^2 - 187.5t + 1125\}$ m
12–59. $s_T = 980$ m
12–61. When $t = 5$ s, $s_B = 62.5$ m.
When $t = 10$ s, $v_A = (v_A)_{max} = 40$ m/s and
$s_A = 200$ m.
When $t = 15$ s, $s_A = 400$ m and $s_B = 312.5$ m.
$\Delta s = s_A - s_B = 87.5$ m
12–62. $v = \{5 - 6t\}$ ft/s, $a = -6$ ft/s^2
12–63. For $0 \le t < 5$ s, $s = \{2t^2\}$ m and $a = 4$ m/s^2.
For 5 s $< t < 20$ s, $s = \{20t - 50\}$ m and $a = 0$.
For 20 s $< t \le 30$ s, $s = \{2t^2 - 60t + 750\}$ m
and $a = 4$ m/s^2.
12–65. $v = 354$ ft/s, $t = 5.32$ s
12–66. When $s = 100$ m, $t = 10$ s.
When $s = 400$ m, $t = 16.9$ s.
$a|_{s=100\text{ m}} = 4$ m/s^2, $a|_{s=400\text{ m}} = 16$ m/s^2
12–67. At $s = 100$ s, a changes from $a_{max} = 1.5$ ft/s^2
to $a_{min} = -0.6$ ft/s^2.
12–69. $a = 5.31$ m/s^2, $\alpha = 53.0°$
$\beta = 37.0°$, $\gamma = 90.0°$
12–70. $\Delta\mathbf{r} = \{6\mathbf{i} + 4\mathbf{j}\}$ m
12–71. (4 ft, 2 ft, 6 ft)
12–73. (5.15 ft, 1.33 ft)
12–74. $\mathbf{r} = \{11\mathbf{i} + 2\mathbf{j} + 21\mathbf{k}\}$ ft
12–75. $(v_{sp})_{avg} = 4.28$ m/s
12–77. $v = 8.55$ ft/s, $a = 5.82$ m/s^2
12–78. $v = 1003$ m/s, $a = 103$ m/s^2
12–79. $d = 4.00$ ft, $a = 37.8$ ft/s^2
12–81. $(\mathbf{v}_{BC})_{avg} = \{3.88\mathbf{i} + 6.72\mathbf{j}\}$ m/s
12–82. $v = \sqrt{c^2k^2 + b^2}$, $a = ck^2$
12–83. $v = 10.4$ m/s, $a = 38.5$ m/s^2
12–85. $d = 204$ m, $v = 41.8$ m/s $a = 4.66$ m/s^2
12–86. $\theta = 58.3°$, $(v_0)_{min} = 9.76$ m/s
12–87. $\theta = 76.0°$, $v_A = 49.8$ ft/s, $h = 39.7$ ft
12–89. $R_{max} = 10.2$ m, $\theta = 45°$
12–90. $R = 8.83$ m
12–91. (13.3 ft, −7.09 ft)
12–93. $d = 166$ ft
12–94. $t = 3.57$ s, $v_B = 67.4$ ft/s
12–95. $v_A = 36.7$ ft/s, $h = 11.5$ ft
12–97. $v_A = 19.4$ m/s, $v_B = 40.4$ m/s
12–98. $v_A = 39.7$ ft/s, $s = 6.11$ ft
12–99. $v_B = 160$ m/s, $h_B = 427$ m,
$h_C = 1.08$ km, $R = 2.98$ km
12–101. $v_{min} = 0.838$ m/s, $v_{max} = 1.76$ m/s
12–102. $\theta_A = 11.6°$, $t = 0.408$ s, $\theta_B = 11.6°$
12–103. $\theta_A = 78.4°$, $t = 2.00$ s, $\theta_B = 78.4°$
12–105. $t_A = 0.553$ s, $x = 3.46$ m
12–106. $R = 19.0$ m, $t = 2.48$ s
12–107. $\theta_1 = 24.9°$, $\theta_2 = 85.2°$
12–109. $\theta = 76.0°$, $v_A = 49.8$ ft/s, $h = 39.7$ ft
12–110. $v = 63.2$ ft/s
12–111. $v = 38.7$ m/s
12–113. $v = 4.40$ m/s, $a_t = 5.04$ m/s^2, $a_n = 1.39$ m/s^2
12–114. $a_t = 8.66$ ft/s^2, $\rho = 1280$ ft
12–115. $v = 97.2$ ft/s, $a = 42.6$ ft/s^2
12–117. When cars A and B are side by side, $t = 55.7$ s.
When cars A and B are 90° apart, $t = 27.8$ s.
12–118. $t = 66.4$ s
12–119. $h = 5.99$ Mm
12–121. $a = 2.75$ m/s^2
12–122. $a = 1.68$ m/s^2
12–123. $v = 1.5$ m/s, $a = 0.117$ m/s^2
12–125. $v = 43.0$ m/s, $a = 6.52$ m/s^2
12–126. $v = 105$ ft/s, $a = 22.7$ ft/s^2
12–127. $a_t = 3.62$ m/s^2, $\rho = 29.6$ m
12–129. $t = 7.00$ s, $s = 98.0$ m
12–130. $a = 7.42$ ft/s^2
12–131. $a = 2.36$ m/s^2
12–133. $a = 3.05$ m/s^2
12–134. $a = 0.763$ m/s^2
12–135. $a = 0.952$ m/s^2
12–137. $y = -0.0766x^2$, $v = 8.37$ m/s,
$a_n = 9.38$ m/s^2, $a_t = 2.88$ m/s^2
12–138. $v_B = 19.1$ m/s, $a = 8.22$ m/s^2, $\phi = 17.3°$
up from negative$-t$ axis
12–139. $a_{min} = 3.09$ m/s^2
12–141. $(a_n)_A = g = 32.2$ ft/s^2, $(a_t)_A = 0$,
$\rho_A = 699$ ft, $(a_n)_B = 14.0$ ft/s^2,
$(a_t)_B = 29.0$ ft/s^2, $\rho_B = 8.51(10^3)$ ft
12–142. $t = 1.21$ s
12–143. $a_{max} = \dfrac{v^2 a}{b^2}$
12–145. $d = 11.0$ m, $a_A = 19.0$ m/s^2, $a_B = 12.8$ m/s^2
12–146. $t = 2.51$ s, $a_A = 22.2$ m/s^2, $a_B = 65.1$ m/s^2
12–147. $\theta = 10.6°$
12–149. $a = 0.511$ m/s^2
12–150. $a = 0.309$ m/s^2
12–151. $a = 322$ mm/s^2, $\theta = 26.6°$
12–153. $v_n = 0$, $v_t = 7.21$ m/s,
$a_n = 0.555$ m/s^2, $a_t = 2.77$ m/s^2
12–154. $a = 7.48$ ft/s^2
12–155. $a = 14.3$ in./s^2
12–157. $v_r = 5.44$ ft/s, $v_\theta = 87.0$ ft/s,
$a_r = -1386$ ft/s^2, $a_\theta = 261$ ft/s^2
12–158. $v = 464$ ft/s, $a = 43.2(10^3)$ ft/s^2
12–159. $\mathbf{v} = \{-14.2\mathbf{u}_r - 24.0\mathbf{u}_z\}$ m/s
$\mathbf{a} = \{-3.61\mathbf{u}_r - 6.00\mathbf{u}_z\}$ m/s^2

12–161. $v_r = -2\sin t,\ v_\theta = \cos t,$
$a_r = -\frac{5}{2}\cos t,\ a_\theta = -2\sin t$
12–162. $v_r = ae^{at},\ v_\theta = e^{at},$
$a_r = e^{at}(a^2-1),\ a_\theta = 2ae^{at}$
12–163. $v_r = 0,\ v_\theta = 10$ ft/s,
$a_r = -0.25$ ft/s^2, $a_\theta = -3.20$ ft/s^2
12–165. $\mathbf{a} = (\dddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta})\mathbf{u}_r$
$+ (3\dot{r}\ddot{\theta} + \dot{r}\ddot{\theta} + 3\ddot{r}\dot{\theta} - r\dot{\theta}^3)\mathbf{u}_\theta + (\dddot{z})\mathbf{u}_z$
12–166. $a = 48.3$ in./s^2
12–167. $v_r = 1.20$ m/s, $v_\theta = 1.26$ m/s,
$a_r = -3.77$ m/s^2, $a_\theta = 7.20$ m/s^2
12–169. $v_r = 1.20$ m/s, $v_\theta = 1.50$ m/s,
$a_r = -4.50$ m/s^2, $a_\theta = 7.20$ m/s^2
12–170. $v_r = 16.0$ ft/s, $v_\theta = 1.94$ ft/s,
$a_r = 7.76$ ft/s^2, $a_\theta = 1.94$ ft/s^2
12–171. $v = 4.24$ m/s, $a = 17.6$ m/s^2
12–173. $a = 27.8$ m/s^2
12–174. $v_r = 0,\ v_\theta = 12$ ft/s,
$a_r = -216$ ft/s^2, $a_\theta = 0$
12–175. $v = 12.6$ m/s, $a = 83.2$ m/s^2
12–177. $v_r = -1.84$ m/s, $v_\theta = 19.1$ m/s,
$a_r = -2.29$ m/s^2, $a_\theta = 4.60$ m/s^2
12–178. $v_r = -24.2$ ft/s, $v_\theta = 25.3$ ft/s
12–179. $v_r = 0,\ v_\theta = 4.80$ ft/s,
$v_z = -0.664$ ft/s, $a_r = -2.88$ ft/s^2
$a_\theta = 0,\ a_z = -0.365$ ft/s^2
12–181. $v = 10.7$ ft/s, $a = 24.6$ ft/s^2
12–182. $v = 10.7$ ft/s, $a = 40.6$ ft/s^2
12–183. $\dot{\theta} = 0.333$ rad/s, $a = 6.67$ m/s^2
12–185. $v = 1.32$ m/s
12–186. $a = 8.66$ m/s^2
12–187. $\dot{\theta} = 0.0178$ rad/s
12–189. $v_r = 32.0$ ft/s, $v_\theta = 50.3$ ft/s,
$a_r = -201$ ft/s^2, $a_\theta = 256$ ft/s^2
12–190. $v_r = 32.0$ ft/s, $v_\theta = 50.3$ ft/s,
$a_r = -161$ ft/s^2, $a_\theta = 319$ ft/s^2
12–191. $v = 5.95$ ft/s, $a = 3.44$ ft/s^2
12–193. $v_r = 0.242$ m/s, $v_\theta = 0.943$ m/s,
$a_r = -2.33$ m/s^2, $a_\theta = 1.74$ m/s^2
12–194. $\dot{\theta} = 10.0$ rad/s
12–195. $v_B = 0.5$ m/s
12–197. $v = 24$ ft/s
12–198. $v_B = 1.67$ m/s
12–199. $\Delta s_B = 1.33$ ft $\rightarrow$
12–201. $t = 3.83$ s
12–202. $v_B = 0.75$ m/s
12–203. $t = 5.00$ s
12–205. $v_{B/C} = 39$ ft/s $\uparrow$
12–206. $v_B = 1.50$ m/s
12–207. $v_A = 1.33$ m/s
12–209. $v_B = 8$ ft/s$\downarrow$, $a_B = 6.80$ ft/s^2 $\uparrow$
12–210. $v_A = 2.5$ ft/s$\uparrow$, $a_A = 2.44$ ft/s$^2\uparrow$
12–211. $v_B = 2.40$ m/s $\uparrow$, $a_B = 3.25$ m/s^2 $\uparrow$
12–213. $v_A = 4$ ft/s
12–214. $v_{A/B} = 13.4$ m/s, $\theta_v = 31.7°$
$a_{A/B} = 4.32$ m/s^2, $\theta_a = 79.0°$
12–215. $v_A = 10.0$ m/s $\leftarrow$, $a_A = 46.0$ m/s^2 $\leftarrow$
12–217. $v_C = 1.2$ m/s$\uparrow$, $a_C = 0.512$ m/s$^2\uparrow$
12–218. $v_{B/A} = 1044$ km/h, $\theta = 54.5°$
12–219. $v_{B/A} = 28.5$ mi/h, $\theta_v = 44.5°$,
$a_{B/A} = 3418$ mi/h^2, $\theta_a = 80.6°$
12–221. $v_B = 13.5$ ft/s, $\theta = 84.8°$, $t = 1.85$ min
12–222. $v_w = 58.3$ km/h, $\theta = 59.0°$
12–223. $v_{A/B} = 15.7$ m/s, $\theta = 7.11°$, $t = 38.1$ s
12–225. $v_{A/B} = 98.4$ ft/s, $\theta_v = 67.6°$,
$a_{A/B} = 19.8$ ft/s^2, $\theta_a = 57.4°$
12–226. $v_{r/m} = 16.6$ km/h, $\theta = 25.0°$
12–227. $v_{B/A} = 20.5$ m/s, $\theta_v = 43.1°$
$a_{B/A} = 4.92$ m/s^2, $\theta_a = 6.04°$
12–229. $v_r = 34.6$ km/h$\downarrow$
12–230. $v_m = 4.87$ ft/s, $t = 10.3$ s
12–231. $v_{w/s} = 19.9$ m/s, $\theta = 74.0°$
12–233. Yes, he can catch the ball.
12–234. $v_B = 5.75$ m/s, $v_{C/B} = 17.8$ m/s,
$\theta = 76.2°$, $a_{C/B} = 9.81$ m/s$^2\downarrow$
12–235. $v_{B/A} = 11.2$ m/s, $\theta = 50.3°$

Chapter 13

13–1. $s = 97.4$ ft
13–2. $T = 5.98$ kip
13–3. $v = 3.36$ m/s, $s = 5.04$ m
13–5. $F = 6.37$ N
13–6. $v = 59.8$ ft/s
13–7. $v = 60.7$ ft/s
13–9. $t = 2.04$ s
13–10. $s = 8.49$ m
13–11. $t = 0.249$ s
13–13. $a_A = 9.66$ ft/s$^2\leftarrow$, $a_B = 15.0$ ft/s$^2\rightarrow$
13–14. $T = 11.25$ kN, $F = 33.75$ kN
13–15. $A_x = 685$ N, $A_y = 1.19$ kN, $M_A = 4.74$ kN·m
13–17. $a = \frac{1}{2}(1 - \mu_k)\,g$
13–18. $R = 5.30$ ft, $t_{AC} = 1.82$ s
13–19. $R = 5.08$ ft, $t_{AC} = 1.48$ s
13–21. $\theta = 22.6°$
13–22. $v_B = 5.70$ m/s $\uparrow$
13–23. $v = 3.62$ m/s$\uparrow$
13–25. $R = 2.45$ m, $t_{AB} = 1.72$ s
13–26. $R = \{150t\}$ N
13–27. $t = 2.11$ s
13–29. $v = 2.01$ ft/s

13–30. $v = 0.301\text{ m/s}$
13–31. $T = 1.63\text{ kN}$
13–33. $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$, $a = \left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)g$
13–34. $v = 2.19\text{ m/s}$
13–35. $t = 5.66\text{ s}$
13–37. $t = 0.519\text{ s}$
13–38. $s = 16.7\text{ m}$
13–39. $v = \dfrac{1}{m}\sqrt{1.09F_0^2t^2 + 2F_0tmv_0 + m^2v_0^2}$, $x = \dfrac{y}{0.3} + v_0\left(\sqrt{\dfrac{2m}{0.3F_0}}\right)y^{1/2}$
13–41. $x = d$, $v = \sqrt{\dfrac{kd^2}{m_A + m_B}}$
13–42. $x = d$ for separation.
13–43. $v = \sqrt{\dfrac{mg}{k}}\left[\dfrac{e^{2t\sqrt{mg/k}} - 1}{e^{2t\sqrt{mg/k}} + 1}\right]$, $v_t = \sqrt{\dfrac{mg}{k}}$
13–45. $v = 32.2\text{ ft/s}$
13–46. $P = 2mg\tan\theta$
13–47. $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$
13–49. $a_B = 7.59\text{ ft/s}^2$
13–50. $v = 5.13\text{ m/s}$
13–51. $d = \dfrac{(m_A + m_B)g}{k}$
13–53. $r = 1.36\text{ m}$
13–54. $v = 10.5\text{ m/s}$
13–55. $N = 6.18\text{ kN}$
13–57. $v = 1.63\text{ m/s}$, $N = 7.36\text{ N}$
13–58. $v = 0.969\text{ m/s}$
13–59. $v = 1.48\text{ m/s}$
13–61. $v = 9.29\text{ ft/s}$, $T = 38.0\text{ lb}$
13–62. $v = 2.10\text{ m/s}$
13–63. $T = 0$, $T = 10.6\text{ lb}$
13–65. $v = 6.30\text{ m/s}$, $F_n = 283\text{ N}$, $F_t = 0$, $F_b = 490\text{ N}$
13–66. $v = 22.1\text{ m/s}$
13–67. $\theta = 26.7^\circ$
13–69. $F_f = 1.11\text{ kN}$, $N = 6.73\text{ kN}$
13–70. $v_C = 19.9\text{ ft/s}$, $N_C = 7.91\text{ lb}$, $v_B = 21.0\text{ ft/s}$
13–71. $N = 277\text{ lb}$, $F = 13.4\text{ lb}$
13–73. $v = \sqrt{gr}$, $N = 2mg$
13–74. $v = 49.5\text{ m/s}$
13–75. $a_t = g\left(\dfrac{x}{\sqrt{1 + x^2}}\right)$, $v = \sqrt{v_0^2 + gx^2}$, $N = \dfrac{m}{\sqrt{1 + x^2}}\left[g - \dfrac{v_0^2 + gx^2}{1 + x^2}\right]$
13–77. $F_s = 4.90\text{ lb}$
13–78. $v = 40.1\text{ ft/s}$
13–79. $N_P = 2.65\text{ kN}$, $\rho = 68.3\text{ m}$
13–81. $\theta = 37.7^\circ$
13–82. $N_B = 80.4\text{ N}$, $a_t = 1.92\text{ m/s}^2$
13–85. $F_A = 4.46\text{ lb}$
13–86. $F = 210\text{ N}$
13–87. $F = 1.60\text{ lb}$
13–89. $F_r = -29.4\text{ N}$, $F_\theta = 0$, $F_z = 392\text{ N}$
13–90. $F_r = 102\text{ N}$, $F_z = 375\text{ N}$, $F_\theta = 79.7\text{ N}$
13–91. $N = 4.90\text{ N}$, $F = 4.17\text{ N}$
13–93. $F_{OA} = 12.0\text{ lb}$
13–94. $F = 5.07\text{ kN}$, $N = 2.74\text{ kN}$
13–95. $F = 17.0\text{ N}$
13–97. $(N)_{max} = 36.0\text{ N}$, $(N)_{min} = 4.00\text{ N}$
13–98. $N_s = 3.72\text{ N}$, $F_r = 7.44\text{ N}$
13–99. $F_r = -900\text{ N}$, $F_\theta = -200\text{ N}$, $F_z = 1.96\text{ kN}$
13–101. $\theta = \tan^{-1}\left(\dfrac{4r_c\dot{\theta}_0^2}{g}\right)$
13–102. $N = 0.883\text{ N}$, $F = 3.92\text{ N}$
13–103. $N = 2.95\text{ N}$
13–105. $F_r = 1.78\text{ N}$, $N_s = 5.79\text{ N}$
13–106. $F_r = 2.93\text{ N}$, $N_s = 6.37\text{ N}$
13–107. $F = 0.163\text{ lb}$
13–109. $F_r = 25.6\text{ N}$, $F_{OA} = 0$
13–110. $F_r = 20.7\text{ N}$, $F_{OA} = 0$
13–111. $r = 0.198\text{ m}$
13–113. $v_o = 30.4\text{ km/s}$, $\dfrac{1}{r} = 0.348\,(10^{-12})\cos\theta + 6.74\,(10^{-12})$
13–114. $h = 35.9\text{ mm}$, $v_s = 3.07\text{ km/s}$
13–115. $v_0 = 7.45\text{ km/s}$
13–118. $v_B = 7.71\text{ km/s}$, $v_A = 4.63\text{ km/s}$
13–119. $v_A = 6.67(10^3)\text{ m/s}$, $v_B = 2.77(10^3)\text{ m/s}$
13–121. $v_A = 7.47\text{ km/s}$
13–122. $r_0 = 11.1\text{ Mm}$, $\Delta v_A = 814\text{ m/s}$
13–123. $(v_A)_C = 5.27(10^3)\text{ m/s}$, $\Delta v = 684\text{ m/s}$
13–125. (a) $r = 194\,(10^3)$ mi
(b) $r = 392\,(10^3)$ mi
(c) $194\,(10^3)\text{ mi} < r < 392\,(10^3)$ mi
(d) $r > 392\,(10^3)$ mi
13–126. $v_A = 4.89(10^3)\text{ m/s}$, $v_B = 3.26(10^3)\text{ m/s}$
13–127. $v_A = 11.5\text{ Mm/h}$, $d = 27.3\text{ Mm}$
13–129. $v_A = 2.01(10^3)\text{ m/s}$
13–130. $v_{A'} = 521\text{ m/s}$, $t = 21.8\text{ h}$
13–131. $v_A = 7.01(10^3)\text{ m/s}$

Chapter 14

14–1. $v = 10.7\text{ m/s}$
14–2. $x_{max} = 3.24\text{ ft}$

14–3. $s = 1.35$ m
14–5. $h = 39.3$ m, $\rho = 26.2$ m
14–6. $d = 12$ m
14–7. Observer A: $v_2 = 6.08$ m/s,
Observer B: $v_2 = 4.08$ m/s
14–9. $x_{max} = 0.173$ m
14–10. $s = 20.5$ m
14–11. $v = 4.08$ m/s
14–13. $v_B = 31.5$ ft/s, $d = 22.6$ ft, $v_C = 54.1$ ft/s
14–14. $v_A = 7.18$ ft/s
14–15. $v_A = 3.52$ ft/s
14–17. $v_B = 27.8$ ft/s
14–18. $y = 3.81$ ft
14–19. $v_B = 3.34$ m/s
14–21. $v_A = 0.771$ ft/s
14–22. $s_{Tot} = 3.88$ ft
14–23. $x = 0.688$ m
14–25. $s = 0.0735$ ft
14–26. $v_A = 28.3$ m/s
14–27. $v_B = 18.0$ m/s, $N_B = 12.5$ kN
14–29. $s = 0.730$ m
14–30. $s = 3.33$ ft
14–31. $R = 2.83$ m, $v_C = 7.67$ m/s
14–33. $d = 36.2$ ft
14–34. $s = 1.90$ ft
14–35. $v_B = 42.2$ ft/s, $N = 50.6$ lb, $a_t = 26.2$ ft/s^2
14–37. $h_A = 22.5$ m, $h_C = 12.5$ m
14–38. $v_B = 14.9$ m/s, $N = 1.25$ kN
14–39. $v_B = 5.42$ m/s
14–41. $l_0 = 2.77$ ft
14–42. $\theta = 47.2°$
14–43. $P_i = 4.20$ hp
14–45. $P = 8.32\,(10^3)$ hp
14–46. $t = 46.2$ min
14–47. $P = 12.6$ kW
14–49. $P_{max} = 113$ kW, $P_{avg} = 56.5$ kW
14–50. $P_o = 4.36$ hp
14–51. $P = 92.2$ hp
14–53. $P_i = 483$ kW
14–54. $P_i = 622$ kW
14–55. $P_i = 22.2$ kW
14–57. $P = 0.0364$ hp
14–58. $P = 0.231$ hp
14–59. $P = 12.6$ kW
14–61. $P = \{400(10^3)t\}$ W
14–62. $P = \{160\,t - 533t^2\}$ kW, $U = 1.69$ kJ
14–63. $P_{max} = 10.7$ kW
14–65. $P = 58.1$ kW
14–66. $F = 227$ N
14–67. $h = 133$ in.
14–69. $N = 694$ N
14–70. $\theta = 48.2°$
14–71. $v_C = 17.7$ ft/s
14–73. $N_B = 0$, $h = 18.75$ m, $N_C = 17.2$ kN
14–74. $v_A = 1.54$ m/s, $v_B = 4.62$ m/s
14–75. $s_B = 5.70$ m
14–77. $h = 23.75$ m, $v_C = 21.6$ m/s
14–78. $v_B = 15.5$ m/s
14–79. $l = 2.77$ ft
14–81. $\theta = 118°$
14–83. $F = GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$
14–85. $v_B = 34.8$ Mm/h
14–86. $s = 130$ m
14–87. $s_B = 0.638$ m, $s_A = 1.02$ m
14–89. $\theta = 22.3°$, $s = 0.587$ m
14–90. $N = 78.6$ N
14–91. $y = 5.10$ m, $N = 15.3$ N, $a = 9.32$ m/s^2 ↘
14–93. $v = 1.68$ m/s
14–94. $v_2 = \sqrt{\frac{2}{\pi}(\pi - 2)gr}$
14–95. $v = 6.97$ m/s
14–97. $d = 1.34$ m

Chapter 15

15–1. $v = 1.75$ N·s
15–2. $v = 29.4$ ft/s
15–3. $F = 24.8$ kN
15–5. $I = 5.68$ N·s
15–6. $F = 19.4$ kN, $T = 12.5$ kN
15–7. $F_{AB} = 16.7$ lb, $v = 13.4$ ft/s
15–9. $v = 6.62$ m/s
15–10. $P = 205$ N
15–11. $v = 60.0$ m/s
15–13. $\mu_k = 0.340$
15–14. $I = 15$ kN·s in both cases.
15–15. $v = 4.05$ m/s
15–17. $v = 8.81$ m/s, $s = 24.8$ m
15–18. $v|_{t=3\,s} = 5.68$ m/s ↓, $v|_{t=6\,s} = 21.1$ m/s ↑
15–19. $v = 4.00$ m/s
15–21. $T = 14.9$ kN, $F = 24.8$ kN
15–22. $v_{max} = 108$ m/s, $s = 1.83$ km
15–23. $v = 10.1$ ft/s
15–25. $v = 7.21$ m/s ↑
15–26. Observer A: $v = 7.40$ m/s,
Observer B: $v = 5.40$ m/s
15–27. $v = 5.07$ m/s
15–29. $t = 1.02$ s, $I = 162$ N·s
15–30. $v = 16.1$ m/s
15–31. $(v_A)_2 = 10.5$ ft/s →
15–33. $v = 7.65$ m/s
15–34. $v = 0.6$ ft/s ←
15–35. $v = 18.6$ m/s →

15–37. $v = 5.21\ \text{m/s} \leftarrow$, $\Delta T = -32.6\ \text{kJ}$
15–38. $v = 0.5\ \text{m/s}$, $\Delta T = -16.9\ \text{kJ}$
15–39. $v = 733\ \text{m/s}$
15–41. $v_B = 3.48\ \text{ft/s}$, $d = 0.376\ \text{ft}$
15–42. $v_B = 3.48\ \text{ft/s}$, $N_{avg} = 504\ \text{lb}$, $t = 0.216\ \text{s}$
15–43. $s = 4.00\ \text{m}$
15–45. $v_2 = \sqrt{v_1^2 + 2gh}$, $\theta_2 = \sin^{-1}\left(\dfrac{v_1 \sin\theta}{\sqrt{v_1^2 + 2gh}}\right)$
15–46. $\theta = \phi = 9.52°$
15–47. $s_{max} = 481\ \text{mm}$
15–49. $x = 0.364\ \text{ft} \leftarrow$
15–50. $x = 1.58\ \text{ft} \rightarrow$
15–51. $s_B = 6.67\ \text{m} \rightarrow$
15–53. $s_B = 71.4\ \text{mm} \rightarrow$
15–54. $s_B = 71.4\ \text{mm} \rightarrow$
15–55. $v_c = 5.04\ \text{m/s} \leftarrow$
15–57. $d = 6.87\ \text{mm}$
15–59. $e = 0.75$, $\Delta T = -9.65\ \text{kJ}$
15–61. $x_{max} = 0.839\ \text{m}$
15–63. $v_C = 0.1875v \rightarrow$, $v_D = 0.5625v \rightarrow$,
$v_B = 0.8125v \rightarrow$, $v_A = 0.4375v \rightarrow$
15–65. $t = 0.226\ \text{s}$
15–66. $(v_B)_2 = \dfrac{1}{3}\sqrt{2gh}(1 + e)$
15–67. $(v_A)_2 = 1.04\ \text{ft/s}$, $(v_B)_3 = 0.964\ \text{ft/s}$,
$(v_C)_3 = 11.9\ \text{ft/s}$
15–69. $v'_B = 22.2\ \text{m/s}$, $\theta = 13.0°$
15–70. $(v_B)_2 = \dfrac{e(1 + e)}{2}v_0$
15–71. $v_A = 29.3\ \text{ft/s}$, $v_{B2} = 33.1\ \text{ft/s}$, $\theta = 27.7°$ ∠
15–73. $v_A = 1.35\ \text{m/s} \rightarrow$, $v_B = 5.89\ \text{m/s}$, $\theta = 32.9°$ ∠
15–74. $e = 0.0113$
15–75. $h = 1.57\ \text{m}$
15–77. $(v_B)_3 = 3.24\ \text{m/s}$, $\theta = 43.9°$
15–78. $v'_B = 31.8\ \text{ft/s}$
15–79. $(v_A)_2 = 3.80\ \text{m/s} \leftarrow$,
$(v_B)_2 = 6.51\ \text{m/s}$, $(\theta_B)_2 = 68.6°$
15–81. (a) $(v_B)_1 = 8.81\ \text{m/s}$, $\theta = 10.5°$ ∠,
(b) $(v_B)_2 = 4.62\ \text{m/s}$, $\phi = 20.3°$ ∠,
(c) $s = 3.96\ \text{m}$
15–82. $s = 0.456\ \text{ft}$
15–83. $(v_A)_2 = 42.8\ \text{ft/s} \leftarrow$, $F = 2.49\ \text{kip}$
15–85. $\mu_k = 0.25$
15–86. $(v_B)_2 = 1.06\ \text{m/s} \leftarrow$, $(v_A)_2 = 0.968\ \text{m/s}$,
$(\theta_A)_2 = 5.11°$ ∠
15–87. $(v_A)_2 = 4.06\ \text{ft/s}$, $(v_B)_2 = 6.24\ \text{ft/s}$
15–89. $(v_A)_2 = 12.1\ \text{m/s}$, $(v_B)_2 = 12.4\ \text{m/s}$
15–90. $d = 1.15\ \text{ft}$, $h = 0.770\ \text{ft}$
15–91. $(v_B)_3 = 1.50\ \text{m/s}$
15–93. $(v_A)_2 = 8.19\ \text{m/s}$, $(v_B)_2 = 9.38\ \text{m/s}$
15–94. $\{-9.17\mathbf{i} - 6.12\mathbf{k}\}\ \text{slug}\cdot\text{ft}^2/\text{s}$
15–95. $\{-9.17\mathbf{i} + 4.08\mathbf{j} - 2.72\mathbf{k}\}\ \text{slug}\cdot\text{ft}^2/\text{s}$
15–97. $(H_A)_P = \{-52.8\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$,
$(H_B)_P = \{-118\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
15–98. $\{-21.5\mathbf{i} + 21.5\mathbf{j} + 37.6\}\ \text{kg}\cdot\text{m}^2/\text{s}$
15–99. $\{21.5\mathbf{i} + 21.5\mathbf{j} + 59.1\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
15–101. $v = 20.2\ \text{ft/s}$, $h = 6.36\ \text{ft}$
15–102. $t = 11.9\ \text{s}$
15–103. $v_2 = 9.22\ \text{ft/s}$, $\Sigma U_{1-2} = 3.04\ \text{ft}\cdot\text{lb}$
15–105. $v = 9.50\ \text{m/s}$
15–107. $v = 3.33\ \text{m/s}$
15–109. $v_C = 44.0\ \text{ft/s}$, $H_A = 8.19\ \text{slug}\cdot\text{ft}^2/\text{s}$.
The cord will not unstretch.
15–110. $v_2 = 4.03\ \text{m/s}$, $\Sigma U_{1-2} = 725\ \text{J}$
15–111. $v_B = 10.8\ \text{ft/s}$, $U_{AB} = 11.3\ \text{ft}\cdot\text{lb}$
15–113. $v_B = 10.2\ \text{km/s}$, $r_B = 13.8\ \text{Mm}$
15–114. $T = 40.1\ \text{kN}$
15–115. $C_x = 4.97\ \text{kN}$, $D_x = 2.23\ \text{kN}$, $D_y = 7.20\ \text{kN}$
15–117. $F = 303\ \text{lb}$
15–118. $F = 50.0\ \text{lb}$
15–119. $F_x = 9.87\ \text{lb}$, $F_y = 4.93\ \text{lb}$
15–121. $F_x = 19.5\ \text{lb}$, $F_y = 1.96\ \text{lb}$
15–122. $F = 20.0\ \text{lb}$
15–123. $F = 22.4\ \text{lb}$
15–125. $T = 82.8\ \text{N}$, $N = 396\ \text{N}$
15–126. $F = 6.24\ \text{N}$, $P = 3.12\ \text{N}$
15–127. $d = 2.56\ \text{ft}$
15–129. $C_x = 4.26\ \text{kN}$, $C_y = 2.12\ \text{kN}$, $M_C = 5.16\ \text{kN}\cdot\text{m}$
15–130. $v = \left\{\dfrac{8000}{2000 + 50t}\right\}\ \text{m/s}$
15–131. $A_y = 4.18\ \text{kN}$, $B_x = 65.0\ \text{N} \rightarrow$,
$B_y = 3.72\ \text{kN} \uparrow$
15–133. $a = 0.125\ \text{m/s}^2$, $v = 4.05\ \text{m/s}$
15–134. $v_{max} = 2.07\ (10^3)\ \text{ft/s}$
15–135. 452 Pa
15–137. $R = \{20t + 2.48\}\ \text{lb}$
15–138. $a_i = 133\ \text{ft/s}^2$, $a_f = 200\ \text{ft/s}^2$
15–139. $v_{max} = 580\ \text{ft/s}$
15–141. $v = \sqrt{\dfrac{2}{3}g\left(\dfrac{y^3 - h^3}{y^2}\right)}$
15–142. $F_D = 11.5\ \text{kN}$
15–143. $a = 37.5\ \text{ft/s}^2$
15–145. $a = 0.0476\ \text{m/s}^2$
15–146. $v_{max} = 2.07(10^3)\ \text{ft/s}$
15–147. $F = \{7.85t + 0.320\}\ \text{N}$
15–149. $F = m'v^2$

Chapter 16

16–1. $v_A = 2.60\ \text{m/s}$, $a_A = 9.35\ \text{m/s}^2$
16–2. $v_A = 22.0\ \text{m/s}$,
$(a_A)_t = 12.0\ \text{m/s}^2$, $(a_A)_n = 968\ \text{m/s}^2$
16–3. $v_A = 26.0\ \text{m/s}$,
$(a_A)_t = 10.0\ \text{m/s}^2$, $(a_A)_n = 1352\ \text{m/s}^2$

16–5. $\theta = 5443$ rev, $\omega = 740$ rad/s, $\alpha = 8$ rad/s^2
16–6. $\theta = 3.32$ rev, $t = 1.67$ s
16–7. $t = 6.98$ s, $\theta_D = 34.9$ rev
16–9. $a_B = 29.0$ m/s^2
16–10. $a_B = 16.5$ m/s^2
16–11. $\alpha = 60$ rad/s^2, $\omega = 90.0$ rad/s, $\theta = 90.0$ rad
16–13. $\omega_B = 180$ rad/s, $\omega_C = 360$ rad/s
16–14. $\omega = 42.7$ rad/s, $\theta = 42.7$ rad
16–15. $a_t = 2.83$ m/s^2, $a_n = 35.6$ m/s^2
16–17. $\omega_B = 21.9$ rad/s ↺
16–18. $\omega_B = 31.7$ rad/s ↺
16–19. $\omega_B = 156$ rad/s
16–21. $v_A = 8.10$ m/s,
$(a_A)_t = 4.95$ m/s^2, $(a_A)_n = 437$ m/s^2
16–22. $\omega_D = 4.00$ rad/s, $\alpha_D = 0.400$ rad/s^2
16–23. $\omega_D = 12.0$ rad/s, $\alpha_D = 0.600$ rad/s^2
16–25. $v_P = 2.42$ ft/s, $a_P = 34.4$ ft/s^2
16–26. $\omega_C = 1.68$ rad/s, $\theta_C = 1.68$ rad
16–27. $\omega = 148$ rad/s
16–29. $r_A = 31.8$ mm, $r_B = 31.8$ mm,
1.91 canisters per minute
16–30. $(\omega_B)_{max} = 8.49$ rad/s, $(v_C)_{max} = 0.6$ m/s
16–31. $s_W = 2.89$ m
16–33. $\omega_B = 312$ rad/s, $\alpha_B = 176$ rad/s^2
16–34. $v_E = 3$ m/s,
$(a_E)_t = 2.70$ m/s^2, $(a_E)_n = 600$ m/s^2
16–35. $\mathbf{v}_C = \{-4.8\mathbf{i} - 3.6\mathbf{j} - 1.2\mathbf{k}\}$ m/s,
$\mathbf{a}_C = \{38.4\mathbf{i} - 64.8\mathbf{j} + 40.8\mathbf{k}\}$ m/s^2
16–37. $v_C = 2.50$ m/s, $a_C = 13.1$ m/s^2
16–38. $v = 7.21$ ft/s, $a = 91.2$ ft/s^2
16–39. $\omega = \dfrac{r v_A}{y\sqrt{y^2 - r^2}}$, $\alpha = \dfrac{r v_A^2 (2y^2 - r^2)}{y^2 (y^2 - r^2)^{3/2}}$
16–41. $\omega = 8.70$ rad/s, $\alpha = -50.5$ rad/s^2
16–42. $\omega = -19.2$ rad/s, $\alpha = -183$ rad/s^2
16–43. $\omega_{AB} = 0$
16–45. $v = -\left(\dfrac{r_1^2 \omega \sin 2\theta}{2\sqrt{r_1^2 \cos^2\theta + r_2^2 + 2r_1 r_2}} + r_1 \omega \sin\theta\right)$
16–46. $v = \omega d\left(\sin\theta + \dfrac{d \sin 2\theta}{2\sqrt{(R + r)^2 - d^2 \sin^2\theta}}\right)$
16–47. $v = -r\omega \sin\theta$
16–49. $v_C = L\omega\uparrow$, $a_C = 0.577\,L\omega^2\uparrow$
16–50. $\omega = \dfrac{2v_0}{r}\sin^2\theta/2$, $\alpha = \dfrac{2v_0^2}{r^2}(\sin\theta)(\sin^2\theta/2)$
16–51. $v_B = \left(\dfrac{h}{d}\right)v_A$
16–53. $\dot{\theta} = \dfrac{v \sin\phi}{L\cos(\phi - \theta)}$
16–54. $\omega = \dfrac{v}{2r}$
16–55. $\omega' = \dfrac{(R + r)\omega}{r}$, $\alpha' = \dfrac{(R + r)\alpha}{r}$

16–57. $v_B = 12.6$ in./s, $65.7°$ ∡
16–58. $\omega_{AB} = 2.00$ rad/s
16–59. $v_C = 1.06$ m/s $\leftarrow$, $\omega_{BC} = 0.707$ rad/s ↻
16–61. $\omega_{BC} = 2.31$ rad/s ↺, $\omega_{AB} = 3.46$ rad/s ↺
16–62. $\omega_A = 32.0$ rad/s
16–63. $\omega_{CB} = 2.45$ rad/s ↺, $v_C = 2.20$ ft/s $\leftarrow$
16–65. $\omega = 20$ rad/s, $v_A = 2$ ft/s $\rightarrow$
16–66. $\omega = 3.11$ rad/s, $v_O = 0.667$ ft/s $\rightarrow$
16–67. $v_A = 5.16$ ft/s, $\theta = 39.8°$ ∡
16–69. $v_C = 24.6$ m/s $\downarrow$
16–70. $\omega_{BC} = 10.6$ rad/s ↺, $v_C = 29.0$ m/s $\rightarrow$
16–71. $v_P = 4.88$ m/s $\leftarrow$
16–73. $v_E = 4.00$ m/s, $\theta = 52.7°$ ∡
16–74. $\omega_B = 90$ rad/s ↻, $\omega_A = 180$ rad/s ↺
16–75. $\omega_{CD} = 4.03$ rad/s
16–77. $\omega_P = 5$ rad/s, $\omega_A = 1.67$ rad/s
16–78. $\omega_D = 105$ rad/s ↻
16–79. $v_D = 7.07$ m/s
16–82. $\omega_{AB} = 1.24$ rad/s
16–83. $\omega_{BC} = 6.79$ rad/s
16–85. $v_A = 2$ ft/s $\rightarrow$, $v_B = 10$ ft/s $\leftarrow$.
The cylinder slips.
16–86. $v_B = 14$ in./s$\downarrow$,
$v_A = 10.8$ in./s, $\theta = 21.8°$ ∡
16–87. $\omega_{BC} = 8.66$ rad/s ↺, $\omega_{AB} = 4.00$ rad/s ↻
16–89. $v_A = \omega(r_2 - r_1)$
16–90. $v_C = 2.50$ ft/s $\leftarrow$,
$v_D = 9.43$ ft/s, $\theta = 55.8°$ ∡
16–91. $v_C = 2.50$ ft/s $\leftarrow$,
$v_E = 7.91$ ft/s, $\theta = 18.4°$ ∡
16–93. $\omega_{BPD} = 3.00$ rad/s ↻, $v_P = 1.79$ m/s $\leftarrow$
16–94. $\omega_B = 6.67$ rad/s
16–95. $v_A = 60.0$ ft/s $\rightarrow$, $v_C = 220$ ft/s $\leftarrow$,
$v_B = 161$ ft/s, $\theta = 60.3°$ ∡
16–97. $\omega_S = 57.5$ rad/s ↺, $\omega_{OA} = 10.6$ rad/s ↺
16–98. $\omega_S = 15.0$ rad/s, $\omega_R = 3.00$ rad/s
16–99. $\omega_{CD} = 57.7$ rad/s ↺
16–101. $\omega_R = 4$ rad/s
16–102. $\omega_R = 4$ rad/s
16–103. $v_C = 3.86$ m/s $\leftarrow$, $a_C = 17.7$ m/s^2 $\leftarrow$
16–105. $\alpha = 0.0962$ rad/s^2 ↻, $a_A = 0.385$ ft/s^2 $\rightarrow$
16–106. $a_C = 13.0$ m/s^2 $\swarrow$, $\alpha_{BC} = 12.4$ rad/s^2 ↻
16–107. $\omega = 6.67$ rad/s ↺, $v_B = 4.00$ m/s $\searrow$
$\alpha = 15.7$ rad/s^2 ↻, $a_B = 24.8$ m/s^2 $\nwarrow$
16–109. $\omega_{BC} = 0$, $\omega_{CD} = 4.00$ rad/s ↻,
$\alpha_{BC} = 6.16$ rad/s^2 ↻, $\alpha_{CD} = 21.9$ rad/s^2 ↻
16–110. $\omega_C = 20.0$ rad/s ↺, $\alpha_C = 127$ rad/s ↻
16–111. $\alpha_{AB} = 4.62$ rad/s^2 ↺,
$a_B = 13.3$ m/s^2, $\theta = 37.0°$ ∡
16–113. $v_A = 0.424$ m/s, $\theta_v = 45°$ ∡,
$a_A = 0.806$ m/s^2, $\theta_a = 7.13°$ ∡

16–114. $v_B = 0.6\ \text{m/s} \downarrow$,
$a_B = 1.84\ \text{m/s}^2$, $\theta = 60.6°$ ∡

16–115. $v_B = 4v \rightarrow$,
$v_A = 2\sqrt{2}v$, $\theta = 45°$ ∡,
$a_B = \dfrac{2v^2}{r} \downarrow$, $a_A = \dfrac{2v^2}{r} \rightarrow$

16–117. $a_C = 10.0\ \text{m/s}^2$, $\theta = 2.02°$ ∡

16–118. $\alpha = 40.0\ \text{rad/s}^2$, $a_A = 2.00\ \text{m/s}^2 \leftarrow$

16–119. $v_B = 1.58\omega a$, $a_B = 1.58\alpha a - 1.77\omega^2 a$

16–121. $\omega_{AC} = 0$, $\omega_F = 10.7\ \text{rad/s}$ ↻,
$\alpha_{AC} = 28.7\ \text{rad/s}^2$ ↻

16–122. $\omega_{CD} = 7.79\ \text{rad/s}$ ↺, $\alpha_{CD} = 136\ \text{rad/s}^2$ ↻

16–123. $v_C = 1.56\ \text{m/s} \leftarrow$,
$a_C = 29.7\ \text{m/s}^2$, $\theta = 24.1°$ ∡

16–125. $\omega = 4.73\ \text{rad/s}$ ↺, $\alpha = 131\ \text{rad/s}^2$ ↻

16–126. $\omega_{AB} = 7.17\ \text{rad/s}$ ↻, $\alpha_{AB} = 23.1\ \text{rad/s}^2$ ↺

16–127. $\alpha_{AB} = 3.70\ \text{rad/s}^2$ ↻

16–129. $\mathbf{v}_B = \{0.6\mathbf{i} + 2.4\mathbf{j}\}\ \text{m/s}$,
$\mathbf{a}_B = \{-14.2\mathbf{i} + 8.40\mathbf{j}\}\ \text{m/s}^2$

16–130. $v_B = 1.30\ \text{ft/s}$, $a_B = 0.6204\ \text{ft/s}^2$

16–131. $\mathbf{v}_m = \{7.5\mathbf{i} - 5\mathbf{j}\}\ \text{ft/s}$, $\mathbf{a}_m = \{5\mathbf{i} + 3.75\mathbf{j}\}\ \text{ft/s}^2$

16–133. $\mathbf{v}_A = \{-17.2\mathbf{i} + 12.5\mathbf{j}\}\ \text{m/s}$,
$\mathbf{a}_A = \{349\mathbf{i} + 597\mathbf{j}\}\ \text{m/s}^2$

16–134. $\mathbf{a}_A = \{-5.60\mathbf{i} - 16\mathbf{j}\}\ \text{m/s}^2$

16–135. $v_C = 2.40\ \text{m/s}$, $\theta = 60°$ ∡

16–137. $(\mathbf{v}_{B/A})_{xyz} = \{31.0\mathbf{i}\}\ \text{m/s}$,
$(\mathbf{a}_{B/A})_{xyz} = \{-14.0\mathbf{i} - 206\mathbf{j}\}\ \text{m/s}^2$

16–138. $v_B = 7.7\ \text{m/s}$, $a_B = 201\ \text{m/s}^2$

16–139. $\omega_{CB} = 1.33\ \text{rad/s}$ ↺, $\alpha_{CD} = 3.08\ \text{rad/s}^2$ ↻

16–141. $\omega_{CD} = 3.00\ \text{rad/s}$ ↻, $\alpha_{CD} = 12.0\ \text{rad/s}^2$ ↻

16–142. $\mathbf{v}_C = \{-0.944\mathbf{i} + 2.02\mathbf{j}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-11.2\mathbf{i} - 4.15\mathbf{j}\}\ \text{m/s}^2$

16–143. $\omega_{AB} = 5\ \text{rad/s}$ ↻, $\alpha_{AB} = 2.5\ \text{rad/s}^2$ ↻

16–145. $\mathbf{v}_C = \{-7.00\mathbf{i} + 17.3\mathbf{j}\}\ \text{ft/s}$,
$\mathbf{a}_C = \{-34.6\mathbf{i} - 15.5\mathbf{j}\}\ \text{ft/s}^2$

16–146. $\mathbf{v}_C = \{-7.00\mathbf{i} + 17.3\mathbf{j}\}\ \text{ft/s}$,
$\mathbf{a}_C = \{-38.8\mathbf{i} - 6.84\mathbf{j}\}\ \text{ft/s}^2$

16–147. $\omega_{AB} = 0.667\ \text{rad/s}$ ↺, $\alpha_{AB} = 3.08\ \text{rad/s}^2$ ↻

16–149. $(\mathbf{v}_{\text{rel}})_{xyz} = \mathbf{0}$, $(\mathbf{a}_{\text{rel}})_{xyz} = \{1\mathbf{i}\}\ \text{m/s}^2$

16–150. $\omega_{DC} = 2.96\ \text{rad/s}$ ↻

16–151. $\omega_{AC} = 0$, $\alpha_{AC} = 14.4\ \text{rad/s}^2$ ↻

Chapter 17

17–1. $I_y = \dfrac{1}{3} m l^2$

17–2. $m = \pi h R^2\left(k + \dfrac{aR^2}{2}\right)$, $I_z = \dfrac{\pi h R^4}{2}\left[k + \dfrac{2aR^2}{3}\right]$

17–3. $I_z = mR^2$

17–5. $k_x = 1.20\ \text{in.}$

17–6. $I_x = \dfrac{2}{5} m r^2$

17–7. $I_x = \dfrac{93}{70} mb^2$

17–9. $I_y = \dfrac{m}{6}(a^2 + h^2)$

17–10. $k_O = 2.17\ \text{m}$

17–11. $I_O = 1.36\ \text{kg} \cdot \text{m}^2$

17–13. $I_A = 7.67\ \text{kg} \cdot \text{m}^2$

17–14. $I_A = 222\ \text{slug} \cdot \text{ft}^2$

17–15. $I_O = 6.23\ \text{kg} \cdot \text{m}^2$

17–17. $I_G = 0.230\ \text{kg} \cdot \text{m}^2$

17–18. $I_O = 0.560\ \text{kg} \cdot \text{m}^2$

17–19. $I_G = 118\ \text{slug} \cdot \text{ft}^2$

17–21. $\bar{y} = 1.78\ \text{m}$, $I_G = 4.45\ \text{kg} \cdot \text{m}^2$

17–22. $I_x = 3.25\ \text{g} \cdot \text{m}^2$

17–23. $I_{x'} = 7.19\ \text{g} \cdot \text{m}^2$

17–25. $F = 5.96\ \text{lb}$, $N_B = 99.0\ \text{lb}$, $N_A = 101\ \text{lb}$

17–26. $A_y = 72.6\ \text{kN}$, $B_y = 71.6\ \text{kN}$, $a_G = 0.250\ \text{m/s}^2$

17–27. $N_A = 1393\ \text{lb}$, $N_B = 857\ \text{lb}$, $t = 2.72\ \text{s}$

17–29. $a = 2.74\ \text{m/s}^2$, $T = 25.1\ \text{kN}$

17–30. $N = 29.6\ \text{kN}$, $V = 0$, $M = 51.2\ \text{kN} \cdot \text{m}$

17–31. $h = 3.12\ \text{ft}$

17–33. $P = 579\ \text{N}$

17–34. $a = 4\ \text{m/s}^2 \rightarrow$, $N_B = 1.14\ \text{kN}$, $N_A = 327\ \text{N}$

17–35. $a_G = 13.3\ \text{ft/s}^2$

17–37. $P = 785\ \text{N}$

17–38. $P = 314\ \text{N}$

17–39. $N = 0.433wx$, $V = 0.25wx$, $M = 0.125wx^2$

17–41. $B_x = 73.9\ \text{lb}$, $B_y = 69.7\ \text{lb}$, $N_A = 120\ \text{lb}$

17–42. $a = 2.01\ \text{m/s}^2$.
The crate slips.

17–43. $a = 2.68\ \text{ft/s}^2$, $N_A = 26.9\ \text{lb}$, $N_B = 123\ \text{lb}$

17–45. $T = 15.7\ \text{kN}$, $C_x = 8.92\ \text{kN}$, $C_y = 16.3\ \text{kN}$

17–46. $a = 9.81\ \text{m/s}^2$, $C_x = 12.3\ \text{kN}$, $C_y = 12.3\ \text{kN}$

17–47. $h_{\max} = 3.16\ \text{ft}$, $F_A = 248\ \text{lb}$, $N_A = 400\ \text{lb}$

17–49. $F_{AB} = 112\ \text{N}$, $C_x = 26.2\ \text{N}$, $C_y = 49.8\ \text{N}$

17–50. $P = 765\ \text{N}$

17–51. $T = 1.52\ \text{kN}$, $\theta = 18.6°$

17–53. $\alpha = 9.67\ \text{rad/s}^2$

17–54. $F_C = 16.1\ \text{lb}$, $N_C = 159\ \text{lb}$

17–55. $\alpha = 2.62\ \text{rad/s}^2$

17–57. $\omega = 56.2\ \text{rad/s}$, $A_x = 0$, $A_y = 98.1\ \text{N}$

17–58. $\alpha = 14.7\ \text{rad/s}^2$, $A_x = 88.3\ \text{N}$, $A_y = 147\ \text{N}$

17–59. $F_A = \dfrac{3}{2} mg$

17–61. $\alpha = 0.694\ \text{rad/s}^2$

17–62. $\omega = 10.9\ \text{rad/s}$

17–63. $\omega = 9.45\ \text{rad/s}$

17–65. $M = 0.233\ \text{lb} \cdot \text{ft}$

17–67. $\alpha = 8.68\ \text{rad/s}^2$, $A_n = 0$, $A_t = 106\ \text{N}$

17–69. $\alpha = 7.28\ \text{rad/s}^2$

17–70. $F = 22.1\ \text{N}$

17–71. $\omega = 0.474\ \text{rad/s}$

17–73. $t = 6.71$ s
17–74. $\alpha = 14.2\ \text{rad/s}^2$
17–75. $A_x = 89.2$ N, $A_y = 66.9$ N, $t = 1.25$ s
17–77. $t = 1.09$ s
17–78. $v = 4.88$ ft/s
17–79. $a = 2.97\ \text{m/s}^2$
17–81. $A_x = 0$, $A_y = 289$ N, $\alpha = 23.1\ \text{rad/s}^2$
17–82. $N_A = 177$ kN, $V_A = 5.86$ kN, $M_A = 50.7$ kN·m
17–83. $M = 0.3gml$
17–85. $N = wx\left[\frac{\omega^2}{g}\left(L - \frac{x}{2}\right) + \cos\theta\right]$,
$V = wx\sin\theta$, $M = \frac{1}{2}wx^2\sin\theta$
17–86. $\alpha = 12.5$ rad/s ↻, $a_G = 18.75\ \text{m/s}^2$ ↓
17–87. $N_B = 2.89$ kN,
$A_x = 0$, $A_y = 2.89$ kN
17–89. $\omega = 800$ rad/s
17–91. $\alpha = 5.62\ \text{rad/s}^2$, $T = 196$ N
17–93. $\alpha = 2.45\ \text{rad/s}^2$ ↻, $N_B = 2.23$ N, $N_A = 33.3$ N
17–94. $\alpha = 4.32\ \text{rad/s}^2$
17–95. $\theta = 46.9°$
17–97. $\alpha = 0.500\ \text{rad/s}^2$
17–98. $\alpha = 15.6\ \text{rad/s}^2$
17–99. $a_A = 26.7\ \text{m/s}^2$ →
17–101. $F = 42.3$ N
17–102. $\alpha = 4.01\ \text{rad/s}^2$
17–103. $A_y = 15.0$ lb, $A_x = 0.776$ lb, $\alpha = 1.67\ \text{rad/s}^2$
17–105. $\alpha = 18.9\ \text{rad/s}^2$, $P = 76.4$ lb
17–106. $\alpha = \frac{6P}{mL}$, $a_B = \frac{2P}{m}$
17–107. $\alpha = \frac{6(P - \mu_k mg)}{mL}$, $a_B = \frac{2(P - \mu_k mg)}{m}$
17–109. $\alpha = 3\ \text{rad/s}^2$
17–110. $\alpha = 14.5\ \text{rad/s}^2$, $t = 0.406$ s
17–111. The disk does not slip.
17–113. $a_G = \mu_k g$ ←, $\alpha = \frac{2\mu_k g}{r}$ ↻
17–114. $\omega = \frac{1}{3}\omega_0$, $t = \frac{\omega_0 r}{3\mu_k g}$
17–115. $\alpha_A = 43.6\ \text{rad/s}^2$ ↻, $\alpha_B = 43.6\ \text{rad/s}^2$ ↺, $T = 19.6$ N
17–117. $T_A = \frac{4}{7}W$
17–118. $\alpha = 23.4\ \text{rad/s}^2$, $B_y = 9.62$ lb
17–119. $\alpha = \frac{10g}{13\sqrt{2}\,r}$

Chapter 18

18–2. $\omega = 14.0$ rad/s
18–3. $\omega = 14.1$ rad/s
18–5. $\omega = 2.02$ rad/s
18–6. $\omega = 1.78$ rad/s
18–7. $T = 283$ ft·lb
18–9. $\omega = 21.5$ rad/s
18–10. $s = 5.16$ m, $T = 78.5$ N
18–11. $\omega = 14.9$ rad/s
18–13. $\omega = 6.11$ rad/s
18–14. $\omega = 8.64$ rad/s
18–15. $\omega = 3.16$ rad/s
18–17. $\omega = \sqrt{\omega_0^2 + \frac{g}{r^2}s\sin\theta}$
18–18. $v_C = 7.49$ m/s
18–19. $\omega = 6.92$ rad/s
18–21. $s = 0.304$ ft
18–22. $v_C = 19.6$ ft/s
18–23. $\theta = 0.445$ rev
18–25. $s_G = 1.60$ m
18–26. $\omega_2 = 5.37$ rad/s
18–27. $\omega = 44.6$ rad/s
18–29. $v_G = 11.9$ ft/s
18–30. $\omega = 2.50$ rad/s
18–31. $\omega = 5.40$ rad/s
18–33. $\theta = 0.891$ rev, regardless of orientation
18–34. $\omega = 5.74$ rad/s
18–35. $\omega_{AB} = 5.92$ rad/s
18–37. $s_C = 78.0$ mm
18–38. $s = 0.301$ m, $T = 163$ N
18–39. $v_A = 1.29$ m/s
18–41. $s_b = 242$ mm, $T = 67.8$ N
18–42. $v_b = 2.52$ m/s
18–43. $\theta = 48.2°$
18–45. $\omega = 3.78$ rad/s
18–46. $\omega = 3.75$ rad/s
18–47. $\omega = 3.28$ rad/s
18–49. $(\omega_{AB})_2 = (\omega_{BC})_2 = 1.12$ rad/s
18–50. $v_A = 1.40$ m/s
18–51. $\theta_0 = 8.94$ rev
18–53. $\omega_{BC} = 1.34$ rad/s
18–54. $v_b = 15.5$ ft/s
18–55. $v_A = 4.00$ m/s
18–57. $\omega = 12.8$ rad/s
18–58. $k = 18.4$ N/m
18–59. $\omega = 2.67$ rad/s
18–61. $\omega_{AB} = 3.70$ rad/s
18–62. $\omega = 1.80$ rad/s
18–63. $v_A = 21.0$ ft/s
18–65. $\omega = 2.71$ rad/s
18–66. $k = 100$ lb/ft
18–67. $(v_A)_2 = 7.24$ m/s

Chapter 19

19–5. $\int M\,dt = 0.833\ \text{kg}\cdot\text{m}^2/\text{s}$

19–6. $\omega = 0.0178\ \text{rad/s}$

19–7. $v_B = 24.1\ \text{m/s}$

19–9. $\omega_2 = 103\ \text{rad/s}$

19–10. $t = 0.6125\ \text{s}$

19–11. $\omega_2 = 53.7\ \text{rad/s}$

19–13. $y = \frac{2}{3}l$

19–14. $d = \frac{2}{3}l$

19–15. (a) $\omega_{BC} = 68.7\ \text{rad/s}$, (b) $\omega_{BC} = 66.8\ \text{rad/s}$, (c) $\omega_{BC} = 68.7\ \text{rad/s}$

19–17. $v_G = 26.8\ \text{ft/s}$

19–18. $v_G = 2\ \text{m/s}$, $\omega = 3.90\ \text{rad/s}$

19–19. $v_A = 24.1\ \text{m/s}$

19–21. $\omega = 12.7\ \text{rad/s}$

19–22. $\omega_A = 47.3\ \text{rad/s}$

19–23. $t = 1.32\ \text{s}$

19–25. $t = 1.04\ \text{s}$

19–26. $\omega = 9\ \text{rad/s}$

19–27. $v_B = 1.59\ \text{m/s}$

19–29. $\omega = 1.91\ \text{rad/s}$

19–30. $\omega_2 = 0.656\ \text{rad/s}$, $\theta = 18.8^\circ$

19–31. $\omega_2 = 0.577\ \text{rad/s}$, $\theta = 15.8^\circ$

19–33. $\omega_2 = 2.55\ \text{rev/s}$

19–34. $\omega = 0.190\ \text{rad/s}$

19–35. $\omega = 0.0906\ \text{rad/s}$

19–37. $\omega = 22.7\ \text{rad/s}$

19–38. $h_C = 0.500\ \text{ft}$

19–39. $\omega_2 = 1.01\ \text{rad/s}$

19–41. $\theta = 66.9^\circ$

19–42. $\omega_2 = 57\ \text{rad/s}$, $U_F = 367\ \text{J}$

19–43. $\omega_2 = 3.47\ \text{rad/s}$

19–45. $v = 5.96\ \text{ft/s}$

19–46. $h = \frac{7}{5}r$

19–47. $\theta = 50.2^\circ$

19–49. $(v_D)_3 = 1.54\ \text{m/s}$, $\omega_3 = 0.934\ \text{rad/s}$

19–50. $\omega_1 = 7.17\ \text{rad/s}$

19–51. $\theta = \tan^{-1}\left(\sqrt{\frac{7}{5}e}\right)$

19–53. $\omega_3 = 2.73\ \text{rad/s}$

19–54. $\omega = \sqrt{7.5\frac{g}{L}}$

19–55. $h_B = 0.980\ \text{ft}$

19–57. $(v_G)_{y2} = e(v_G)_{y1}\uparrow$, $(v_G)_{x2} = \frac{5}{7}\left((v_G)_{x1} - \frac{2}{5}\omega_1 r\right)\leftarrow$

19–58. $\theta_1 = 39.8^\circ$

Chapter 20

20–1. (a) $\boldsymbol{\alpha} = \omega_s\omega_t\,\mathbf{j}$, (b) $\boldsymbol{\alpha} = -\omega_s\omega_t\,\mathbf{k}$

20–2. $\mathbf{v}_A = \{-0.225\mathbf{i}\}\ \text{m/s}$, $\mathbf{a}_A = \{-0.1125\mathbf{j} - 0.130\mathbf{k}\}\ \text{m/s}^2$

20–3. $\mathbf{v}_A = \{-5.20\mathbf{i} - 12\mathbf{j} + 20.8\mathbf{k}\}\ \text{ft/s}$, $\mathbf{a}_A = \{-24.1\mathbf{i} - 13.3\mathbf{j} - 7.20\mathbf{k}\}\ \text{ft/s}^2$

20–5. $(\omega_C)_{DE} = 40\ \text{rad/s}$, $(\omega_{DE})_y = 5\ \text{rad/s}$

20–6. $\boldsymbol{\omega} = \{-8.24\mathbf{j}\}\ \text{rad/s}$, $\boldsymbol{\alpha} = \{24.7\mathbf{i} - 5.49\mathbf{j}\}\ \text{rad/s}^2$

20–7. $\mathbf{v}_A = \{-7.79\mathbf{i} - 2.25\mathbf{j} + 3.90\mathbf{k}\}\ \text{ft/s}$, $\mathbf{a}_A = \{8.30\mathbf{i} - 35.2\mathbf{j} + 7.02\mathbf{k}\}\ \text{ft/s}^2$

20–9. $\mathbf{v}_B = \{-0.4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\}\ \text{m/s}$, $\mathbf{a}_B = \{-8.20\mathbf{i} + 40.6\mathbf{j} - \mathbf{k}\}\ \text{rad/s}^2$

20–10. $\boldsymbol{\omega} = \{42.4\mathbf{j} + 43.4\mathbf{k}\}\ \text{rad/s}$, $\boldsymbol{\alpha} = \{-42.4\mathbf{i}\}\ \text{rad/s}^2$

20–11. $\boldsymbol{\omega} = \{2\mathbf{i} + 42.4\mathbf{j} + 43.4\mathbf{k}\}\ \text{rad/s}$, $\boldsymbol{\alpha} = \{-42.4\mathbf{i} - 82.9\mathbf{j} + 84.9\mathbf{k}\}\ \text{rad/s}^2$

20–13. $v_B = 0$, $v_C = 0.283\ \text{m/s}$, $a_B = 1.13\ \text{m/s}^2$, $a_C = 1.60\ \text{m/s}^2$

20–14. $\mathbf{v}_C = \{1.8\mathbf{j} - 1.5\mathbf{k}\}\ \text{m/s}$, $\mathbf{a}_C = \{-36.6\mathbf{i} + 0.45\mathbf{j} - 0.9\mathbf{k}\}\ \text{m/s}^2$

20–15. $\mathbf{v}_A = \{-8.66\mathbf{i} + 8.00\mathbf{j} - 13.9\mathbf{k}\}\ \text{ft/s}$, $\mathbf{a}_A = \{-24.8\mathbf{i} + 8.29\mathbf{j} - 30.9\mathbf{k}\}\ \text{ft/s}^2$

20–17. $\mathbf{v}_A = \{-1.80\mathbf{i}\}\ \text{ft/s}$, $\mathbf{a}_A = \{-0.750\mathbf{i} - 0.720\mathbf{j} - 0.831\mathbf{k}\}\ \text{ft/s}^2$

20–18. $\boldsymbol{\omega}_P = \{-40\mathbf{j}\}\ \text{rad/s}$, $\boldsymbol{\alpha}_B = \{-6400\mathbf{i}\}\ \text{rad/s}^2$

20–19. $\boldsymbol{\omega} = \{4.35\mathbf{i} + 12.7\mathbf{j}\}\ \text{rad/s}$, $\boldsymbol{\alpha} = \{-26.1\mathbf{k}\}\ \text{rad/s}^2$

20–21. $\boldsymbol{\omega} = \{30\mathbf{j} - 5\mathbf{k}\}\ \text{rad/s}$, $\boldsymbol{\alpha} = \{150\mathbf{i}\}\ \text{rad/s}^2$

20–22. $\mathbf{v}_A = \{10\mathbf{i} + 14.7\mathbf{j} - 19.6\mathbf{k}\}\ \text{ft/s}$, $\mathbf{a}_A = \{-6.12\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\}\ \text{ft/s}^2$

20–23. $\omega_A = 47.8\ \text{rad/s}$, $\omega_B = 7.78\ \text{rad/s}$

20–25. $\boldsymbol{\omega}_{BC} = \{0.204\mathbf{i} - 0.612\mathbf{j} + 1.36\mathbf{k}\}\ \text{rad/s}$, $\mathbf{v}_B = \{-0.333\mathbf{j}\}\ \text{m/s}$

20–26. $\boldsymbol{\omega}_{AB} = \{-1.00\mathbf{i} - 0.500\mathbf{j} + 2.50\mathbf{k}\}\ \text{rad/s}$, $\mathbf{v}_B = \{-2.50\mathbf{j} - 2.50\mathbf{k}\}\ \text{m/s}$

20–27. $\boldsymbol{\alpha}_{AB} = \{-7.9\mathbf{i} - 3.95\mathbf{j} + 4.75\mathbf{k}\}\ \text{rad/s}^2$, $\mathbf{a}_B = \{-19.75\mathbf{j} - 19.75\mathbf{k}\}\ \text{m/s}^2$

20–29. $\mathbf{a}_B = \{-37.6\mathbf{j}\}\ \text{ft/s}^2$

20–30. $\mathbf{v}_B = \{-1.92\mathbf{j} + 2.56\mathbf{k}\}\ \text{m/s}$

20–31. $v_B = 5.00\ \text{m/s}$, $\boldsymbol{\omega}_{AB} = \{-4.00\mathbf{i} - 0.600\mathbf{j} - 1.20\mathbf{k}\}\ \text{rad/s}$

20–33. $\boldsymbol{\omega}_{BD} = \{-1.20\mathbf{j}\}\ \text{rad/s}$

20–34. $\boldsymbol{\alpha}_{BD} = \{-8.00\mathbf{j}\}\ \text{rad/s}^2$
20–35. $\boldsymbol{\omega}_{AB} = \{-0.500\mathbf{i} + 0.667\mathbf{j} - 1.00\mathbf{k}\}\ \text{rad/s}$
$\mathbf{v}_B = \{-7.50\mathbf{j}\}\ \text{ft/s}$
20–37. $\mathbf{v}_C = \{-1.00\mathbf{i} + 5.00\mathbf{j} + 0.800\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\}\ \text{m/s}^2$
20–38. $\mathbf{v}_C = \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-28.2\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\}\ \text{m/s}^2$
20–39. $\mathbf{v}_B = \{-2.75\mathbf{i} - 2.50\mathbf{j} + 3.17\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{2.50\mathbf{i} - 2.24\mathbf{j} - 0.00389\mathbf{k}\}\ \text{ft/s}^2$
20–41. $\mathbf{v}_C = \{3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-13.0\mathbf{i} + 28.5\mathbf{j} - 10.2\mathbf{k}\}\ \text{m/s}^2$
20–42. $\mathbf{v}_B = \{-17.8\mathbf{i} - 3\mathbf{j} + 5.20\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{9\mathbf{i} - 29.4\mathbf{j} - 1.5\mathbf{k}\}\ \text{m/s}^2$
20–43. $\mathbf{v}_B = \{-17.8\mathbf{i} - 3\mathbf{j} + 5.20\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{3.05\mathbf{i} - 30.9\mathbf{j} + 1.10\mathbf{k}\}\ \text{m/s}^2$
20–45. $\mathbf{v}_P = \{-0.849\mathbf{i} + 0.849\mathbf{j} + 0.566\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_P = \{-5.09\mathbf{i} - 7.35\mathbf{j} + 6.79\mathbf{k}\}\ \text{m/s}^2$
20–46. $\mathbf{v}_A = \{-8.66\mathbf{i} + 2.26\mathbf{j} + 2.26\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_A = \{-22.6\mathbf{i} - 47.8\mathbf{j} + 45.3\mathbf{k}\}\ \text{m/s}^2$
20–47. $\mathbf{v}_A = \{-8.66\mathbf{i} + 2.26\mathbf{j} + 2.26\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_A = \{-26.1\mathbf{i} - 44.4\mathbf{j} + 7.92\mathbf{k}\}\ \text{m/s}^2$
20–49. $\mathbf{v}_P = \{-9.80\mathbf{i} + 14.4\mathbf{j} + 48.0\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{-160\mathbf{i} + 5.16\mathbf{j} - 13\mathbf{k}\}\ \text{ft/s}^2$
20–50. $\mathbf{v}_P = \{-25.5\mathbf{i} - 13.4\mathbf{j} + 20.5\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{161\mathbf{i} - 249\mathbf{j} - 39.6\mathbf{k}\}\ \text{ft/s}^2$
20–51. $\mathbf{v}_P = \{-25.5\mathbf{i} - 13.4\mathbf{j} + 20.5\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{161\mathbf{i} - 243\mathbf{j} - 33.9\mathbf{k}\}\ \text{ft/s}^2$
20–53. $\mathbf{v}_A = \{-8.66\mathbf{i} + 8\mathbf{j} - 13.9\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_A = \{-17.9\mathbf{i} + 8.29\mathbf{j} - 30.9\mathbf{k}\}\ \text{ft/s}^2$,
20–54. $\mathbf{v}_C = \{-1.73\mathbf{i} - 5.77\mathbf{j} + 7.06\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_C = \{9.88\mathbf{i} - 72.8\mathbf{j} + 0.365\mathbf{k}\}\ \text{ft/s}^2$

Chapter 21

21–2. $I_{\bar{y}} = \dfrac{3m}{80}(h^2 + 4a^2),\ I_{y'} = \dfrac{m}{20}(2h^2 + 3a^2)$
21–3. $I_y = 2614\ \text{slug}\cdot\text{ft}^2$
21–5. $I_{yz} = \dfrac{m}{6}ah$
21–6. $I_{xy} = \dfrac{m}{12}a^2$
21–7. $I_{xy} = 636\rho$
21–9. $I_{z'z'} = 0.0961\ \text{slug}\cdot\text{ft}^2$
21–10. $k_y = 2.35\ \text{ft},\ k_x = 1.80\ \text{ft}$
21–11. $I_{aa} = \dfrac{m}{12}(3a^2 + 4h^2)$
21–13. $I_{yz} = 0$
21–14. $I_{xy} = 0.32\ \text{kg}\cdot\text{m}^2,\ I_{yz} = 0.08\ \text{kg}\cdot\text{m}^2,\ I_{xz} = 0$
21–15. $I_{z'} = 0.0595\ \text{kg}\cdot\text{m}^2$
21–17. $\bar{y} = 0.5\ \text{ft},\ \bar{x} = -0.667\ \text{ft},\ I_{x'} = 0.0272\ \text{slug}\cdot\text{ft}^2$,
$I_{y'} = 0.0155\ \text{slug}\cdot\text{ft}^2,\ I_{z'} = 0.0427\ \text{slug}\cdot\text{ft}^2$
21–18. $I_x = 0.455\ \text{slug}\cdot\text{ft}^2$
21–19. $I_{aa} = 1.13\ \text{slug}\cdot\text{ft}^2$
21–21. $I_z = 0.0880\ \text{slug}\cdot\text{ft}^2$
21–25. $\mathbf{H} = \{-477(10^{-6})\,\mathbf{i} + 198(10^{-6})\,\mathbf{j} + 0.169\,\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21–26. $\omega_2 = 61.7\ \text{rad/s}$
21–27. $\omega_2 = 87.2\ \text{rad/s}$
21–29. $\omega_x = 19.7\ \text{rad/s}$
21–30. $h = 2.24\ \text{in.}$
21–31. $T = 0.0920\ \text{ft}\cdot\text{lb}$
21–33. $\omega_p = 4.82\ \text{rad/s}$
21–34. $\mathbf{H}_A = \{-2000\mathbf{i} - 55\,000\mathbf{j} + 22\,500\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21–35. $T = 37.0\ \text{MJ}$
21–37. $\boldsymbol{\omega} = \{-0.750\mathbf{j} + 1.00\mathbf{k}\}\ \text{rad/s}$
21–38. $T = 1.14\ \text{J}$
21–39. $H_z = 0.4575\ \text{kg}\cdot\text{m}^2/\text{s}$
21–41. $\Sigma M_x = (I_x\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z)$,
$-\ \Omega_z(I_y\omega_y - I_{yz}\omega_z - I_{yx}\omega_x)$,
$+\ \Omega_y(I_z\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)$
Similarly for ΣM_y and ΣM_z.
21–43. $B_z = 4\ \text{lb},\ A_x = -2.00\ \text{lb},\ A_y = 0.627\ \text{lb}$,
$B_x = 2.00\ \text{lb},\ B_y = -1.37\ \text{lb}$
21–45. $A_Z = 1.46\ \text{lb},\ B_Z = 13.5\ \text{lb},\ A_X = A_Y = B_X = 0$,
21–46. $\dot{\omega}_x = -14.7\ \text{rad/s}^2,\ B_z = 77.7\ \text{N},\ B_y = 3.33\ \text{N}$,
$A_x = 0,\ A_y = 6.67\ \text{N},\ A_z = 81.75\ \text{N}$
21–47. $\dot{\omega}_x = 9.285\ \text{rad/s}^2,\ B_z = 97.7\ \text{N},\ B_y = 3.33\ \text{N}$,
$A_x = 0,\ A_y = 6.67\ \text{N},\ A_z = 122\ \text{N}$
21–49. $\dot{\omega}_z = 200\ \text{rad/s}^2,\ D_y = -12.9\ \text{N},\ D_x = -37.5\ \text{N}$,
$C_x = -37.5\ \text{N},\ C_y = -11.1\ \text{N},\ C_z = 36.8\ \text{N}$
21–50. $T_B = 47.1\ \text{lb},\ M_y = 0,\ M_z = 0,\ A_x = 0$,
$A_y = -93.2\ \text{lb},\ A_z = 57.1\ \text{lb}$
21–51. $\dot{\omega}_y = -102\ \text{rad/s}^2,\ A_x = B_x = 0,\ A_y = 0$,
$A_z = 297\ \text{N},\ B_z = -143\ \text{N}$
21–53. $M_z = 0,\ A_x = 0,\ M_y = 0,\ \theta = 64.1°$,
$A_y = 1.30\ \text{lb},\ A_z = 20.2\ \text{lb}$
21–54. $N = 148\ \text{N},\ F_f = 0$
21–55. $(M_0)_x = 72.0\ \text{N}\cdot\text{m},\ (M_0)_z = 0$
21–57. $M_x = -\dfrac{4}{3}ml^2\,\omega_s\omega_p\cos\theta$,
$M_y = \dfrac{1}{3}ml^2\,\omega_p{}^2\sin 2\theta,\ M_z = 0$
21–58. $B_x = 0,\ B_y = -3.90\ \text{lb},\ A_y = -1.69\ \text{lb}$,
$A_z = B_z = 7.5\ \text{lb}$
21–59. $\Sigma M_x = 0,\ \Sigma M_y = (-0.036\sin\theta)\ \text{N}\cdot\text{m}$,
$\Sigma M_z = (0.003\sin 2\theta)\ \text{N}\cdot\text{m}$
21–61. $\alpha = 69.3°,\ \beta = 128°,\ \gamma = 45°$. No, the orientation will not be the same for any order. Finite rotations are not vectors.
21–62. $\omega_P = 27.9\ \text{rad/s}$
21–63. $\omega_R = 368\ \text{rad/s}$
21–65. $\omega_P = 1.19\ \text{rad/s}$
21–66. $M_x = 328\ \text{N}\cdot\text{m}$

21–67. $\dot{\phi} = \left(\dfrac{2g\cos\theta}{a + r\cos\theta}\right)^{1/2}$
21–69. $\omega_s = 3.63(10^3)$ rad/s
21–70. $\theta = 68.1°$
21–71. $\dot{\phi} = 81.7$ rad/s, $\dot{\psi} = 212$ rad/s, regular precession
21–74. $\dot{\psi} = 2.35$ rev/h
21–75. $\alpha = 90°$, $\beta = 9.12°$, $\gamma = 80.9°$
21–77. $H_G = 12.5$ Mg·m²/s
21–78. $\dot{\phi} = 3.32$ rad/s

Chapter 22

22–1. $\ddot{y} + 56.1\,y = 0$, $y|_{t=0.22\text{ s}} = 0.192$ m
22–2. $x = -0.05\cos(20t)$
22–3. $y = 0.107\sin(7.00t) + 0.100\cos(7.00t)$, $\phi = 43.0°$
22–5. $\omega_n = 49.5$ rad/s, $\tau = 0.127$ s
22–6. $x = \{-0.126\sin(3.16t) - 0.09\cos(3.16t)\}$ m, $C = 0.155$ m
22–7. $\omega_n = 19.7$ rad/s, $C = 1$ in. $y = (0.0833\cos 19.7t)$ ft
22–9. $\omega_n = 8.16$ rad/s, $x = -0.05\cos(8.16t)$, $C = 50$ mm
22–10. $\tau = 2\pi\sqrt{\dfrac{2mL}{3mg + 6kL}}$
22–11. $\tau = 1.45$ s
22–13. $\tau = 2\pi\sqrt{\dfrac{k_G^2 + d^2}{gd}}$
22–14. $k = 90.8$ lb·ft/rad
22–15. $k = 1.36$ N/m, $m_B = 3.58$ kg
22–17. $k_1 = 2067$ N/m, $k_2 = 302$ N/m, or vice versa
22–18. $m_B = 21.2$ kg, $k = 609$ N/m
22–19. $y = 503$ mm
22–21. $x = 0.167\cos 6.55\,t$
22–22. $\omega_n = \sqrt{\dfrac{3g(4R^2 - l^2)^{1/2}}{6R^2 - l^2}}$
22–23. $\tau = 1.66$ s
22–25. $f = 0.900$ Hz
22–26. $\tau = 2\pi k_O\sqrt{\dfrac{m}{C}}$
22–27. $\omega_n = 3.45$ rad/s
22–29. $\tau = 2\pi\sqrt{\dfrac{l}{2g}}$
22–30. $\ddot{x} + 333x = 0$
22–31. $\tau = 1.52$ s
22–33. $\tau = 0.774$ s
22–34. $\ddot{\theta} + 468\theta = 0$
22–35. $\tau = 0.487$ s
22–37. $E = 0.175\dot{\theta}^2 + 10\,\theta^2$, $\tau = 0.830$ s
22–38. $\tau = \pi\sqrt{\dfrac{m}{k}}$
22–39. $f = 1.28$ Hz
22–41. $x = A\sin\omega_n t + B\cos\omega_n t + \; + \dfrac{F_0/k}{1 - (\omega/\omega_n)^2}\cos\omega t$
22–42. $y = A\sin\omega_n t + B\cos\omega_n t + \dfrac{F_O}{k}$
22–43. $y = \{-0.0232\sin 8.97\,t + 0.333\cos 8.97\,t + 0.0520\sin 4t\}$ ft
22–45. $y = A\sin\omega_n t + B\cos\omega_n t + \dfrac{\delta_0}{1 - (\omega/\omega_n)^2}\sin\omega t$
22–46. $y = (361\sin 7.75t + 100\cos 7.75t, - 350\sin 8t)$ mm
22–47. $C = \dfrac{3F_O}{\frac{3}{2}(mg + Lk) - mL\omega^2}$
22–49. $(x_p)_{\max} = 29.5$ mm
22–50. $\ddot{\theta} + \dfrac{4c}{m}\dot{\theta} + \dfrac{k}{m}\dot{\theta} = 0$
22–51. $(v_p)_{\max} = 0.3125$ m/s
22–53. $\omega = 14.0$ rad/s
22–54. $(x_p)_{\max} = 14.6$ mm
22–55. $(x_p)_{\max} = 35.5$ mm
22–57. $\omega = 19.7$ rad/s
22–58. $C = 0.490$ in.
22–59. $\omega = 19.0$ rad/s
22–61. $(x_p)_{\max} = 4.53$ mm
22–62. $Y = \dfrac{mr\omega^2L^3}{48EI - M\omega^2L^3}$
22–63. $\omega = 12.2$ rad/s, $\omega = 7.07$ rad/s
22–65. $\phi' = 9.89°$
22–66. MF = 0.997
22–67. $y = \{-0.0702\,e^{-3.57t}\sin(8.540)\}$ m
22–69. $F = 2c\dot{y}$, $c_c = 2m\sqrt{\dfrac{k}{m}}$, $c < \sqrt{mk}$
22–71. $\omega = 21.1$ rad/s
22–73. $1.55\ddot{\theta} + 540\dot{\theta} + 200\theta = 0$, $(c_{dp})_c = 3.92$ lb·s/ft
22–74. $c_c = \sqrt{8(m + M)k}$, $x_{\max} = \left[\dfrac{m}{e}\sqrt{\dfrac{1}{2k(m + M)}}\right]v_0$
22–75. $x_{\max} = \dfrac{2mv_0}{\sqrt{8k(m + M) - c^2}}\,e^{-\pi c/(2\sqrt{8k(m+M)-c^2})}$
22–77. $Lq + Rq + \left(\dfrac{1}{C}\right)q = E_0\cos\omega t$
22–78. $L\ddot{q} + R\dot{q} + \left(\dfrac{2}{C}\right)q = 0$
22–79. $L\ddot{q} + R\dot{q} + \dfrac{1}{C}q = 0$

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Fundamental Equations of Dynamics

KINEMATICS

Particle Rectilinear Motion

Variable a	*Constant $a = a_c$*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_c t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$v = \dot{s}$	$a_t = \dot{v} = v\dfrac{dv}{ds}$
	$a_n = \dfrac{v^2}{\rho}$ $\quad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{\lvert d^2y/dx^2 \rvert}$

Relative Motion

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ $\qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Rigid Body Motion About a Fixed Axis

Variable α	*Constant $\alpha = \alpha_c$*
$\alpha = \dfrac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \dfrac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\,d\omega = \alpha\,d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For Point P

$s = \theta r \quad v = \omega r \quad a_t = \alpha r \quad a_n = \omega^2 r$

Relative General Plane Motion—Translating Axes

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A(\text{pin})}$

Relative General Plane Motion—Trans. and Rot. Axis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

KINETICS

Mass Moment of Inertia $\quad I = \int r^2\,dm$

Parallel-Axis Theorem $\quad I = I_G + md^2$

Radius of Gyration $\quad k = \sqrt{\dfrac{I}{m}}$

Equations of Motion

Particle	$\Sigma \mathbf{F} = m\mathbf{a}$
Rigid Body (Plane Motion)	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$

Principle of Work and Energy

$T_1 + \Sigma U_{1-2} = T_2$

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

Work

Variable force	$U_F = \int F\cos\theta\,ds$
Constant force	$U_F = (F_c\cos\theta)\,\Delta s$
Weight	$U_W = -W\,\Delta y$
Spring	$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right)$
Couple moment	$U_M = M\Delta\theta$

Power and Efficiency

$P = \dfrac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{U_{\text{out}}}{U_{\text{in}}}$

Conservation of Energy Theorem

$T_1 + V_1 = T_2 + V_2$

Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = +\frac{1}{2}ks^2$

Principle of Linear Impulse and Momentum

Particle	$m\mathbf{v}_1 + \Sigma\int \mathbf{F}\,dt = m\mathbf{v}_2$
Rigid Body	$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$

Conservation of Linear Momentum

$\Sigma(\text{syst. } m\mathbf{v})_1 = \Sigma(\text{syst. } m\mathbf{v})_2$

Coefficient of Restitution

$e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

Principle of Angular Impulse and Momentum

Particle	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = (d)(mv)$
Rigid Body (Plane motion)	$(\mathbf{H}_G)_1 + \Sigma\int \mathbf{M}_G\,dt = (\mathbf{H}_G)_2$ where $H_G = I_G\omega$ $(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = I_O\omega$

Conservation of Angular Momentum

$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$

SI Prefixes

Multiple	*Exponential Form*	*Prefix*	*SI Symbol*
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (FPS) to (SI)

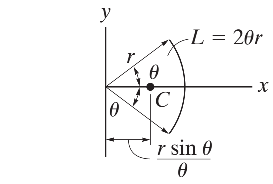
Quantity	*Unit of Measurement (FPS)*	*Equals*	*Unit of Measurement (SI)*
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

Conversion Factors (FPS)

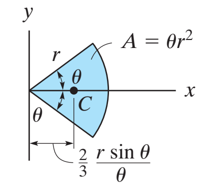
1 ft = 12 in. (inches)
1 mi. (mile) = 5280 ft
1 kip (kilopound) = 1000 lb
1 ton = 2000 lb

Geometric Properties of Line and Area Elements

Centroid Location	Centroid Location	Area Moment of Inertia



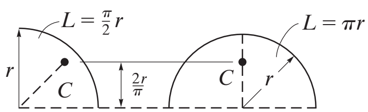
Circular arc segment



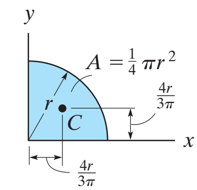
Circular sector area

$$I_x = \frac{1}{4} r^4 \left(\theta - \frac{1}{2} \sin 2\theta\right)$$

$$I_y = \frac{1}{4} r^4 \left(\theta + \frac{1}{2} \sin 2\theta\right)$$



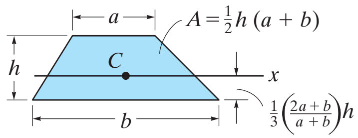
Quarter and semicircle arcs



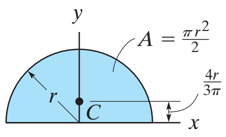
Quarter circle area

$$I_x = \frac{1}{16}\pi r^4$$

$$I_y = \frac{1}{16}\pi r^4$$



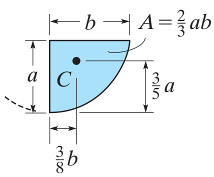
Trapezoidal area



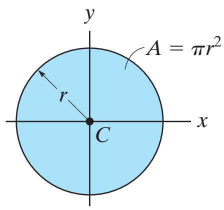
Semicircular area

$$I_x = \frac{1}{8}\pi r^4$$

$$I_y = \frac{1}{8}\pi r^4$$



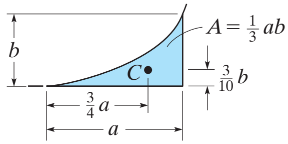
Semiparabolic area



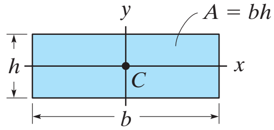
Circular area

$$I_x = \frac{1}{4}\pi r^4$$

$$I_y = \frac{1}{4}\pi r^4$$



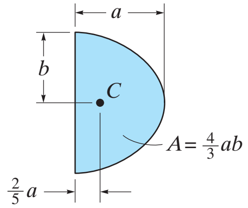
Exparabolic area



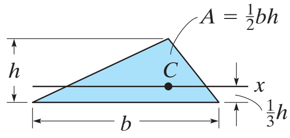
Rectangular area

$$I_x = \frac{1}{12}bh^3$$

$$I_y = \frac{1}{12}hb^3$$



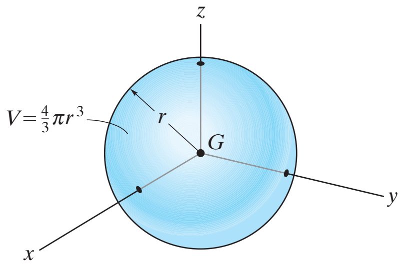
Parabolic area



Triangular area

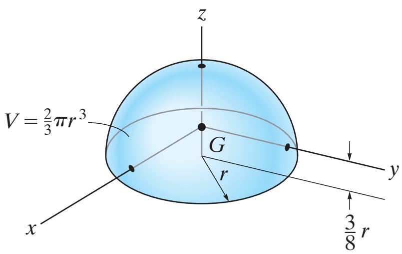
$$I_x = \frac{1}{36}bh^3$$

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



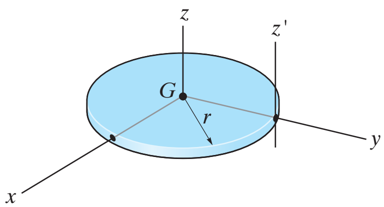
Sphere

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$$



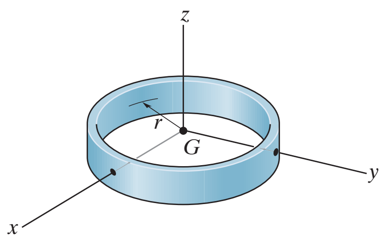
Hemisphere

$$I_{xx} = I_{yy} = 0.259\, mr^2 \qquad I_{zz} = \frac{2}{5} mr^2$$



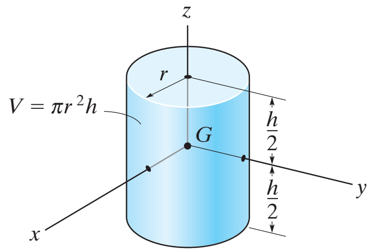
Thin Circular disk

$$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \qquad I_{zz} = \frac{1}{2} mr^2 \qquad I_{z'z'} = \frac{3}{2} mr^2$$



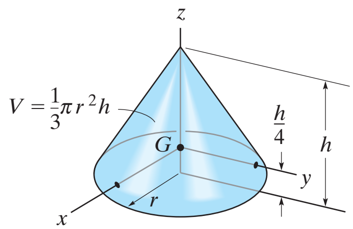
Thin ring

$$I_{xx} = I_{yy} = \frac{1}{2} mr^2 \qquad I_{zz} = mr^2$$



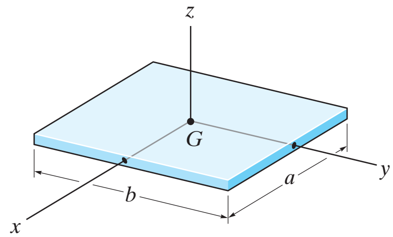
Cylinder

$$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2) \qquad I_{zz} = \frac{1}{2} mr^2$$



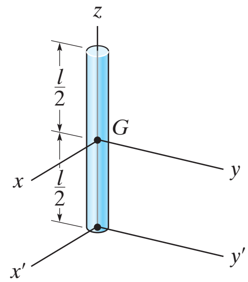
Cone

$$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2) \qquad I_{zz} = \frac{3}{10} mr^2$$



Thin plate

$$I_{xx} = \frac{1}{12} mb^2 \qquad I_{yy} = \frac{1}{12} ma^2 \qquad I_{zz} = \frac{1}{12} m(a^2 + b^2)$$



Slender Rod

$$I_{xx} = I_{yy} = \frac{1}{12} ml^2 \qquad I_{x'x'} = I_{y'y'} = \frac{1}{3} ml^2 \qquad I_{z'z'} = 0$$